kzn education
Department:
Education
KWAZULU-NATAL

MATHEMATICS

MATRIC INTERVENTION PROGRAMME

GRADE 12

2019

Contents

1. CALCULUS

- Investigate and use instantaneous rate of change of a variable when interpreting models of situations:
- Demonstrating an intuitive understanding of the limit concept in the context of approximating the rate of change or gradient at a point;
- Establishing the derivatives of the following functions from first principles:

$f(x) = b$; $f(x) = ax + b$; $f(x) = ax^2 + bx + c$; $f(x) = ax^3$; $f(x) = \frac{a}{x}$ a, b and c are $\in R$ and

then generalise to the derivative of $f(x) = x^n$.

- Differentiate by using the power rule (If $f(x) = ax^n$, then $f'(x) = an.x^{n-1}$)

Use the following rules of differentiation:

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] \; ; \; \frac{d}{dx}\left[k.f(x)\right] = k.\frac{d}{dx}[f(x)]$$

Examples:

Differentiate :

(a) $f(x) = \sqrt[3]{x^2} + \frac{1}{2x^4} - 1$

(b) $y = \frac{t^2 - 3t + 2}{t - 2}$

(c) $f(x) = (x^2 - 2)(x^{\frac{1}{3}} + 1)$

(d) $y = \frac{2x^5 - 3x + 1}{x}$

NOTE:
The following notations can be used:
1. $f'(x)$
2. D_x
3. $\frac{dy}{dx}$
4. y'

Applications of Calculus:

- Determine the equations of tangents to curves.
- Generate sketch graphs of cubic functions using differentiation to determine the stationary points mxima, minima and points of inflection) and use the factor theorem or other techniques (synthetic division) to determine the intercepts with the *x*-axis.
- Candidates are further expected to be able to interpret cubic functions

a. By determining the equation of a cubic function from a given graph.

b. Using the second derivative to determine a point of inflection where applicable.

c. Integration with transformation of curves and points.

Candidates are expected to interpret the graph of the derivative of a function

KEY QUESTIONS

✔rate of change or gradient of a function

✔derivatives from 1st principles

✔using differentiation rules

✔equations of tangents/Normal

✔cubic graphs – Sketching and interpretation, Derivative graphs

✔maxima and minima – Areas/Volumes/Perimeter/Word Problems

HINTS TO LEARNERS:

1. Differentiate by using the power rule (If $f(x) = ax^n$, then $f'(x) = an.x^{n-1}$)
2. Don't forget, derivative notation, $f'(x)$
3. Simplify the expression first, removing any surds, any quotients etc.
4. The following simplification rule will prove helpful : $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ and

that $xy + 4y = 2x^2 + 5x - 12$ *is the same as* $y(x+4) = (2x-3)(x+4)$ *hence* $y = 2x-3,\ x \neq 4$ *also*
$y = \frac{2x^3 + 2\sqrt[3]{x} - 6}{2x} = \frac{2x^3}{2x} + \frac{2\sqrt[3]{x}}{2x} - \frac{6}{2x} + x^{\frac{-2}{3}} - \frac{6}{2x} = x^2 + x^{\frac{-2}{3}} - 3x^{-1}$

5. It useful to know, $\sqrt[3]{x^5} = x^{\frac{5}{3}}$
6. Cubic Graph
7. **Point of Inflection:**

We can use the fact that the x-co-ordinate of the point of inflection is half way between the two critical values of the graph of f. So $x = \frac{x_A + x_B}{2}$, if A and B are turning points of $f(x)$. So we first find the x-intercept of T(critical value), then substitute this in f(x) and find the y value. The x value of the point of inflection is obtained by solving the equation $f''(x) = 0$;

Eg, $f'(x) = -6x^2 - 6x + 12$; $f''(x) = -12x - 6; \quad 12x + 6 = 0; \quad x = -\frac{1}{2}$

$$f''(x) < 0 \qquad f''(x) > 0$$

$$x = -\frac{1}{2}$$

$f''(x)$ changes sign at $x = -\frac{1}{2}$ ∴ point of inflection at $x = -\frac{1}{2}$

8. **Concavity**

For all x such that $f''(x) > 0$, a graph will be concave up and for all x such that $f''(x) < 0$, a graph will be concave down.

9. At the point of inflection of a cubic function the concavity changes.

CALCULUS - EXERCISES

Question 1

1.1 Determine $f'(x)$ from first principles if $f(x)=2x^2-5$. (5)

1.2 Evaluate $\frac{dy}{dx}$ if $y=x^{-4}+2x^3-\frac{x}{5}$. (3)

1.3 Given: $g(x)=\frac{x^2+x-2}{x-1}$

1.3.1 Calculate $g'(x)$ for $x \neq 1$. (2)

1.3.2 Explain why it is not possible to determine $g'(1)$. (1)

Question 2

2.1 Use the definition to differentiate $f(x)=1-3x^2$. (Use first principles.) (4)

2.2 Calculate $D_x\left[4-\frac{4}{x^3}-\frac{1}{x^4}\right]$. (3)

2.3 Determine $\frac{dy}{dx}$ if $y=\left(1+\sqrt{x}\right)^2$. (3)

Question 3

3.1 Differentiate f by first principle where $f(x) = x^2 - 2x$ (5)

3.2 Evaluate:

3.2.1 $D_x[(x^3-3)^2]$ (3)

3.2.2 $\frac{dy}{dx}$ if $y = \frac{4}{\sqrt{x}} - \frac{x^3}{9}$ (3)

Question 4

4.1 Determine $f'(x)$ from first principle if $f(x) = x^2 - 5$ (5)

4.2 Determine the derivative of : $g(x) = 5x^2 - \frac{2x}{x^3}$ (3)

4.3 Given: $h(x) = ax^2, x > 0$

Determine the value of a if it is given that $h^{-1}(8) = h'(4)$ (4)

Question 5

5.1 Differentiate f from first principle: $f(x) = -\frac{3}{x}$ (5)

5.2 If $f(x) = (x+3)^2$, determine:

5.2.1 $f'(x)$ (3)

5.2.2 the gradient of the tangent to $f(x)$ at $x = 3$ (2)

5.3 Determine: $D_x[6\sqrt[3]{x^2} + \frac{4}{x} - \pi x^3]$ (4)

5.4 It is given that $f(x) = ax^3 - 24x + b$ has a local minimum at $(-2; 17)$. Calculate the values of a and b. (4)

Question 6

6.1 Given: $f(x) = -\frac{x^2}{2} - 3$

Determine $f'(x)$ from first principle (5)

6.2 Determine:

6.2.1 $D_x[(x^2 - 2)\left(\frac{1}{x^2} + 3\right)]$ (4)

6.2.2 $\frac{dy}{dx}$ if $y = 4\sqrt{x} - \frac{8}{\sqrt{8}} - \pi x^3$ (3)

Question 7

7.1 Determine $f'(x)$ from first principles if (5)+(3)

7.1.1 $f(x) = -2x^3$ 7.1.2 $f(x) = 8$

7.2 Evaluate:

7.2.1 $\frac{dy}{dx}$ if $y = \frac{3}{2x} - \frac{x^2}{2}$ (3)

7.2.2 $f'(1)$ if $f(x) = (7x+1)^2$ (4)

Question 8

8.1 Determine $\frac{dy}{dx}$ if

8.1.1 $y=(2x)^2-\frac{1}{3x}$ (3)

8.1.2 $y=\frac{2\sqrt{x}-5x^2}{\sqrt{x}}$ (4)

8.2 The graph $h(x)=ax^3+px$ passes through the point $(3;-2)$. The gradient of the tangent to h at (0;0) is 3.

8.2.1 Determine the values of a and p. (4)

8.2.2 Determine the gradient of the tangent to h at $x=3$. (2)

Question 9

9.1 Determine $f'(x)$ from first principle if $f(x)=3x^2$ (5)

9.2 John determine $g'(a)$, the derivative of a certain function g at $x=a$, and at the answer:

$\frac{\sqrt{4+h}-2}{h}$.

Write down the equation of g and the value of a (2)

9.3 Determine $\frac{dy}{dx}$ if $y=\sqrt{x^3}-\frac{5}{x^3}$ (4)

9.4 $g(x)=-8x+20$ is a tangent to $f(x)=x^3+ax^2+bx+18$ at $x=1$.

Calculate values of a and b. (5)

APPLICATION OF CALCULUS

Question 10

Given: $g(x) = (x-6)(x-3)(x+2)$

10.1 Calculate the y-intercept of g. (1)

10.2 Write down the x-intercepts of g. (2)

10.3 Determine the turning points of g. (6)

10.4 Sketch the graph of g . (4)

10.5 For which values of x is $g(x) . g^{/}(x) < 0$? (3)

[16]

Question 11

Given: $f(x) = -x^3 + x^2 + 8x - 12$

11.1 Calculate the x-intercepts of the graph of f. (5)

11.2 Calculate the coordinates of the turning points of the graph of f. (5)

11.3 Sketch the graph of f, showing clearly all the intercepts with the axes and turning points. (3)

11.4 Write down the x-coordinate of the point of inflection of f. (2)

11.5 Write down the coordinates of the turning points of $h(x) = f(x) - 3$. (2)

[17]

Question 12

Given: $f(x) = x(x-3)^2$ with $f'(1) = f'(3) = 0$ and $f(1) = 4$

12.1 Show that f has a point of inflection at $x = 2$ (5)

12.2 Sketch the graph of f, clearly indicating the intercepts with the axes and the turning points (4)

12.3 For which values of x will $y = -f(x)$ be concave down? (2)

12.4 Use your graph to answer the following questions:

12.4.1 Determine the coordinates of the local maximum of h if

$h(x) = f(x-2) + 3$ (2)

12.4.2 Claire claims that $f'(2) = 1$.

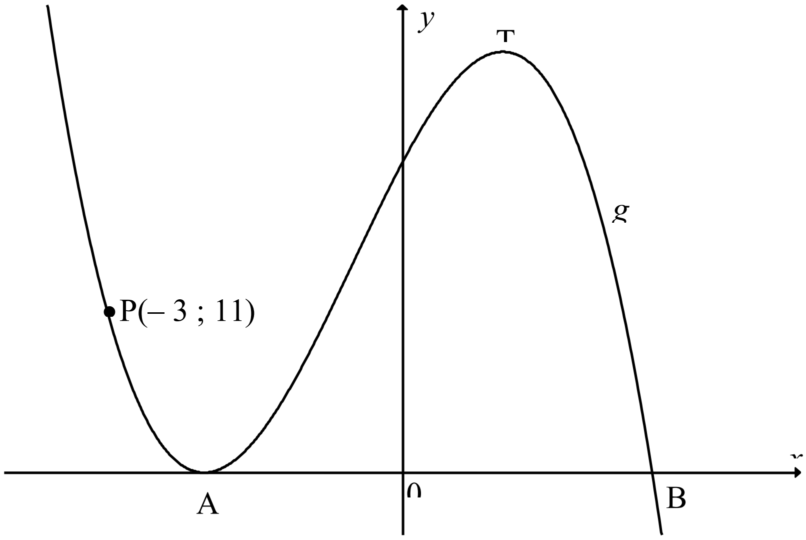
Do you agree with Claire? Justify your answer. (2)

Question 13

Sketched below is the graph of $g(x) = -2x^3 - 3x^2 + 12x + 20 = -(2x-5)(x+2)^2$

A and T are turning points of g. A and B are the x-intercepts of g.

P(– 3 ; 11) is a point on the graph.



13.1 Determine the length of AB. (2)

13.2 Determine the x-coordinate of T (4)

13.3 Determine the equation of the tangent to g at P(– 3 ; 11), in the form $y = \ldots$ (5)

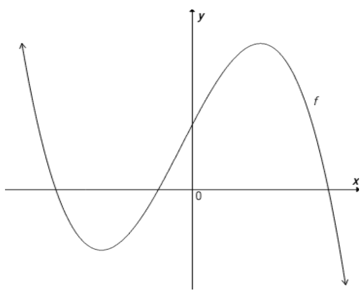
13.4 Determine the value(s) of k for which $-2x^3 - 3x^2 + 12x + 20 = k$ has three distinct roots. (3)

13.5 Determine the x-coordinate of the point of inflection. (4)

[18]

Question 14

14.1 The graph of the function $f(x) = -x^3 - x^2 + 16x + 16$ is sketched below.



14.1.1 Calculate the x-coordinates of the turning points of f. (4)

14.1.2 Calculate the x-coordinates of the point at which $f'(x)$ is a maximum (3)

14.2 Consider the graph of $g(x) = -2x^2 - 9x + 5$.

14.2.1 Determine the equation of the tangent to the graph of g at $x = -1$ (4)

14.2.2 for which values of q will the line $y = -5x + q$ not intersect the parabola?

14.3 Given: $h(x) = 4x^3 + 5x$

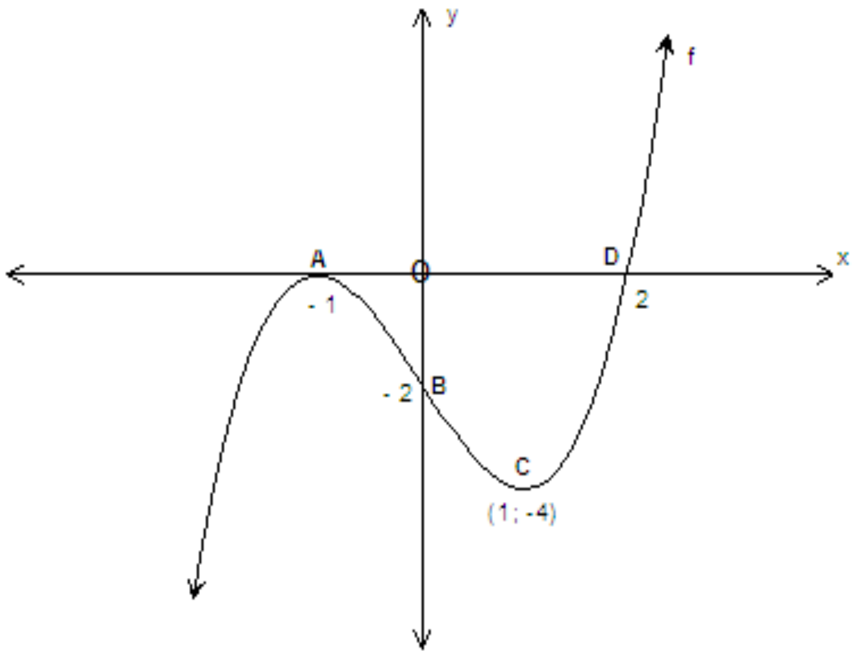
Explain if it is possible to draw a tangent to the graph of h that has a negative

Gradient. Show ALL calculations.

(3)

Question 15

The sketch below has A*(-1 ; 0)* and C(1; -4) of the graph of *f*. B(*0; -2*) and D(*2 : 0*) are the*y*- and the *x*- intercepts



15.1 Without determining the equation of *f* give the value(s) of *x* in each of the following:

(a) *f(x) =0* (2)

(b) *f'(x) = 0* (2)

(c) *f''(x).f(x) ≥ 0 and x*< 0 (3)

15.2 Draw the graph of f''(x) indicating the y- and the x-intercepts. (3)

Question 16

Given: $h(x) = -x^3 + ax^2 + bx$ and $g(x) = -2x$. P and Q(2;10) are the turning points of h. The graph of h

Passes through the origin.

16.1 Show that $a = \frac{3}{2}$ and $b = 6$ (5)

16.2 Calculate the average gradient of h between P and Q if it given that $x = -1$ at P

16.3 Show that the concavity of h changes at $x = \frac{1}{2}$ (3)

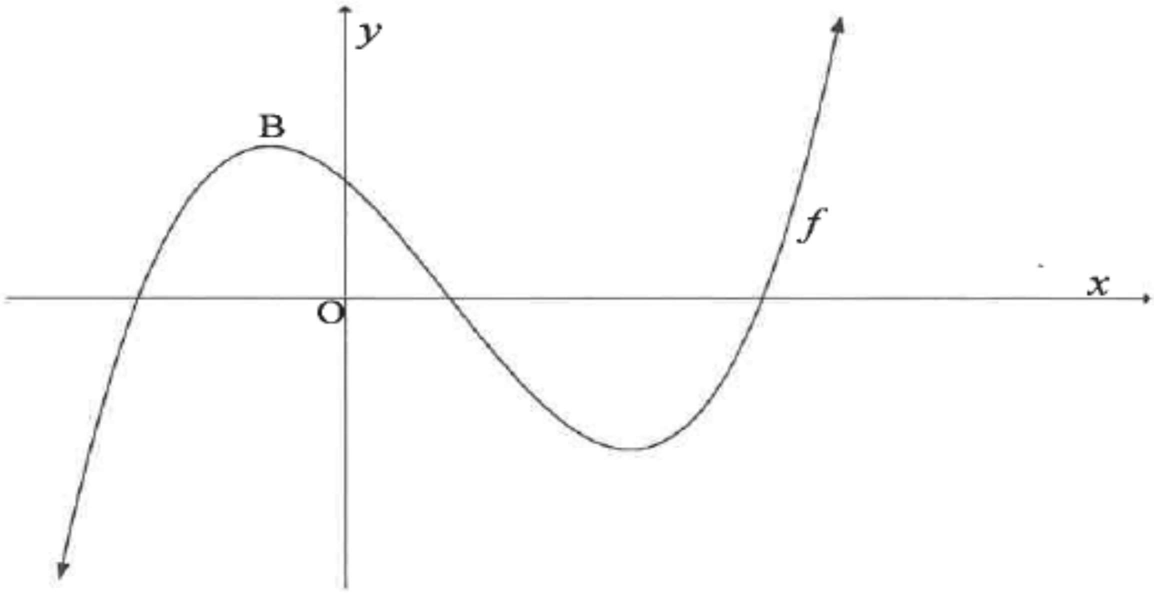
16.4 Explain the significance of the change in Question 16.3 with respect to h (1)

16.5 Determine the value of x, given $x < 0$, at which the tangent to h is parallel to g (4)

Question 17

The sketch below represents the curve of $f(x) = x^3 + bx^2$+cx+d. The solutions of the

Equation $f(x) = 0$ are -2 ; 1 and 4.



17.1 Calculate th values of b, c and d. (4)

17.2 Calculate the x −coordinate of B, the maximum turning point of f. (4)

17.3 Determine an equation for the tangent to the graph of f at $x = -1$ (4)

17.4 Sketch the graph of $f''(x)$. Clearly indicate the $x -$ and $y -$intercepts on

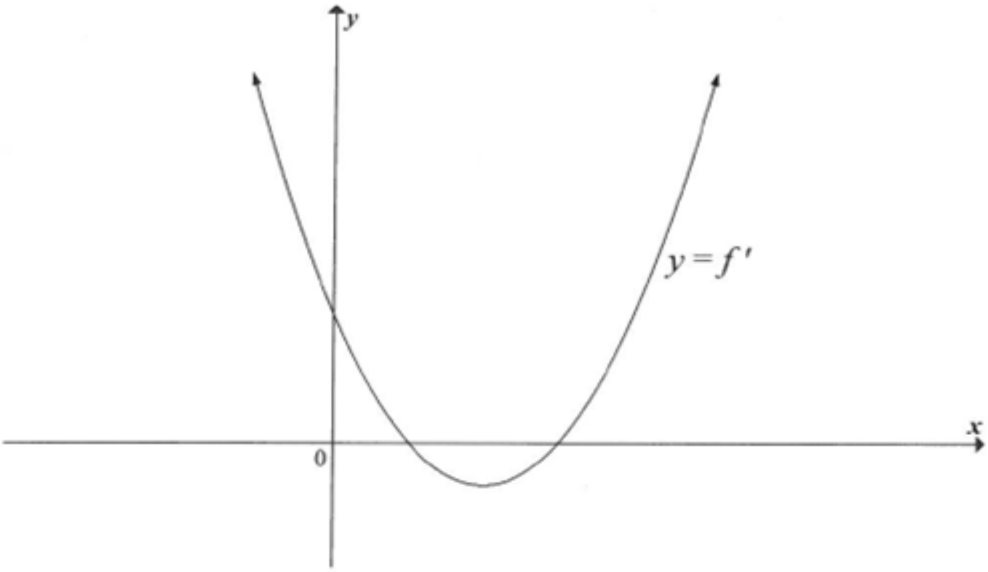
Your sketch. (3)

17.5 For which value(s) of x is $f(x)$ concave upwards? (2)

Question 18

The graph of $y = ax^2 + bx + c$ below represents the derivative of f.

It is given that $f'(1) = 0$, and $f'(3) = 0$ and $f'(0) = 6$.



18.1 Write down the x-coordinates of the stationary points of f. (2)

18.2 For which value(s) of x is f strictly decreasing? (2)

18.3 Explain at which value of x the stationary point of f will be a local minimum. (2)

18.4 Determine the x-coordinate of the point of inflection of f. (1)

18.5 For which value(s) of x is f concave up? (2)

Question 19

For a certain function f, the first derivative is given as $f'(x) = 3x^2 + 8x - 3$

19.1 Calculate the x-coordinates of the the stationary points of f. (3)

19.2 For which values of x is f concave down? (3)

19.3 Determine the values of x for which f is strictly increasing (2)

19.4 If it is further given that $f(x) = ax^3 + bx^2 + cx + d$ and $f(0) = -18$, determine the Equation of f. (5)

Question 20

A particle moves along a straight line. The distance, s, (in metres) of the particle from a fixed point on the line at time t seconds ($t \geq 0$) is given by $s(t) = 2t^2 - 18t + 45$.

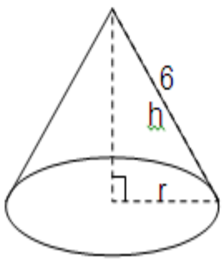
20.1 Calculate the particle's initial velocity. (Velocity is the rate of change of distance.) (3)

20.2 Determine the rate at which the velocity of the particle is changing at t seconds. (1)

20.3 After how many seconds will the particle be closest to the fixed point? (2)

[6]

Question 21



The slant height of a cone is 6cm.If you change the shape of the cone, keeping the slant height at 6cm, the volume changes.

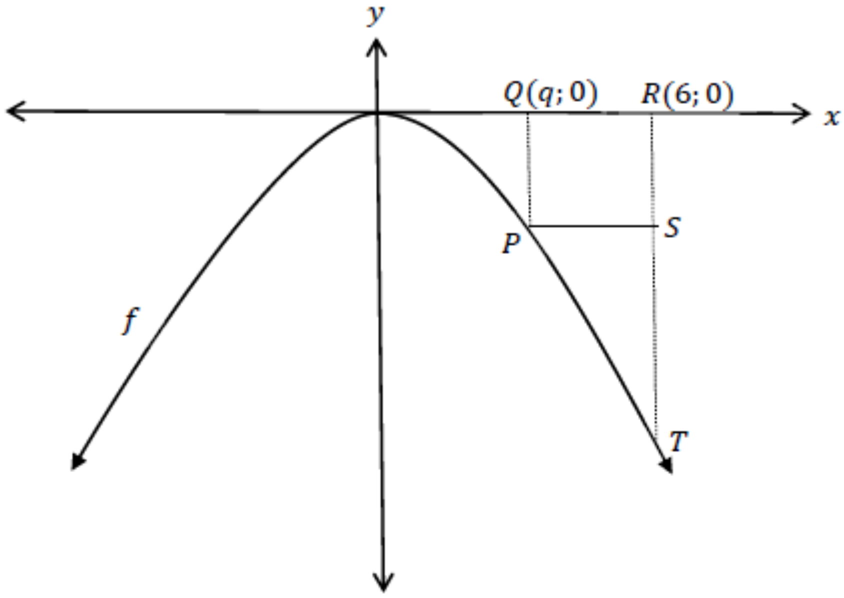
[Volume of a cone = $\frac{1}{3}$ area of base x h]

21.1 Find r^2 in terms of the height. (2)

21.2 Show the volume is given by $V = 12\pi h - \frac{\pi}{3}h^3$ (2)

Question 22

The graph below shows the sketch of $f(x) = -2x^2$. R is the point (6;0) and Q is the Point (q;0). P and T are points on f. RST is parallel to the y-axis and PS is parallel to the x-axis. $PQRS$ is a rectangle



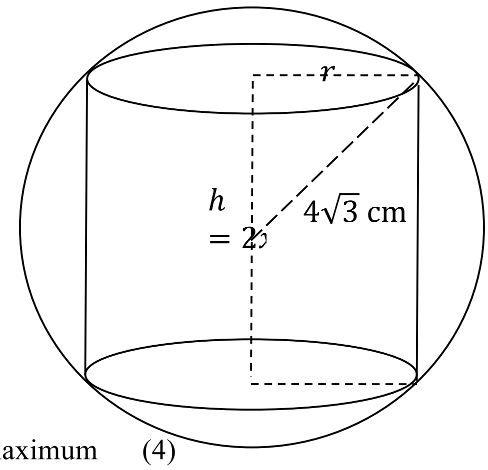
22.1 Write down the coordinates of P in terms of q. (1)

22.2 Show that the area (A) of rectangle PQRS can be expressed as: $A = 12q^2 - 2q^3$. (2)

22.3 Determine the maximum area of rectangle PQRS. (4)

Question 23

A cylinder fits nicely into a sphere of radius $4\sqrt{3}$ cm. Volume of cylinder = $\pi r^2 h$



23.1 If the height of the cylinder is 2x, show that the the radius r of the cylinder is given by r =$\sqrt{48 - x^2}$ cm (2)

23.2 Hence, show that the volume of the cylinder in terms of x is V = 96π - $2\pi x^2$ (3)

23.3 Calculate the height of the cylinder so that the volume is a maximum (4)

QUESTION 24

The number of molecules of a certain drug in the bloodstream t hours after it has been taken is represented by the equation $M(t) = -t^3 + 3t^2 + 72t$, $0 < t < 10$.

24.1 Determine the number of molecules of the drug in the bloodstream 3 hours after the drug was taken. (2)

24.2 Determine the rate at which the number of molecules of the drug in the bloodstream is changing at exactly 2 hours after the drug was taken. (3)

24.3 How many hours after taking the drug will the rate at which the number of molecules of the drug in the bloodstream is changing, be a maximum? (3)

2. NUMBER PATTERNS

QUESTION 3 (March 2009)

The following is an arithmetic sequence:

$$1-p \;\;;\;\; 2p-3 \;\;;\;\; p+5 \;\;;\; \ldots$$

3.1 Calculate the value of p. (3)

3.2 Write down the value of:

3.2.1 The first term of the sequence (1)

3.2.2 The common differe (1)

3.3 Explain why none of the numbers in this arithmetic sequence are perfect squares. (2)

[7]

QUESTION 4 (March 2009)

Consider the sequence: $6 \;;\; 6 \;;\; 2 \;;\; -6 \;;\; -18 \;;\; \ldots$

4.1 Write down the next term of the sequence, if the sequence behaves consistently. (1)

4.2 Determine an expression for the n^{th} term, T_n. (5)

4.3 Show that -6838 is in this sequence. (4)

[10]

QUESTION 3 (Nov 2009)

Given: $\sum_{t=0}^{99} (3t-1)$

3.1 Write down the first THREE terms of the series. (1)

3.2 Calculate the sum of the series. (4)

[5]

QUESTION 4 (Nov 2009)

The following sequence of numbers forms a quadratic sequence:

$-3 ; -2 ; -3 ; -6 ; -11 ; \ldots$

4.1 The first differences of the above sequence also form a sequence. Determine an expression for the general term of the first differences. (3)

4.2 Calculate the first difference between the 35^{th} and 36^{th} terms of the quadratic sequence (2)

4.3 Determine an expression for the n^{th} term of the quadratic sequence. (4)

4.4 Explain why the sequence of numbers will never contain a positive term. (2)

[11]

QUESTION 2 (March 2010)

Consider the following sequence: 399 ; 360 ; 323 ; 288 ; 255 ; 224 ; …

2.1 Determine the n^{th} term T_n in terms of n. (6)

2.2 Determine which term (or terms) has a value of 0. (3)

2.3 Which term in the sequence will have the lowest value? (1)

[10]

QUESTION 2 (Mar 2011)

The sequence 3 ; 9 ; 17 ; 27 ; … is a quadratic sequence.

2.1 Write down the next term. (1)

2.2 Determine an expression for the n^{th} term of the sequence. (4)

2.3 What is the value of the first term of the sequence that is greater than 269? (4)

[9]

QUESTION 4 (Mar 2011)

4.1 The sum to n terms of a sequence of numbers is given as: $\frac{n}{2}(5n+9)$

4.1.1 Calculate the sum to 23 terms of the sequence. **(2)**

4.1.2 Hence calculate the 23^{rd} term of the sequence. **(3)**

QUESTION 2 (Nov2011)

2.1 Given the sequence: 4 ; x ; 32

Determine the value(s) of x if the sequence is:

2.1.1 Arithmetic **(2)**

2.1.2 Geometric **(3)**

2.2 Determine the value of P if $P=\sum_{k=1}^{13} 3^{k-5}$ **(4)**

2.1 $3x + 1$; $2x$; $3x - 7$ are the first three terms of an arithmetic sequence. Calculate the value of x. **(2)**

2.2 The first and second terms of an arithmetic sequence are 10 and 6 respectively.

2.2.1 Calculate the 11^{th} term of the sequence. **(2)**

2.2.2 The sum of the first n terms of this sequence is -560. Calculate n. **(6)**

[10]

QUESTION 3 (Nov 2012)

3.1 Given the geometric sequence: 27 ; 9 ; 3 …

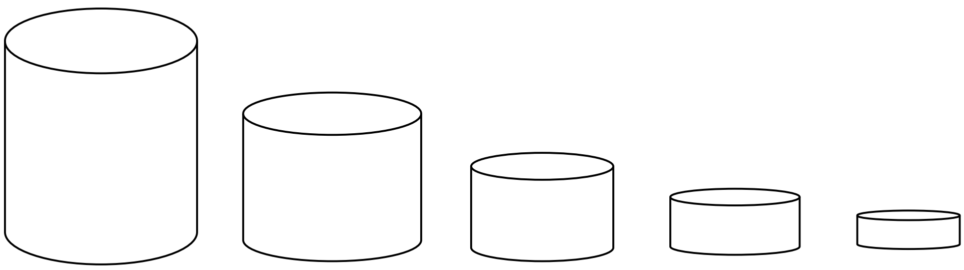
3.1.1 Determine a formula for T_n, the n^{th} term of the sequence. **(2)**

3.1.2 Why does the sum to infinity for this sequence exist? **(1)**

3.1.3 Determine S_∞. **(2)**

3.2

Twenty water tanks are decreasing in size in such a way that the volume of each tank is $\frac{1}{2}$ the volume of the previous tank. The first tank is empty, but the other 19 tanks are full of water.



Would it be possible for the first water tank to hold all the water from the other 19 tanks?

Motivate your answer. **(4)**

3.3 The n^{th} term of a sequence is given by $T_n = -2(n-5)^2 + 18$.

3.3.1 Write down the first THREE terms of the sequence. (3)

3.3.2 Which term of the sequence will have the greatest value? (1)

3.3.3 What is the second difference of this quadratic sequence? (2)

3.3.4 Determine ALL values of n for which the terms of the sequence will be less than -110. (6)

[21]

Question 3 (WC June 2011)

3.1 Given the sequence 2; 5; 8; 11; …

3.1.1 Determine the 250^{th} term. (3)

3.1.2 Which term is 302? (3)

3.1.3 How many terms must be added to obtain a sum of 610? (5)

3.2 Given the geometric series: $5.(3)^4 + 5.(3)^3 + 5.(3)^2 + \ldots\ldots$

3.2.1 Explain why the series converges. (2)

3.2.2 Calculate the sum to infinity of the series. (3)

3.2.3 Calculate the sum of the first 9 terms of the series, correct to **TWO** decimal places. (4)

Question 4 (WC June 2011)

Consider the number pattern: 6 ; 10 ; 16 ; 24 ; 34 ; …

4.1 If the pattern behaves consistently, determine the next TWO terms of the pattern. (2)

4.2 Determine the general term of this pattern. (6)

4.3 Calculate n if the n^{th} term in the pattern is 1264. (5)

[13]

Question 2 (March 2013)

2.1 Given the geometric series: $256 + p + 64 - 32 + \ldots$

2.1.1 Determine the value of p. (

2.1.2 Calculate the sum of the first 8 terms of the series. (

2.1.3 Why does the sum to infinity for this series exist? (

2.1.4 Calculate S_{∞} (

2.2 Consider the arithmetic sequence: $-8 ; -2 ; 4 ; 10 ; \ldots$

2.2.1 Write down the next term of the sequence. (

2.2.2 If the n^{th} term of the sequence is 148, determine the value of n. (

2.2.3 Calculate the smallest value of n for which the sum of the first n terms of the sequence will be greater than 10 140. (

2.3 Calculate $\sum_{k=1}^{30}(3k+5)$ (

[

Question 3 (March 2013)

Consider the sequence: 3 ; 9 ; 27 ; ...

Jacob says that the fourth term of the sequence is 81.
Vusi disagrees and says that the fourth term of the sequence is 57.

3.1 Explain why Jacob and Vusi could both be correct. (2)

3.2 Jacob and Vusi continue with their number patterns.

Determine a formula for the n^{th} term of:

3.2.1 Jacob's sequence (1)

3.2.2 Vusi's sequence (4)
[7]

Question 2 (Nov 2010 IEB)

(a) Given the arithmetic sequence: $21\ ;\ 20\frac{1}{4}\ ;\ 19\frac{1}{2}$

(1) Determine the n^{th} term T_n in terms of n. Simplify your answer. (3)

(2) Show that $T_{29-n} + T_{29+n} = 0$ for all values of n, $n \leq 28$, $n \in N_0$. (3)

(b) Evaluate:

$$\sum_{k=3}^{5} (-1)^k \frac{2}{k}$$ (3)

Question 9 (Nov 2010 IEB)

QUESTION 2

2.1 Prove that in any arithmetic series in which the first term is a and whose constant difference is d, the sum of the first n terms is $S_n = \frac{n}{2}[2a + (n-1)d]$. (4)

2.2 Calculate the value of $\sum_{k=1}^{50}(100 - 3k)$. (4)

2.3 A quadratic sequence is defined with the following properties:

$$T_2 - T_1 = 7$$
$$T_3 - T_2 = 13$$
$$T_4 - T_3 = 19$$

2.3.1 Write down the value of:

(a) $T_5 - T_4$ (1)

(b) $T_{70} - T_{69}$ (3)

2.3.2 Calculate the value of T_{69} if $T_{89} = 23\,594$. (5)

[17

QUESTION 3

Consider the infinite geometric series: $45 + 40{,}5 + 36{,}45 + \dots$

3.1 Calculate the value of the TWELFTH term of the series (correct to TWO decimal places). (3)

3.2 Explain why this series converges. (1)

3.3 Calculate the sum to infinity of the series. (2)

3.4 What is the smallest value of n for which $S_\infty - S_n < 1$? (5)

[11]

RONDENBOSCH JUNE 2015

Question 2

2.1 The sum of the first n terms of a sequence is given by

$S_n = 2n + 3n^2$

2.1.1 Calculate S_{20} (2)

2.1.2 Calculate T_{20} (3)

2.2 Consider the geometric series:

$64(x-2) + 16(x-2)^2 + 4(x-2)^3 + \cdots$

2.2.1 For which value(s) of x will the series converge? (4)

2.2.2 Determine the sum to infinity if $x = 5$. (3)

2.3 Given the first four terms of a quadratic sequence:

$x + 20$; 24 ; $x + 8$ and 14

Calculate T_n. Show **ALL** working. (9)

2.4 Evaluate

$$\sum_{n=2}^{52} 7n + 1$$

(4)

[25]

OAKHILL JUNE 2015

Question 2

2.1 Consider the arithmetic sequence: $4; 1; -2; -5; -8; \ldots$

a. Write down the next term in the sequence.

b. If the n^{th} term of the sequence is -461, determine the value of n. (3)

2.2 Given the geometric series: $3 + 2 + x + \frac{8}{9} + \frac{16}{27} + \cdots$

a. State the value of x. (1)

b. Write this infinite geometric series in sigma notation. (2)

c. Calculate S_∞ (2)

d. If $S_\infty - S_n = \dfrac{256}{729}$, determine the value of n. (5)

2.3 Show that there is no real value of n for which the sequence $1; x + 3; x - 1$ will be a geometric sequence. (4)

2.4 The first two terms of a geometric sequence and an arithmetic sequence a 7 and $7r$. The sum of the first three terms of the geometric sequence is 28 more than the sum of the first three terms of the arithmetic sequence.

a. State the third term of each sequence in terms of r. (2)

b. Determine TWO possible values of r (4)

[24]

QUESTION 2

(a) Write down the n^{th} term of the sequence: $\frac{1}{4}\ ;\ \frac{4}{9}\ ;\ \frac{9}{16}\ ;\ \frac{16}{25}\ ; \ldots$

(b) Given the geometric sequence: $\frac{3}{8}\ ;\ \frac{3}{4}\ ;\ \frac{3}{2} \ldots$

(1) Determine the value of the 10^{th} term.

(2) Determine which term has a value of 12 288.

(c) An arithmetic series has 21 terms. The first term is 3 and the last term is 53. I
the sum of the series.

n

QUESTION 2

(a) Given the sequence: 1,2 ; 12,3 ; 123,4 ; 1234,5 …
Write down the next two terms, assuming the pattern continues.

(b) Evaluate: $\displaystyle\sum_{k=2}^{5} \frac{3^{k-1}}{k}$

(c) In an arithmetic sequence, the sum of the third and fourth terms equals 167 and $T_{21} = -4$.

(1) Determine the value of the constant difference.

(2) Hence calculate the sum of the first 21 terms.

QUESTION 2

(a) Write down the next term of the number pattern:

$\frac{1}{2}; \frac{8}{9}; \frac{27}{28} \ldots$ (2)

(b) Given: 2 ; 6 ; k

Write down the value of k if the sequence is:

(1) arithmetic (1)

(2) geometric (1)

(c) Evaluate the sum of the infinite series: 5,6 + 3,36 + 2,016 + 1,2096 + ... (3)

(d) Given: 0 ; – 1 ; 1 ; 6 ; 14

(1) Show that this sequence has a constant second difference. (2)

(2) Determine a simplified expression for the n^{th} term of the sequence. (4)

(3) Find the 30$^{\text{th}}$ term. (2)

[15]

UV p1 2015

QUESTION 2

(a) Write down the nth term of the sequence $\frac{1}{4};\frac{2}{9};\frac{3}{16};\frac{4}{25};\ldots$ (2)

(b) Given the arithmetic series: 20+18+16+…

Determine:

(1) the 100$^{\text{th}}$ term (2)

(2) the value(s) of n if $S_n = 80$ (4)

The sum of the first n terms of a series is given by the formula $S_n = 3^{n-1} + 9$

(3) Determine the sum of the first 6 terms. (1)

(4) Determine the first 3 terms of the sequence. (3)

QUESTION 2 [6]

The price of a rugby test ticket has increased over the past years as follows :

2009	2010	2011	2012	2013
100	140	200	280	380

(a) If the pattern continues in the same way, what will the price of the ticket be this year, i.e. in 2015? (1)

(b) Consider the number sequence : 100 ; 140 ; 200 ; 280 ; 380 ; .....

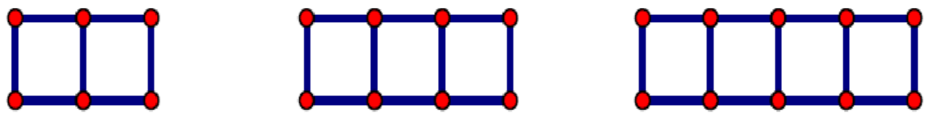
(i) What type of sequence is represented? (1)

(ii) Determine the n^{th} term of the sequence. (4)

QUESTION 2

(a) Given the sequence: 4; 7; 10; 13; 16 …
Assuming that this pattern remains consistent, write down an expression for the n^{th} term of the sequence. (2)

(b) The first three figures of a pattern comprising dots and sticks are shown.



(1) Write down the number of dots (d) and the number sticks (s) in the fourth figure. (2)

(2) Determine expressions in terms of n for each of the number of dots and the number of sticks in the n^{th} figure. (4)

(3) 100 dots are available to make a figure in the sequence. Calculate how many sticks will be needed for this figure. (4)

(c) An athlete runs 20 km on a certain Monday.
Thereafter, he increases the distance by 10% every day.

Calculate the number of kilometres he ran:

(1) on the following Saturday. (2)

(2) altogether over the 6 days. (3)

[17]

QUESTION 3

(a) The first three terms of an arithmetic sequence are:

$10-x\ ;\ 2x+3\ ;\ 4x+1$

(1) Calculate the value of x. (3)

(2) Find S_{23}. (3)

(b) If $S_n = 2n^2 + 3n$ then:

(1) Calculate T_{15}. (3)

(2) Find T_n in its simplest form. (4)

(c) Calculate the value of $\sum_{n=3}^{6}\left(2n^2 - 1\right)$. (3)

(d) In a converging geometric series $S_\infty = \frac{40}{3}$ and $T_2 = \frac{5}{2}$; calculate the possible value(s) of the first term in the series. (6)

[22]

2013 June Hg SCE

QUESTION 5

5.1 Given: 1 ; 4 ; 7 ; 10 ; ... is an arithmetic sequence.

Determine:

5.1.1 The 66^{th} term of the sequence (3)

5.1.2 The sum of the first 33 even terms of the sequence (6)

QUESTION 2

Given the arithmetic series: $2 + 9 + 16 + \ldots$ (to 251 terms).

2.1 Write down the fourth term of the series. (1)

2.2 Calculate the 251^{st} term of the series. (3)

2.3 Express the series in sigma notation. (2)

2.4 Calculate the sum of the series. (2)

2.5 How many terms in the series are divisible by 4? (4)

[12]

QUESTION 3

3.1 Given the quadratic sequence: $-1\,; -7\,; -11\,; p\,; \ldots$

3.1.1 Write down the value of p.

3.1.2 Determine the n^{th} term of the sequence.

3.1.3 The first difference between two consecutive terms of the sequence is
Calculate the values of these two terms.

3.2 The first three terms of a geometric sequence are: $16\,; 4\,; 1$

3.2.1 Calculate the value of the 12^{th} term. (Leave your answer in simpl
exponential form.)

3.2.2 Calculate the sum of the first 10 terms of the sequence.

3.3 Determine the value of: $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots$ up to 98 factors.

QUESTION 5

5.1 The first term of a sequence is 4. The fourth term is $\frac{1}{2}$. Determine the numerical value of the third term in each of the following cases:

5.1.1 The sequence is geometric.

5.1.2 The sequence is arithmetic.

5.2 The first three terms of an arithmetic sequence are $2x-1$; $3x$ and $5x-2$.

5.2.1 Determine the value of x.

5.2.2 Calculate the sum of the first 21 terms of the sequence.

5.3 Solve for n, given that $\sum_{k=1}^{n} 2^{k+1} = 4\,092$.

QUESTION 2

2.1 A geometric sequence has $T_3 = 20$ and $T_4 = 40$.

Determine:

2.1.1 The common ratio

2.1.2 A formula for T_n

2.2 The following sequence has the property that the sequence of numerators is arithme and the sequence of denominators is geometric:

$$\frac{2}{1};\frac{-1}{5};\frac{-4}{25};\ldots$$

2.2.1 Write down the FOURTH term of the sequence.

2.2.2 Determine a formula for the n^{th} term.

2.2.3 Determine the 500^{th} term of the sequence.

2.2.4 Which will be the first term of the sequence to have a NUMERATC which is less than -59?

Dbe HG P1 2012

QUESTION 5

5.1 In an arithmetic sequence the 7^{th} term is 17. The sum of the 1^{st} and 2^{nd} terms is 12.

Determine:

5.1.1 The first three terms of the sequence

5.1.2 The sum of the first 40 terms of the sequence

5.2 Determine the value of n if $\sum_{r=1}^{n}(5r-4)=235$.

5.3 The first term of a geometric sequence is 2 and the sum of the first 4 terms is 10 times the sum of the first 2 terms. The common ratio is greater than 1.

Determine the first three terms of the sequence.

5.4 A geometric sequence has first term 3 and common ratio $\frac{1}{3}$.

Show that for any value of n, $S_\infty - S_n = \frac{1}{2.3^{n-2}}$

QUESTION 3

3.1 Given: $\sum_{p=4}^{21}(-3)^p$

3.1.1 Write down the values of the first three terms of the series. (2)

3.1.2 Write down the value of the constant ratio. (1)

3.1.3 Will $\sum_{p=4}^{\infty}(-3)^p$ converge? Explain your answer. (2)

3.1.4 Calculate $\sum_{p=4}^{21}(-3)^p x$. Give your answer in terms of x. (3)

3.2 $6-x$, 5 and $\sqrt{4x+12}$ are the first three terms of an arithmetic sequence.

3.2.1 Determine the value of x. (5)

3.2.2 Calculate the value of the 10^{th} term of this arithmetic sequence. (3)

[16]

QUESTION 2

The first three terms of an arithmetic sequence are 4; 13 and 22.

2.1 Write down the fourth term of this sequence. (1)

2.2 Determine the general term of the sequence. (2)

2.3 Consider the terms of this sequence which are even.
Calculate the sum of the first 25 terms which are even. (4)

2.4 The original sequence (4; 13 and 22) forms the first differences of a new sequence with a first term equal to -6. Determine a formula for the n^{th} term of this new sequence. (4)

[11

[24]

QUESTION 2

2.1 The following arithmetic sequence is given: 20 ; 23 ; 26 ; 29; … ; 101

2.1.1 How many terms are there in this sequence? (2)

2.1.2 The even numbers are removed from the sequence.
Calculate the sum of the terms of the remaining sequence. (6)

2.2 Study the geometric series: $x + \frac{x^2}{3} + \frac{x^3}{9} + \cdots$ ....

2.2.1 Determine the n-th term in terms of x. (2)

2.2.2 Determine the value(s) of x for which the series will converge. (3)

2.3 The sum of n terms in a sequence is given by $S_n = -n^2 + 5$. Determine the 23rd term. (3)

[16]

QUESTION 3

The sequence 3 ; 9 ; 17 ; 27 ; … is quadratic.

3.1 Determine an expression for the n-th term of the sequence. (4)

3.2 What is the value of the first term of the sequence that is greater than 269? (4)

[8]

(SHARP WHK SHEET 1)

8. An arithmetic and geometric series both have the same first term, a = 9. The fifth term of the arithmetic series is equal to the second term of the geometric series minus 1. The sum of first three terms of the geometric series is equal to the twenty-eighth term of the arithmetic series.

a) Find the ratio and common difference for each of the series (if the common difference is a whole number).

b) Is the geometric series converging or diverging? If it is converging, determine the sum to infinity.

c) Find the sum of the first 15 numbers of the arithmetic series.

d) Will the sum of the first 5 geometric terms be equal to a term in the arithmetic series? If it is, what term is it?

(SHARP WH SHEET 1 CONT….)

10. The sum of the first 5 terms of an arithmetic series is 3. The fifth term is of the series is $\frac{2}{5}$.

a) Determine the first term and the common difference.

b) Which term will be equal to 0?

c) When will the sum of the arithmetic series be equal to 0?

11. If

$$\sum_{i=1}^{n}(5n+3) = 728$$

Determine the value of n.

12. For which values of x will

$$\sum_{i=1}^{x} 4.3^{n} < 4400$$

13. The eleventh term of an arithmetic series is 45 and the 8^{th} term in the arithmetic series is 33. Find the 15^{th} term of the series and the sum of the first 20 terms of the series.

14. A geometric series has a fourth term equal to 128 and a seventh term equal to $\frac{1024}{27}$. Find the sum to infinity of the series.

15. The first 10 terms of an arithmetic series sum to 465 while the sum of the first 20 terms have a sum of 430.

a) Find the first term and the common difference.

b) Which term is equal to -1?

c) For what value of n would the sum of the series be less than 0?

QUESTION 2

2.1 Given the following arithmetic sequence:

$14\ ;\ 9\ ;\ 4\ ;\ \ldots$

2.1.1 Determine the value of the 50th term. (3)

2.1.2 Calculate the sum of the first fifty terms. (2)

2.2 The following represents the first three terms of an arithmetic sequence:

$-24\ ;\ p\ ;\ p^2$

Calculate the value(s) of p. (4)

2.3 Consider the geometric series: $3 + m + \frac{m^2}{3} + \frac{m^3}{9} + \cdots$

2.3.1 For which value(s) of m will the series converge? (3)

2.3.2 It is given that: $3 + m + \frac{m^2}{3} + \frac{m^3}{9} + \cdots = \frac{27}{7}$

Calculate the value of m. (3)

2.4 The sum of the first three terms of a geometric sequence is $31\frac{1}{2}$. The sum of the fourth, fifth and sixth term of the same sequence is $3\frac{15}{16}$. Determine the value of the common ratio (r). (5)

[20]

QUESTION 2

The first two terms of an arithmetic series, A, and an infinite geometric series, B, are the same.

A: $-2 + x +$ .... and

B: $-2 + x +$ .... are given.

2.1 Write down in terms of x

2.1.1 The third term of the geometric series, B. (2)

2.1.2 The third term of the arithmetic series, A. (2)

2.2 If the sum of the first three terms in the arithmetic series A is equal to the third term of the geometric series B, then calculate the value of x. (5)

2.3 If $x = -6$, does the geometric series B converge?
Show calculations to support your answer. (3)

[12]

QUESTION 7 [18]

(a) The 5th term of arithmetic series is 14. The sum of the first 10 terms is 160.
Determine the first term and the common difference of the series. (5)

(b) log 2 and log 4 are the first 2 terms of an arithmetic as well as a geometric sequence.
Determine an expression for the nth term of each sequence. Simplify each answer to one term. (6)

(c) Determine the value of p if : $\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12}(24-3t)$ (7)

QUESTION 2 (NSC Nov 2016)

Given the finite arithmetic sequence: 5 ; 1 ; –3 ; ... ; –83 ; –87

2.1 Write down the fourth term (T_4) of the sequence. (1)

2.2 Calculate the number of terms in the sequence. (3)

2.3 Calculate the sum of all the negative numbers in the sequence. (3)

2.4 Consider the sequence: 5 ; 1 ; –3 ; ... ; –83 ; –87 ; … ; –4 187
Determine the number of terms in this sequence that will be exactly divisible by 5. (4)

[11]

QUESTION 2

Given the geometric sequence: $-\frac{1}{4}\ ;\ b\ ;\ -1\ ;......$

2.1 Calculate the possible values of b. (3)

2.2 If $b=\frac{1}{2}$, calculate the 19^{th} term (T_{19}) of the sequence. (3)

2.3 If $b=\frac{1}{2}$, write the sum of the first 20 positive terms of the sequence in sigma notation. (4)

2.4 Is the geometric series formed in QUESTION 2.3 convergent? Give reasons for your answer. (2)

[12]

QUESTION 3

3.1 The first four terms of a quadratic number pattern are $-1\ ;\ x\ ;\ 3\ ;\ x+8$

3.1.1 Calculate the value(s) of x. (4)

3.1.2 If $x=0$, determine the position of the first term in the quadratic number pattern for which the sum of the first n first differences will be greater than 250. (4)

3. FUNCTIONS, INVERSES AND LOGARITHMS

HINTS TO LEARNERS: FOCUS ON THE FOLLOWING CHARECTERISTICS:

- ✔ Definition of a function, vertical line test
- ✔ Domain and the Range
- ✔ General concept of the inverse of a function
- ✔ How the DOMAIN of the function may need to be restricted (in order to obtain a one-to-one function).
- ✔ Intercepts with the AXES
- ✔ Turning points
- ✔ Minima and Maxima
- ✔ Asymptotes (horizontal and vertical) and line of symmetry
- ✔ Shape of the graph
- ✔ Intervals on which the function INCREASES/DECREASES
- ✔ How f(x) has been transformed to generate f(-x), -f(x), f(x+a), af(x) and x = f(y) where a $\in$ R

HINTS TO LEARNERS:

- ✔ Be able to recognise the graph as being:

Linear : $y = ax + q$ **),**

Quadratic : $y = a(x+p)^2 + q$

Hyperbolic :

$$y = \frac{a}{x+p} + q$$

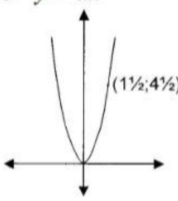
or Exponential

$$y = b^{x+p}$$

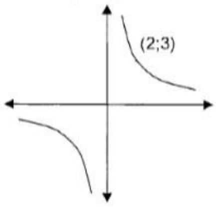
- ✔ Understand the effects of the parameters **a, p and q**
- ✔ Sketch the graphs, indicating all **intercepts** with the axes and **turning points** or **asymptotes**
- ✔ (it is important to be able use and interpret functional notation)
- ✔ Understand the definition of a FUNCTION
- ✔ Understand the INVERSE of a function(swop x and y)
- ✔ Know the inverse of $y = ax + q$, $y = ax^2$, $y = b^x$ where b > 0
- ✔ Be able to write the inverse in the form y = …
- ✔ Understand the definition of a logarithm: $y = \log_b x \leftrightarrow x = b^y$ where b > 1 and b ≠ 1
- ✔ Understand reflection about the y-axis **(**f(x) = f(-x)**)**and about the x-axis **(**f(x) = -f(x)**)** ✔Understand vertical translation **(**f(x) = f(x) + q**)** and horizontal translation **(**f(x) = f(x + p)**)**
- ✔ If given a sketch of a function [f(x)], be able to determine the values of x for which:
  1. f(x) < 0 OR f(x) > 0
  2. f'(x) ≥ 0 OR f'(x) ≤ 0
  3. f^{-1}(x) < 0 OR f^{-1}(x) ≥ 0
- ✔ If given two functions (f(x) and g(x)) on the same set of axes, be able to determine the values of x for which:

1. $f(x) = g(x)$
2. $f(x) < g(x)$ OR $f(x) - g(x) < 0$
3. $f(x) > g(x)$ OR $f(x) - g(x) > 0$
4. f(x).g(x)>0 OR f(x).g(x)<0
5. $\frac{f(x)}{g(x)}>0$
6. g(x).f^1(x)>0
7. $f^{-1}(x) = g^{-1}(x)$
8. $g'(x)>f'(x)$

3.5 $y = ax^2$



3.6 $xy = p$

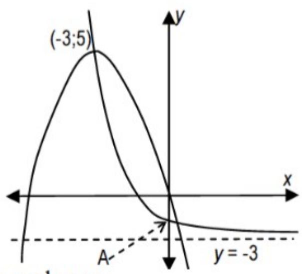


4. $f(x) = \frac{1}{2}.2^{x+2} + 4$ and $g(x) = -x^2 - 5x + 6$
   4.1 Draw graphs of f and g on the same system of axes. Clearly indicate turning points, axes of symmetry and all intercepts with the axes.
   4.2 Use your graphs to find the values of x for which $f(x) = g(x)$.
   4.3 Use your graphs to find the values of x for which $f(x) > g(x)$.
   4.4 Write down the y-asymptote for f.

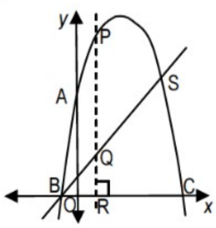
5. On the same system of axes, sketch graphs of $f(x) = \frac{1}{2}x^2 - 4x + 6$ and $g(x) = \dfrac{-2}{x-2} - 1$.
   Indicate all intercepts with the axes and coordinates of the turning points.
   5.1 Write down two asymptotes for g.
   5.2 For which values of x will $f(x) < g(x)$? Use your graph and indicate it on the graph.
   5.3 What is the minimum value of f?
   5.4 For which values of x will both f and g increase as x increases?

6. On the same set of axes sketch the graphs of $g(x) = \dfrac{3}{x+2} + 3$ and $h(x) = 2.2^{-x-2} - 4$.
   Clearly indicate intercepts with the axes, and the asymptotes.
   6.1 Write down the asymptotes for g and h.
   6.2 For which values of x will $h(x) = g(x)$?
   6.3 For which values of x will $g(x) > h(x)$?
   6.4 For which values of x will both g and h decrease as x increases?

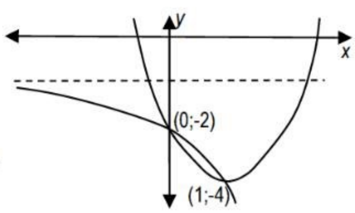
7. The graphs represent the equations:
   $y = a(x-p)^2 + q$ and $y = m^x + k$
   7.1 Give the values of a, p, q, m and k.
   7.2 Give the coordinates of point A.



8. The graphs of $y = -x^2 + 5x + 6$ and $y = x + 1$ are drawn.
   8.1 Calculate the lengths of OA and BC.
   8.2 Calculate the coordinates of point S.
   8.3 If OR = 1, calculate the length of PQ.
   8.4 If PQ = 5, calculate the length of OR.
   8.5 If R lies between B and C, calculate the maximum length of PQ.



9. Sketched are the graphs of $y = ax^2 + bx + c$ and $y = -k^{-x} + q$
   Determine:
   9.1 the values of a, b, c, k and q.
   9.2 the equation of the axis of symmetry of the parabola.
   9.3 The equation of the asymptote of the exponential function.



1. a) Sketch each of the following.
   b) State whether it is a function or not. Give a reason

1.1 f(x) = 4x - 2 f:

1.2 h(x) = x2

1.3 y = - 2x^2 h : y

1.4 g(x) = 3x k: y =

1.5 = (/)x

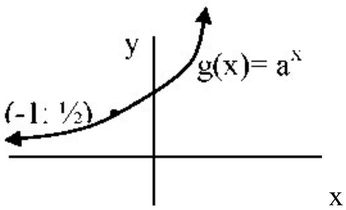
1.6 log 2***x***

2. a) Write down the inverse of the functions above (in the y - form)
   b) Draw a sketch graph of all the inverses in number one.

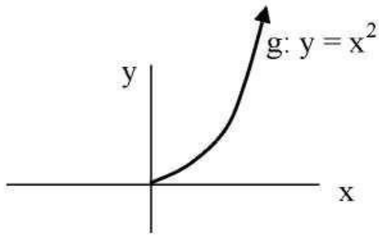
3. a) Sketch the graph of g$^-$ in each of the following.$^{-1}$.

b) Determine the equation of g where necessary and then write down the equation of g$^-$

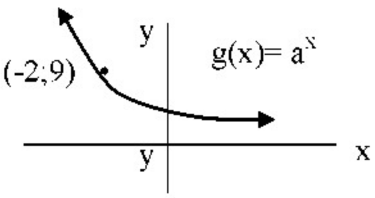
3.1



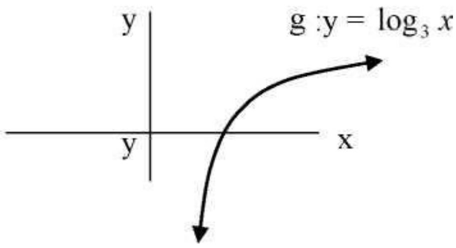
3.2



3.3



3.4



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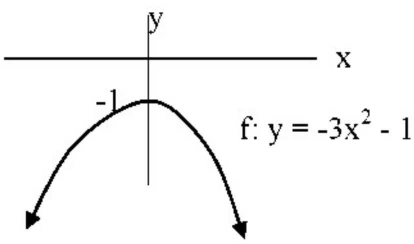


4.1 Give the equation of f^{-1}

4.2 Draw a sketch graph of f^1.

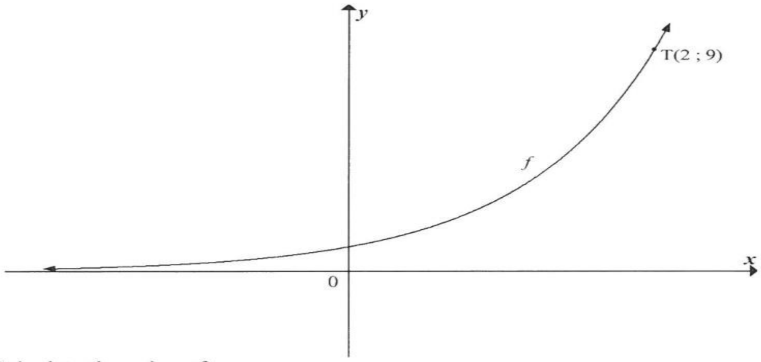
4.3 Is f^{-1} a function? Give a reason.

4.4 Write down how the domain of f should be restricted so that f^{-1} would be a function.

Write down the domain and range of the functions.

QUESTION 5

The graph of $f(x) = a^x$, $a > 1$ is shown below. T(2 ; 9) lies on f.



5.1 Calculate the value of a. (2)

5.2 Determine the equation of $g(x)$ if $g(x) = f(-x)$. (1)

5.3 Determine the value(s) of x for which $f^{-1}(x) \geq 2$. (2)

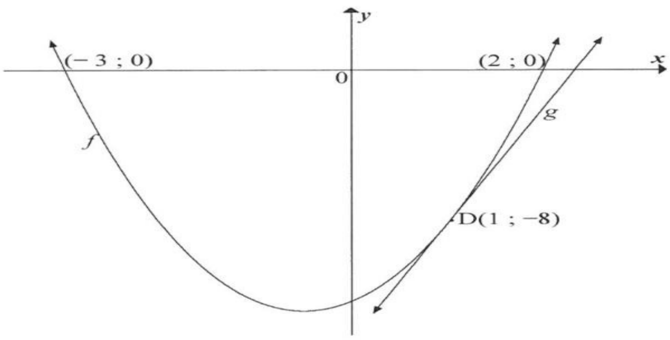
5.4 Is the inverse of f a function? Explain your answer. (2)
[7]

QUESTION 6

The graphs of $f(x) = ax^2 + bx + c$; $a \neq 0$ and $g(x) = mx + k$ are drawn below.

D(1 ; – 8) is a common point on f and g.

- f intersects the x-axis at (– 3 ; 0) and (2 ; 0).
- g is the tangent to f at D.



6.1 For which value(s) of x is $f(x) \leq 0$? (2)

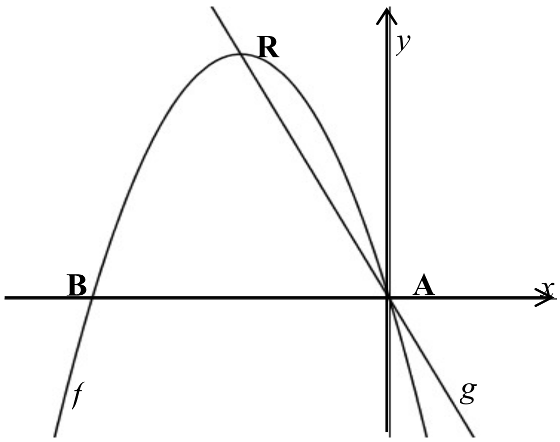
6.2 Determine the values of a, b and c. (5)

6.3 Determine the coordinates of the turning point of f. (3)

6.4 Write down the equation of the axis of symmetry of h if $h(x) = f(x-7) + 2$. (2)

6.5 Calculate the gradient of g. (3)
[15]

QUESTION 4 (WESTERN CAPE SEPTEMBER 2015)



Sketched alongside are the graphs of $f(x) = -(x+2)^2 + 4$ and $g(x) = ax + q$

R is the turning point of f

4.1 Give the coordinates of R. (2)

4.2 Calculate the length of AB. (2)

4.3 Determine the equation of g. (2)

4.4 For which values of x is $g(x) > f(x)$? (2)

4.5 Write down the equation of the axis of symmetry of h if $h(x) = f(-x)$. (2)

4.6 Give the range of p if $p(x) = -f(x)$. (2)

[12] QUESTION 5 (WESTERN CAPE SEPTEMBER 2015)

Given: $h(x) = \frac{2}{x-2} + 1$

5.1 Give the equations of the asymptotes of h (2)

5.2 Determine the x- and y-intercepts of the graph of h. (3)

5.3 Sketch the graph of h using the grid on the DIAGRAM SHEET. (3)

5.4 Give the domain of h. (2)

5.5 Describe the transformation of h to f if $f(x) = h(x+3)$ (2)

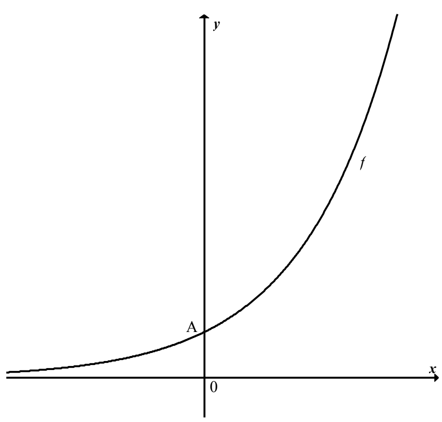
5.6 Determine the equation of the symmetry axis of the hyperbola with a negative gradient. (2)

[14]

QUESTION 6 $f(x) = 3^x$ **(WESTERN CAPE SEPTEMBER 2015)**

alongside.

6.1 The graph of is sketched



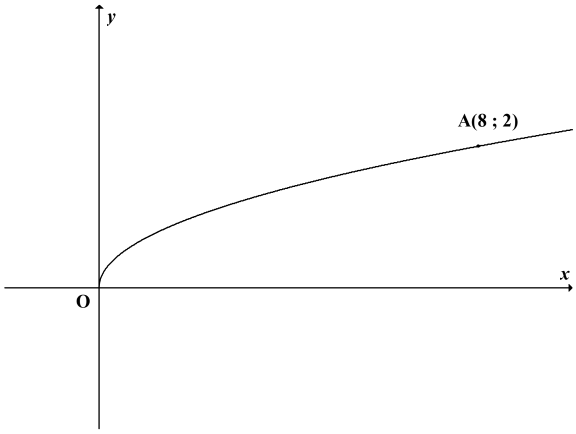
6.1.1 Give the coordinates of A. (1)

6.1.2 Write down the equation of f^{-1} in the form $y = \ldots$ (1)

6.1.3 For which value(s) of x will $f^{-1} \leq 0$? (2)

6.1.4 Write down the equation of the asymptote of $f(x-1)$ (1)

6.2 Sketched is the graph of f, the inverse of a restricted parabola. The point A(8 ; 2) lies on the graph of f.



6.2.1 Determine the equation of f in the form $y = \ldots$ (2)

6.2.2 Hence, write down the equation of f^{-1} in the form $y = \ldots$ (2)

6.2.3 Give the coordinates of the turning point of $g(x) = f^{-1}(x + 3) - 1$. (1)

[10]

Question 4A

4.1 If $f(x) = \log_5 x$

a. State the domain of f. (1)

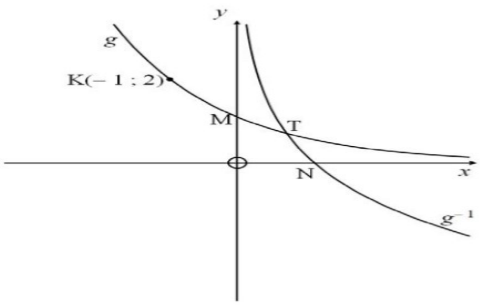
b. State the equation of f^{-1} in the form $y = \ldots$ (2)

c. State the equation of $k(x)$, the line of symmetry of f f^{-1} and (1)

4.2 Sketch k, f and f^{-1} on the same system of axes and label all intercepts with the axes. Indicate the coordinates of two points on each graph. (7)

4.3 If h is the reflection of f^{-1} in the y-axis, state the equation of $h(x)$. (1)

[12]



(1) Show that $b = \frac{1}{2}$. (1)

(2) Determine the equation of $g^{-1}(x)$ in the form $g^{-1}(x) = \ldots$ (2)

(3) The x-coordinate of T is 0,64 (correct to 2 decimal digits). Give the y-coordinate of T (correct to 2 decimal digits). (2)

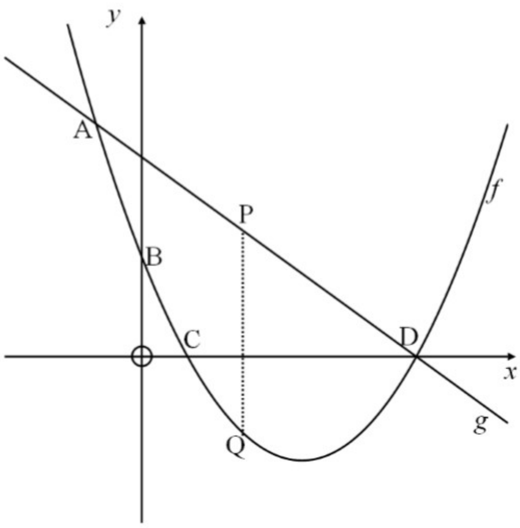
(4) Write down the value(s) of x for which $g^{-1}(x) > 0$. (2)

(5) $h(x)$ is the reflection of $g^{-1}(x)$ about the y-axis. Give the equation of $h(x)$, stating the domain. (2)
[16]

QUESTION 6

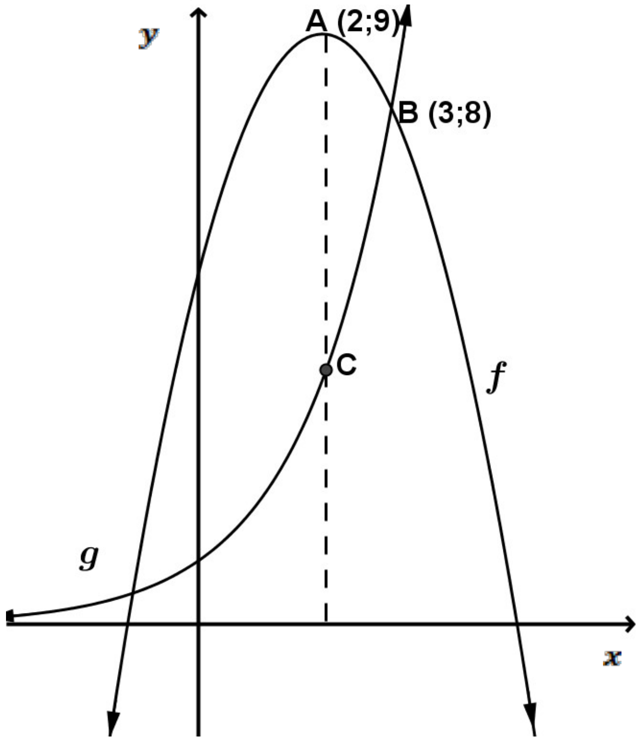
Refer to the figure showing the graphs of $f(x) = \frac{x^2}{2} - \frac{7x}{2} + 3$ and $g(x) = -x + 6$.

D is the x-intercept of both f and g.



QUESTION 3 (WESTERN CAPE JUNE 2015)

The graphs of $f(x)$ and $g(x)$ are shown in the diagram below.



The turning point of $f(x)$ is A (2;9) and the graphs f and g intersects at B (3;8). C is a point on $g(x)$ and is on the axis of symmetry of f.

3.1 Show that the function f can be defined by the equation: $f(x) = -x^2 + 4x + 5$ (4)

3.2 Write down the equation of the axis of symmetry of f. (1)

3.3 The graph $g(x)$, has the equation $y = a^x$. Determine the value of a. (2)

3.4 If it is given that $(-1; 0)$ is one root of f, write down the coordinates of the other root. (1)

3.5 For which value(s) of x will $f(x) < 0$? (2)

3.6 Determine the length of AC. (2)

3.7 Discuss the nature of the roots of $h(x)$ if, $h(x) = f(x) - 9$. (2)

[14]

Question 4

4.1 If $f(x) = \log_5 x$

d. State the domain of f. (1)

e. State the equation of f^{-1} in the form $y = \ldots$ (2)

f. State the equation of $k(x)$, the line of symmetry of f^{-1} and (1)

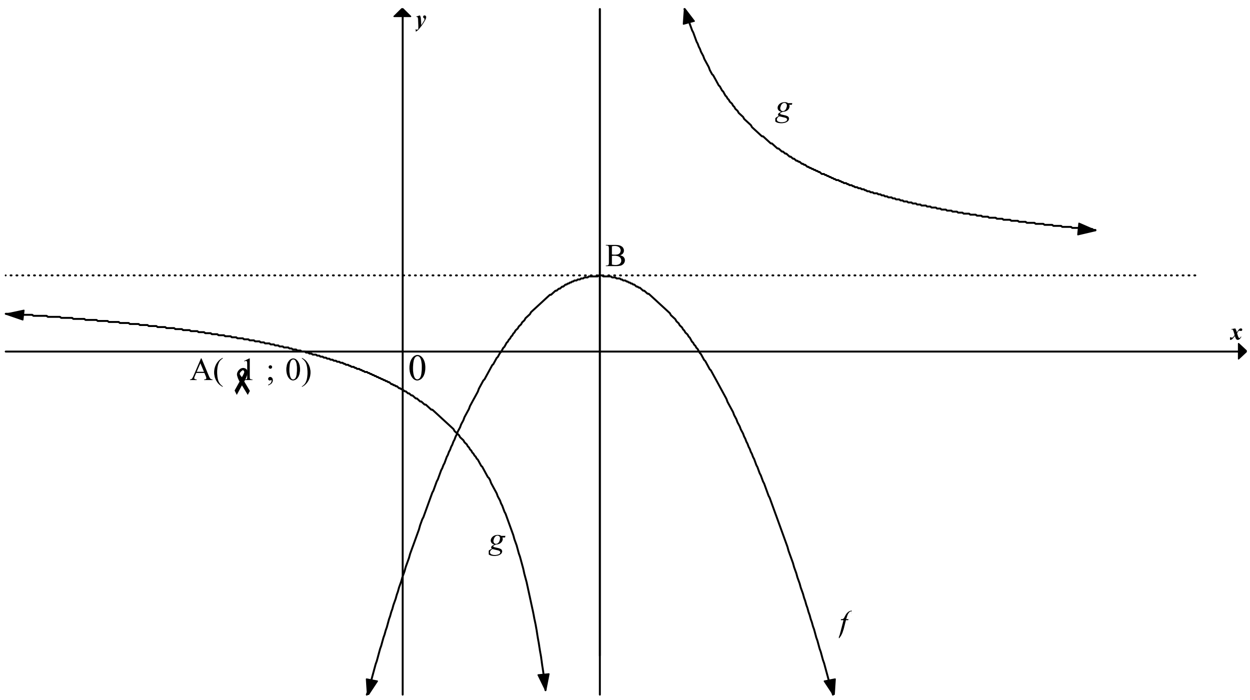
4.2 Sketch k, f and f^{-1} on the same system of axes and label all intercepts with the axes. Indicate the coordinates of two points on each graph. (7)

4.3 If h is the reflection of f^{-1} in the y-axis, state the equation of $h(x)$. (1)

QUESTION 5

Sketched below is the parabola f, with equation $f(x) = -x^2 + 4x - 3$ and a hyperbola g, with equation $(x - p)(y + t) = 3$.

B, the turning point of f, lies at the point of intersection of the asymptotes of g.
A is the x-intercept of g.



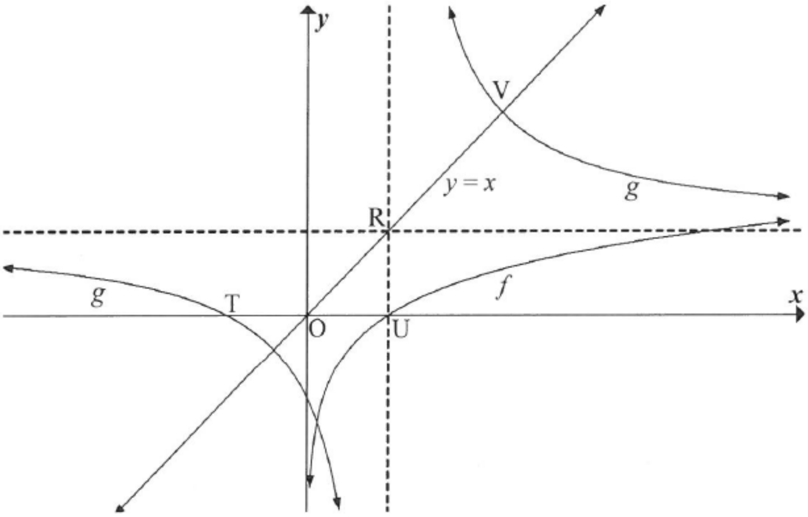
5.1	Show that the coordinates of B are $(2 ; 1)$	(2)
5.2	Write down the range of f.	(1)
5.3	For which value(s) of x will $g(x) \geq 0$?	(2)
5.4	Determine the equation of the vertical asymptote of the graph of h if $h(x) = g(x + 4)$	(1)
5.5	Determine the values of p and t.	(4)
5.6	Write down the values of x for which $f(x).g'(x) \geq 0$	(4)
		[14]

QUESTION (FEB/MAR 2017)

The sketch below shows the graphs of $f(x) = x$ and $g(x) = \frac{2}{x-1} + 1$

T and U are the x- intercepts of g and f respectively

The line $y = x$ intercept the asymptotes of g at R and the graph of g at V



4.1 Write down the coordinates of U. (1)

.

4.2 Write down the equation of the asymptotes of g. (2)

4.3 Determine the coordinates of T. (2)

4.4 Write down the equation of h, the reflection of f in the line y = x, in the form of y =……. (2)

4.5 Write down the equation of the asymptotes of h(x-3). (1)

4.6 Calculate the coordinates of V. (4)

4.7 Determine the coordinate of T' the point which is symmetrical to T about the point R. (2)

[14]

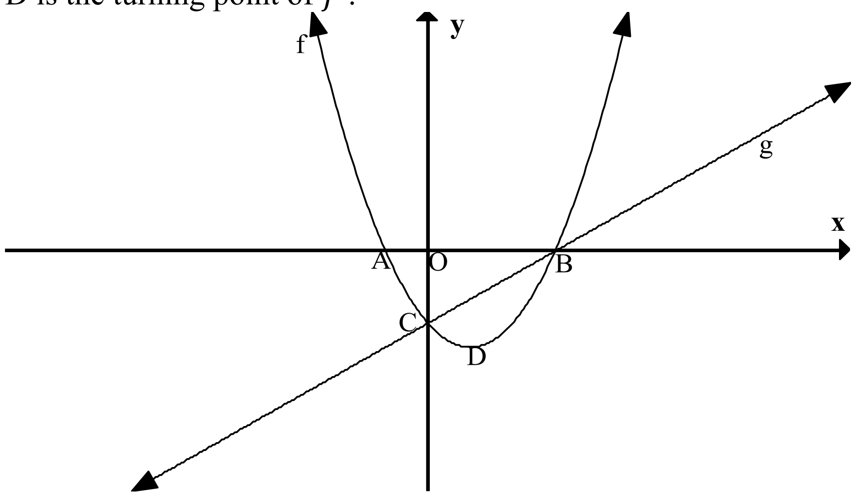
QUESTION 5 (FEB/MAR 2017)

5.1 The sketch below shows the graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = x$-3

A and B are the x- intercepts of f.

The graph of f and g intercept at C and B

D is the turning point of f .



Determine

5.1.1 Determine the coordinates of C. (1)

5.1.2 Calculate the length of AB. (4)

5.1.3 Determine the coordinate of D (2)

5.1.4 Calculate the average gradient of f between C and D. (2)

5.1.5 Calculate the size of $O\hat{C}D$ (2)

5.1.6 Determine the values of k for which f(x) = k will have two unequal positive real roots (3)

5.1.7 For which values of x will F'(x). f''(x) > 0 (3)

5.2 The graph of f parabola has x- intercepts at x = 1 and x= 5. g(x)= 4 is a tangent to f at P, the turning point of f. Sketch the graph of f clearly showing the intercept with the axis and the coordinates of the turning points. (5)

[22]

QUESTION 6 (FEB/MAR 2018)

The function f defined by $f(x) = \frac{a}{x+p} + q$ has the following properties

The range of f is y Є R, y ≠ 1

The graph of f passing through the origin

P (2 +2; 2 +1) lies on the graph of f.

6.1 Write down the value of q (1)

6.2 Calculate the values of a and p (5)

6.3 Sketch a neat graph of the function. Your graph must include asymptotes if any (4)

[10]

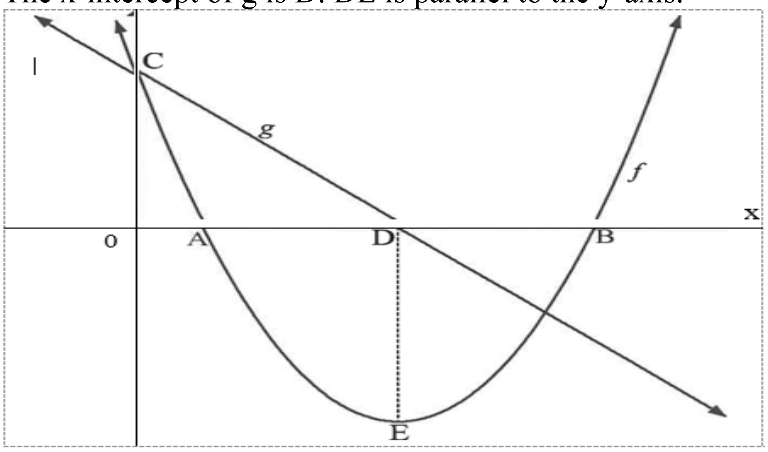
QUESTION 4 (FEB/MAR 2018)

Below are the graphs of f(x) = (x — 9) and a straight-line g.

A and B are the x-intercepts of f and E is the turning point of f.

C is the y-intercept of both f and g

The x-intercept of g is D. DE is parallel to the y-axis.



4.1 Write down the coordinates of E. (2)

4.2 Calculate the coordinates of A. (3)

4.3 M is the reflection of C in the axis of symmetry of f. Write down the coordinates of M. (3)

4.4 Determine the equation of g in the form y = mx + c. (3)

4.5 Write down the equation of g-l in the form y =…… (3)

4.6 For which values of x will x< 0? (4)

[18]

QUESTION 5 (NOV. 2018)

Given: $f(x) = \frac{-1}{x-1}$

4.1 Write down the domain of f. (1)

4.2 Write down the asymptotes of f. (2)

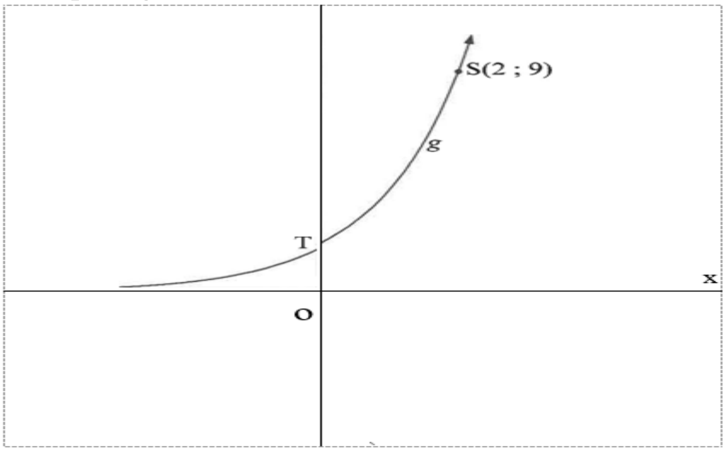
4.3 Sketch the graph of f clearly showing all intercept with the axes and any asymptotes (2)

4.4 For which values of x will $x.f'(x) \geq 0$. (3)

[8]

QUESTION 5 (FEB/MAR 2018)

The graph of g(x) = ax is drawn in the sketch below. The points (2, 9) lies on g. T is the y-intercept of g



4.1 Write down the coordinates of T. (2)

4.2 Calculate the value of a. (2)

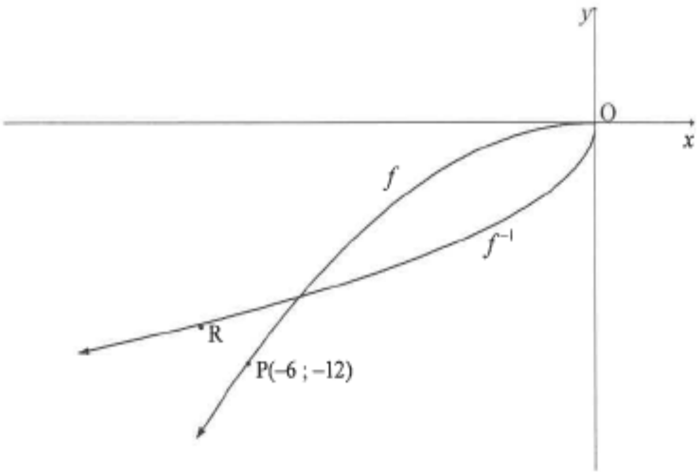
4.3 The graph is obtained by reflecting g in the y-axis. Write down the equation of h. (2)

4.4 Write down the values of x for which $0 < x < 1$ (2)

[8]

QUESTION 4 (NOV. 2018)

In the diagram below the graph of $f(x) = ax^2$ drawn in the interval $x \leq 0$. The graph of f^{-1} is also drawn. P (-6, -12) is a point on f and R is a point on f^{-1}.



4.1 Is f^{-1} a function? Motivate your answer. (2)

4.2 If R is the reflection of P in the line y = x, write down the coordinates of R (1)

4.3 Calculate the value of a. (2)

4.4 Write down the equation of f^{-1} in form of y =…… (3)

TOPIC : EQUATIONS AND ALGEBRA

Question 1

1.1. Solve for x

1.1.1. $x(x-4)=5$ (3)

1.1.2. $4x^2$-20x+1=0 (round off your answer correct to 2 decimal places) (4)

1.1.3. Solve simultaneously for x *and* y:

$y-x+3=0$

$x^2-x=6+y$ (6)

1.1.4. If m and n are rational numbers such that $\sqrt{m}+\sqrt{n}=\sqrt{7+\sqrt{48}}$, calculate a possible value of m^2+n^2 (4)

Question 2

2.1. **Solve for x**

2.1.1. x^2- 5x = -6 (3)

2.1.2. $(3x+1)(x-4)<0$ (3)

2.1.3. $2x+\sqrt{x+1}=1$ (4)

2.1.4. $12^{5+3x}=1$ (2)

2.2. **Solve for x and y**

$2x-y=8$

$x^2-xy+y^2=19$ (7)

2.3. The polynomial $x^{10}-2x^5+c$ is divisible by $x+1$ calculate the value of c

Question 3

3.1. Solve for x

3.1.1. $(x+2)^2=3x\ (x-2)$

Giving your answer correct to one decimal digit (5)

3.1.2. $x^2-9x\geq 36$ (4)

3.1.3. $3^x-3^{x-2}=72$ (4)

3.2. Given (2m – 3) (n + 5) =0

Solve for:

3.2.1. n if m=1 (1)

3.2.2. m if n≠-5 (1)

3.2.3. m if n = -5 (2)

Question 4

4.1. **Solve for x**

4.1.1. $(x-3)(x+1)=5$ (3)

4.1.2. $9^{2x-1} = \frac{3^x}{3}$ (3)

4.1.3. $2\sqrt{2-7x} = \sqrt{-36x}$

4.2. Determine in terms of k, the co-ordinates of the points of intersection of the graphs of $y = kx + k$ and $y = x^2 + 2kx + k$, where $k\epsilon$■ (5)

4.3. Given: $9x^2 + nx + 49 = 0$

4.3.1. Express the roots of the equation in terms of n (2)

4.3.2. For what value(s) of n will the roots be equal? (2)

Question 5

5.1. **Solve for x:**

5.1.1. $10x = 3x^2 - 8$ (3)

5.1.2. $x + \sqrt{n-2} = 4$ (5)

5.1.3. $x(2x-1) \geq 15$ (5)

5.2. **Given: P** $= \frac{4^{x+3}+4^x}{8^{x+2}+8^x}$

5.2.1. Simplify P (3)

5.2.2. Hence solve for x: P=3 (2)

5.3. State whether the following numbers are rational, irrational or non-real.

5.3.1. $\sqrt{3}$ (1)

5.3.2. $\frac{22}{7}$ (1)

5.3.3. The roots of $x^2 + 4 = 0$ (1)

5.4. For which values of m will $x + y$ be a factor of $x^m + y^m$? (2)

Question 6

6.1 Solve for x:

6.1.1 $2x^2 + 11 = x + 21$ (3)

6.1.2 $3x^3 + x^2 - x = 0$ (5)

6.1.3 $2x + p = p(x+2)$, stating any restriction (4)

6.1.4 $x^{-1} - x^{-\frac{1}{2}} = 20$

6.2. Solve for x and y simultaneously in the following equations

$2x^2 - 3xy = -4$ and $4^{x+y} = 2^{y+4}$ (6)

Question 7

7.1. Solve for x. Leave the answer in the simplest surd form where necessary

7.1.1. $(2x+5)(x^2-2)=0$ (3)

7.1.2. $x^2-4 \geq 5$ (4)

7.1.3. $12^{2x}=8.36^x$ (4)

7.2. Solve for x, correct to two decimal places

$2(x+1)^2=9$ (4)

7.3. Solve for x and y simultaneously:

$y=-2x+7$ and $\frac{y+5}{x-1}=\frac{1}{2}$ (4)

7.4. Given that $f(x)=bx^2+3x+4$ and $g(x)=-x-1$, calculate the value(s) of b for which the graph of g will intersect the graph of f. (4)

7.5. Determine the values of p, for which the equation $3^x=1-p$ will have real solutions (2)

Question 8

8.1. Given $x^2+2x=0$

8.1.1. Solve for x (2)

8.1.2. Hence, determine the positive values of x for which $x^2 \geq -2x$ (3)

8.2. Solve for x:

$2x^2-3x-7=0$ (correct to two decimal places) (4)

8.3. Given $k+5=\frac{14}{k}$

8.3.1. Solve for k (3)

8.3.2. Hence, or otherwise, solve for x

If $\sqrt{x+5}+5=\frac{14}{\sqrt{x+5}}$ (3)

8.4. Solve for x and y simultaneously if:

$x-2y-3=0$ and

$4x^2-5xy+y^2=0$ (7)

8.5. The roots of a quadratic equation is given by $x=\frac{-2\pm\sqrt{4-20k}}{2}$

Determine the values of k for which the equation will have real roots (2)

8.6 $x=\frac{a^2+a-2}{a-1}$ if $a=888\ 888\ 888\ 888$

8.7 Determine the range of the function $y=x+\frac{1}{x}$, $x \neq 0$ and x is real. (6)

Probability and the Counting Principle

You must have learnt the following in earlier Grades

Basic Probability

dependent and independent events;

the product rule for independent events: $P(A \text{ and } B) = P(A) \times P(B)$;

the sum rule for mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$;

the identity: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$;

the complementary rule: $P(\text{not } A) = 1 - P(A)$.

Solving of probability problems (where events are not necessarily independent) by using Venn-diagrams, tree diagrams, two-way contingency tables and other techniques.

In Grade 12 you learnt the following:

The fundamental counting principle;

application of the Basic Counting Principle to solve probability problems;

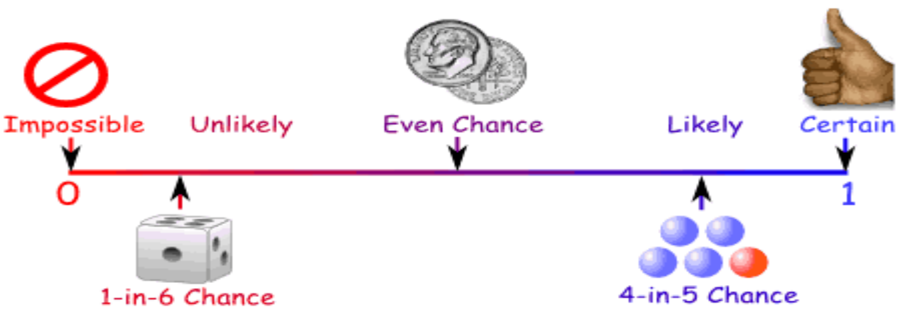
Factorial: n! = n x (n-1) (n-2) x…x3 x 2 x 1 e.g. 4! = 4 x 3 x 2 x1

Basic Probability

Probability = $\frac{Number\ of\ ways\ an\ event\ can\ happen}{Total\ number\ of\ outcomes}$

Probability ranges between 0 (impossible event) and 1(certainty) and can be expressed in fractional form($\frac{a}{b}$), decimal or as a percentage e.g. $\frac{1}{4} = 0.25 = 50\%$

The probability line



Example 1: There are 4 King cards in a deck of 52 cards. What is the probability of choosing a King?

Probability of choosing a King (K) = P(K) = $\frac{4}{52} = \frac{1}{13}$

Example 2

I toss a coin. What is the probability of getting a head?

P(H) = $\frac{1}{2}$

Example 3

What is the probability of obtaining:

1. exactly two heads in three spins of a coin?
2. a 4 when throwing a dice?
3. a sum of 5 when throwing two dice?

1. $E_2 = \{HHT; HTH; THH\}$ P(2 heads) $= \frac{n(E_2)}{n(S)} = \frac{3}{8}$

2. $E_3 = \{4\}$ P(4) $= \frac{n(E_3)}{n(S)} = \frac{1}{6}$

3. $E_4 = \{(1;4);(2;3);(3;2);(4;1)\}$ P(sum of 5) $= \frac{4}{36} = \frac{1}{9}$

Independent events

If the occurrence of event **A** cannot affect the outcome of event **B** then **A** and **B** are independent events, e.g. spinning a coin twice, the outcome of the second spin has nothing to do with the outcome of the first spin.

For independent events A and B: $\boxed{P(A \text{ and } B) = P(A) \times P(B)}$

Example A coin is spun and a dice is thrown. What is the probability that we obtain a head and a 4?

P(head and 4) $=$ P(head) $\times$ P(4) $= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Mutually Exclusive Events

Mutually exclusive events are events which exclude one another, i.e. if **A** occurs then **B** cannot occur and vice versa.

e.g. turning left and turning right also sitting down and standing up.

If A and B are mutually exclusive then **n(A or B)** $=$ **n(A)** $+$ **n(B)** hence **P(A or B)** $=$ **P(A)** $+$ **P(B)**

Otherwise the general rule applies ie . **P(A or B)** $=$ **P(A)** $+$ **P(B)-P(A) x P(B)**

Complement Rule

If a set U= {1; 2; 3; 4 ; 5; 6} and A{1; 2; 3} then the complement of A= A' ={2; 4 ; 6}

Obviously **P(A') = 1 - P(A)** $= \frac{3}{6} =$ also P(A) = 1 – P(A')

The Fundamental Counting Principle

It states that if there are **m** ways an event can occur and **n** ways another event can occur then both events can occur in

m x n ways in which the events can occur.

Example 1

If ice cream comes in 5 different flavours and there are 4 toppings. How many different recipes /combinations can you make?

Answer: 5 x 4 = 20

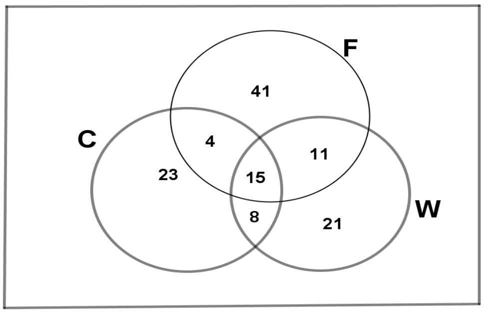
Example 2

You want to buy a Toyota car and the advertisement says there are 5 different colours (white, blue, silver, black and red), 3 models (1.6 Prestige, 1.4 Esteem and 1.6 Sprinter) and 2 body types (hatchback & sedan). How many different combinations/options can you choose from?

Answer: 5 x 3 x 2 = 30

Question 1: Venn Diagrams

A school organised a dance for their 150 Grade 12 learners. The learners were asked to indicate their preference for the theme. They had to choose from Casino(C), France (F) and Winter Wonderland (W). The information collected is shown in the Venn diagram below



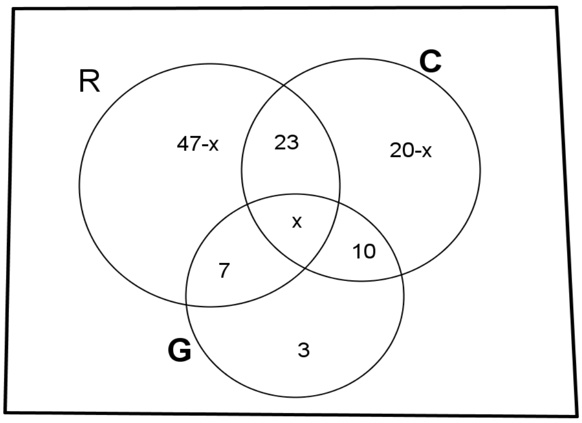
1.1 Calculate the probability that a learner, chosen at random

1.1.1 Does not prefer the Casino or the France theme. (2)

1.1.2 Prefers only two of the given theme choices. (3)

1.1.3 Show with all working ,whether the events preferring Casino(C) and preferring France (F) are independent or not. (5)

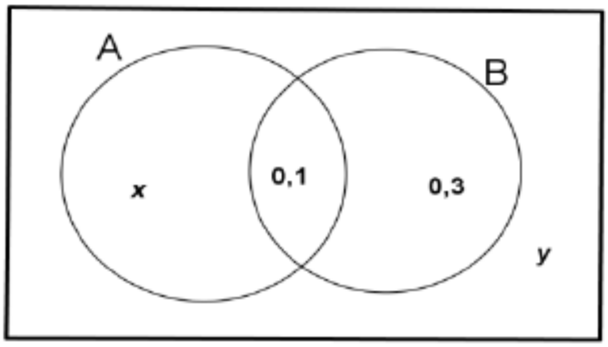
1.2 There are 115 people in a group. The Venn diagram shows the number of people who enjoy listening to radio (R), enjoy gardening (G) and /enjoy cooking(C).There are x people who enjoy all three activities. There are y people who do not enjoy any of the activities.



1.2.1 If there are 28 people who enjoy gardening , calculate the value of x. (1)

1.2.2 Hence determine the value of y. (1)

1.4 A local club has facilities that include tennis courts and a golf course.



A survey of the club members indicated that 72% regularly use the golf course and 48% regularly use tennis courts. Some members regularly use both while 8% use neither of the facilities.
The club has 700 members

1.4.1 Determine the number of members that regularly use at least one of the facilities. You may find that the given Venn diagram is a useful aid. (2)

1.4.2 Suppose we randomly select a member of the club. What is the probability that this person uses exactly (only) one facility? (3)

1.4.3 P (using the golf course) X P (using tennis courts) = 0, 72 X 0, 48 =03,456. Validate statistically whether these events are independent of not (2)

[7]

1.5 A study was done to determine the effect of 3 drugs A, B and C in relieving headache pain. 80 patients were given the chance to use all 3 drugs. The following results were obtained:
40 reported relief from drug A
35 reported relief from drug B.
40 reported relief from drug C.
21reported relief from both drug A and C.
18 reported relief from drug B and C.
68 reported relief from at least one of the drugs.
7 reported relief from all graphs.

1.5.1. How many of the patients got relief from none of the drugs? (1)
1.5.2 How many patients got relief from drug A and B but not C? (2)
1.5.3 What is the probability that a randomly chosen subject got relief from at list two of the drugs? (2)

Question 2: Tree diagrams

2.1 A bag contains 12 blue balls,10 red balls and 18 green balls. 2 balls are then drawn without replacement. Use the information provided to answer the questions that follow.

2.1.1 Draw a Tree diagram to represent the information (3)
2.1.2 Determine the probability that:
(a) Both balls are green. (2)
(b) One blue and one red ball is chosen. (2)
(c) A green ball is chosen given that the 1^{st} ball was blue. (2)
(d) There are t orange balls and 2 yellow balls in a bag. Craig randomly selects one ball from the bag, records his choice and returns the ball to the bag. He then randomly (6)

selects a second ball from the bag, records his choice and returns it to bag. It is known that the probability that Craig will select two balls of the same colour from the bag is 52%. Calculate how many orange balls are in the bag.

Question 3: Contingency Tables

3.1 A survey was conducted asking 60 people with which hand they write and what colour hair they have. The results are summarised in the table below.

		HANDS USED TO WRITE WITH		
		Right	**Left**	**Total**
HAIR COLOUR	**Light**	a	B	20
	Dark	c	D	40
	Total	48	12	60

The survey concluded that the 'hand used for writing' and 'hair colour' are independent events
Calculate the values of a,b,c (5)

A rare kidney disease affects only 1 in 1000 people and the test for this disease has a 99% accuracy rate.

3.1.1 Draw a two-way contingency table showing the results if 100 000 of the general population are tested. (4)

3.1.2 Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease correct to two decimal places. (2)

3.2 The table below summarises of all driving tests taken at a Test Centre in Bloemfontein during winter holidays.

	Male	**Female**	**Totals**
Pass	32	43	75
Fail	8	15	23
Totals	40	58	98

3.2.1 A person has been chosen at random from those who took their test. Find the probability that a person was a female who failed.

3.2.2 The person chosen is a male. Find the probability that he passes the test. (2)

3.2.3 Is being male and passing the test independent or not? (2)

3.3 Each passenger on a certain Banana Airways flight chose exactly one beverage from tea, coffee or fruit juice. The results are shown in the table below.

	MALE	FEMALE	TOTAL
Tea	20	40	60
Coffee	b	C	80
Fruit juice	d	E	20
TOTAL	60	100	A

3.3.1 Write down the value of a. (2)

3.3.2 What is the probability that a randomly selected passenger is male? (2)

3.3.3 Given that the event of a passenger choosing coffee is independent of being a male, calculate the value of b. (3)

[15]

Question 4: Counting principles

4.1 Riana packs his suitcase for his holiday with 3 caps, 5shirts, 3 pairs of jeans and 2 pairs of takkies.

4.1.1. How many different outfits can he put together, if when he dresses, he must wear a shirt, a pair of jeans, a pair of takkies and a cap? (3)

4.1.2 Riana hangs all 5 shirts and the 3 pairs of jeans (each item separately) on a different hanger, on the rail in the cupboard. How many different arrangements are there of his clothing? (1)

4.1.3 What is the probability that the shirts are all hanging together next to each other in the cupboard? (1)

4.2. Find the number of ways I can arrange 4 girls and 5 boys in a row so that

4.2.1 They sit in any order.

4.2.2 The girls sit together and the boys sit together.

4.3. Two friends, Albert and James, take part in the swimming Gala. There are inn total 10 competitors and 10 lanes. Only one swimmer is allowed in each lane. If all swimmers take part , calculate the possible arrangements in the starting line up under the following conditions:

4.3.1 Albert and James are placed next to each other. (2)

4.3.2 Albert is placed in the first lane and James is NOT placed next to Albert (2)

4.3.3 Calculate the probability of the situation where James and Albert are not next to each other. (3)

4.4. We use 10 digits 0,1,2,3,4,5,6,7,8,9 to write numbers.

4.4.1 How many different 3 –digits numbers can be written?

4.4.2 How many different 3- digit numbers can be written whose first digit is not 0, whose second digit is 0, and whose third digit can be anything?

4.5 How many 3-digit codes can be formed from the digits 1;2;3;4;5;6;7;8;9 if:

4.5.1 Repetition is allowed (2)

4.5.2 Repetition is not allowed (2)

4.6 How many 6 digit number can be formed if:

4.6.1 No digit may be repeated but the number may start with a zero. (2)

4.6.2 The only number that is not allowed is 000000 (2)

4.6.3 The number must start with an even number and must then alternate between even and odd(the number 1 may not be used) (2)

4.6.4 The number must start with even, alternate between odds and evens, and no digit may be repeated. (2)

Question 5: (Counting principles cont…)

5.1 How many ways can you arrange the letters ANN? (2)

5.2 How many ways can you arrange the letters POPO? (2)

5.3 How many ways can you arrange the letters MUMMY? (2)

5.4 If you arrange the letters of the word POTRAY in any order, what is that probability the word will start with a P and end with a Y? (3)

5.5 The digits 1 to 7 are used to create a four-digit code to enter a locked room. How many different codes are possible if the digits may not be repeated and the code must be an even number bigger than 5 000? (3)

TYPICAL EXAM QUESTIONS FROM PREVIOUS PAPERS

DBE/ Nov 2010

QUESTION 1

A school organised a camp for their 103 Grade 12 learners. The learners were asked to indicate their food preferences for the camp. They had to choose from chicken, vegetables and fish.

The following information was collected:

- 2 learners do not eat chicken, fish or vegetables
- 5 learners eat only vegetables
- 2 learners only eat chicken
- 21 learners do not eat fish
- 3 learners eat only fish
- 66 learners eat chicken and fish
- 75 learners eat vegetables and fish

Let the number of learners who eat chicken, vegetables and fish be x.

1.1	Draw an appropriate Venn diagram to represent the information.		(7)
1.2	Calculate x.		(2)
1.3	Calculate the probability that a learner, chosen at random:		
	1.3.1	Eats only chicken and fish, and no vegetables.	(2)
	1.3.2	Eats any TWO of the given food choices: chicken, vegetables and fish.	(2) **[13]**

DBE /Nov 2011

QUESTION 5

The digits 0, 1, 2, 3, 4, 5 and 6 are used to make 3 digit codes.

5.1 How many unique codes are possible if digits can be repeated? (2)

5.2 How many unique codes are possible if the digits cannot be repeated? (2)

5.3 In the case where digits may be repeated, how many codes are numbers that are greater than 300 and exactly divisible by 5? (3)
[7]

QUESTION 6

Complaints about a restaurant fell into three main categories: the menu (M), the food (F) and the service (S). In total 173 complaints were received in a certain month. The complaints were as follows:

- 110 complained about the menu.
- 55 complained about the food.
- 67 complained about the service.
- 20 complained about the menu and the food, but not the service.
- 11 complained about the menu and the service, but not the food.
- 16 complained about the food and the service, but not the menu.
- The number who complained about all three is unknown.

6.1 Draw a Venn diagram to illustrate the above information. (6)

6.2 Determine the number of people who complained about ALL THREE categories. (3)

6.3 Determine the probability that a complaint selected at random from those received, complained about AT LEAST TWO of the categories (that is, menu, food and service). (3)
[12]

DBE/Nov 2016

The probability of events A and B occurring are denoted by P(A) and P(B) respectively.

For any two events A and B it is given that:

- $P(B') = 0{,}28$
- $P(B) = 3P(A)$
- $P(A \text{ or } B) = 0{,}96$

Are events A and B mutually exclusive? Justify your answer. (4)

Question 11

A survey was conducted among 100 boys and 60 girls to determine how many of them watched TV in the period during which examinations were written. Their responses are shown in the partially completed table below.

	WATCHED TV DURING EXAMINATIONS	DID NOT WATCH TV DURING EXAMINATIONS	TOTALS
Male	80	a	
Female	48	12	
Totals	b	32	160

11.1 Calculate the values of a and b. (2)

11.2 Are the events 'being a male' and 'did not watch TV during examinations' mutually exclusive? Give a reason for your answer. (2)

11.3 If a learner who participated in this survey is chosen at random, what is the probability that the learner:

11.3.1 Watched TV in the period during which the examinations were written? (2)

11.3.2 Is not a male and did not watch TV in the period during which examinations were written? (2)

[8]

DBE/Nov 2013

QUESTION 4

4.1 A survey of 80 students at a local library indicated the reading preferences below:

44 read the *National Geographic* magazine
33 read the *Getaway* magazine
39 read the *Leadership* magazine
23 read both *National Geographic* and *Leadership* magazines
19 read both *Getaway* and *Leadership* magazines
9 read all three magazines
69 read at least one magazine

4.1.1 How many students did not read any magazine? (1)

4.1.2 Let the number of students who read *National Geographic* and *Getaway*, but not *Leadership*, be represented by x. Draw a Venn diagram to represent reading preferences. (5)

4.1.3 Hence show that $x = 5$. (3)

4.1.4 What is the probability, correct to THREE decimal places, that a student selected at random will read at least two of the three magazines? (3)

4.2 A smoke detector system in a large warehouse uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0,95. The probability that it will be detected by device B is 0,98 and the probability that it will be detected by both devices simultaneously is 0,94.

4.2.1 If smoke is present, what is the probability that it will be detected by device A or device B or both devices? (3)

4.2.2 What is the probability that the smoke will not be detected? (1)

[16]

QUESTION 10

A survey was conducted among 100 Grade 12 learners about their use of Instagram (I), Twitter (T) and WhatsApp (W) on their cell phones. The survey revealed the following:

- 8 use all three.
- 12 use Instagram and Twitter.
- 5 use Twitter and WhatsApp, but not Instagram.
- x use Instagram and WhatsApp, but not Twitter.
- 61 use Instagram.
- 19 use Twitter.
- 73 use WhatsApp.
- 14 use none of these applications.

10.1 Draw a Venn diagram to illustrate the information above. (4)

10.2 Calculate the value of x. (2)

10.3 Calculate the probability that a learner, chosen randomly, uses only ONE of these applications. (2)
[8]

QUESTION 11

A company uses a coding system to identify its clients. Each code is made up of two letters and a sequence of digits, for example AD108 or RR 45789.

The letters are chosen from A; D; R; S and U. Letters may be repeated in the code.

The digits 0 to 9 are used, but NO digit may be repeated in the code.

11.1 How many different clients can be identified with a coding system that is made up of TWO letters and TWO digits? (3)

11.2 Determine the least number of digits that is required for a company to uniquely identify 700 000 clients using their coding system. (3)
[6]

DBE/ Feb-March 2017

Question 10

10.1 The events S and T are independent.

- $P(S \text{ and } T) = \frac{1}{6}$
- $P(S) = \frac{1}{4}$

10.1.1 Calculate $P(T)$. (2)

10.1.2 Hence, calculate $P(S \text{ or } T)$. (2)

10.2 A FIVE-digit code is created from the digits 2 ; 3 ; 5 ; 7 ; 9.

How many different codes can be created if:

10.2.1 Repetition of digits is NOT allowed in the code (2)

10.2.2 Repetition of digits IS allowed in the code (1)

10.3 A group of 3 South Africans, 2 Australians and 2 Englishmen are staying at the same hotel while on holiday. Each person has his/her own room and the rooms are next to each other in a straight corridor.

If the rooms are allocated at random, determine the probability that the 2 Australians will have adjacent rooms and the 2 Englishmen will also have adjacent rooms. (4)

[11]

Question 11

The success rate of the Fana soccer team depends on a number of factors. The fitness of the players is one of the factors that influence the outcome of a match.

- The probability that all the players are fit for the next match is 70%
- If all the players are fit to play the next match, the probability of winning the next match is 85%
- If there are players that are not fit to play the next match, the probability of winning the match is 55%

Based on fitness alone, calculate the probability that the Fana soccer team will win the next match. **[5]**

DBE Exemplar 2014

QUESTION 11

11.1 Events A and B are mutually exclusive. It is given that:

- P(B) = 2P(A)
- P(A or B) = 0,57

Calculate P(B). (3)

11.2 Two identical bags are filled with balls. Bag A contains 3 pink and 2 yellow balls. Bag B contains 5 pink and 4 yellow balls. It is equally likely that Bag A or Bag B is chosen. Each ball has an equal chance of being chosen from the bag. A bag is chosen at random and a ball is then chosen at random from the bag.

11.2.1 Represent the information by means of a tree diagram. Clearly indicate the probability associated with each branch of the tree diagram and write down all the outcomes. (4)

11.2.2 What is the probability that a yellow ball will be chosen from Bag A? (1)

11.2.3 What is the probability that a pink ball will be chosen? (3)

[11]

QUESTION 12

Consider the word M A T H S.

12.1 How many different 5-letter arrangements can be made using all the above letters? (2)

12.2 Determine the probability that the letters S and T will always be the first two letters of the arrangements in QUESTION 12.1. (3)

[5]

DBE November 2018

QUESTION 11

Given the digits: 3 ; 4 ; 5 ; 6 ; 7 ; 8 and 9

11.1 Calculate how many unique 5-digit codes can be formed using the digits above, if:

11.1.1 The digits may be repeated (2)

11.1.2 The digits may not be repeated (2)

11.2 How many unique 3-digit codes can be formed using the above digits, if:

- Digits may be repeated
- The code is greater than 400 but less than 600
- The code is divisible by 5 (3)

[7]

QUESTION 12

12.1 Given: $P(A) = 0{,}45$; $P(B) = y$ and $P(A \text{ or } B) = 0{,}74$

Determine the value(s) of y if A and B are mutually exclusive. (3)

12.2 An organisation decided to distribute gift bags of sweets to a Grade R class at a certain school. There is a mystery gift in exactly $\frac{1}{4}$ of the total number of bags.

Each learner in the class may randomly select two gift bags of sweets, one after the other. The probability that a learner selects two bags of sweets with a mystery gift is $\frac{7}{118}$. Calculate the number of gift bags of sweets with a mystery gift inside. (6)

[9]

FINANCIAL MATHEMATICS

Some terminology

1. **Book value** – the value of an asset after depreciation has taken place.
2. **Scrap value** – the book value of an asset at the end of its useful life.
   Normally, the scrap value of an asset is always used as a part payment on the new asset.
3. **A sinking fund** – a fund set up to replace an asset at the end of its useful life.
4. **Annuity** – a series of equal investment payments or loan repayments made at regular intervals of time subject to a rate of interest.
5. **Future value annuity** – Money is invested at regular intervals in order to save money for the future.
6. **Present value annuity** – Regular payments are made to repay a loan from a bank with interest.
7. **Amortised** – when a loan from a bank has been paid together with interest.

FUTURE VALE ANNUITIES(F_V)

$$F = \frac{x[(1+i)^n - 1]}{i}$$ where:

x = payment per period.
n = number of payments.
i = interest rate.
Please note the following, **ITS IMPORTANT**

1. It is normally acceptable in financial mathematics that if an investment takes place on a monthly basis for a period of n years, there will be $12 \times n$ payments, unless otherwise stated.
2. If a payment is made **IMMEDIATELY**, then there will be $n + 1$payments!!!!!
3. Monthly payments – and the money is withdrawn at the end of the last month – what happens??????
   - The last payment accumulates the interest for one month and –
   - The second last payment accumulates the interest for two months and –
   - The third last payment accumulates the interest for three months and so on...............
4. If the money is withdrawn at the beginning of the last month – what happens?????
   - ✓ The last payment does not accumulate interest!!!!!!
   - ✓ The second last payment accumulates the interest for one month and –
   - ✓ The third last payment accumulates the interest for two months and so on..............

1. Precious invests R300 per month into an account that pays 14% p.a. compounded monthly for a period of five years. Her first payment is immediately, and her last payment is on the day the investment matures. How much does she have in this account at the end of 5 years? 6

2. Zinhle wants to save up some money so that she can give her unborn daughter R16 000 on her 16th birthday. On the day that her daughter is born, she starts making equal monthly payments into an account that pays 8% p.a. compounded monthly. Her last payment into this account is due one month before her daughter turns 16.

   2.1 Calculate the size of the monthly payment.

   2.2 How much should Zinhle invest monthly if she wants to give her daughter R21 000 on her 21st birthday, instead of the R16 000 on her 16th birthday? Her last payment into this account is due one month before her daughter turns 21.

3. Which investment will be the better option: monthly deposits of R100into an account paying 12%p.a. compounded monthly or quarterly investments of R300 into an account paying 4% per quarter. Both investments run for one year.

4. On 31 January 2008, Princess deposited R200 in an account that paid 9% interest per annum, compounded monthly. She continued to deposit R200 on the last day of every month until 31 December 2009. She then decided to stop the payments and leave the money in the account until 31 December 2010. Calculate how much money Princess will have in her account on 31 December 2010.

5. A hydraulic lifter costs R550 000 and is expected to have a useful lifetime of 8 years. It depreciates at 10% p.a. on the reducing balance basis. The cost of a replacement lifter is expected to escalate at 18% p.a. effective. A sinking fund is set up to finance the replacement hydraulic lifter in 8years' time. Find, at the time of purchase of the new hydraulic lifter:

   5.1. The scrap value of the old hydraulic lifter.

   5.2. The expected cost of a new hydraulic lifter.

   5.3. The value that the sinking fund must attain, if the scrap value of the old hydraulic lifter is used to defray expenses.

   5.4. The value of the monthly instalments that are made into the sinking fund if payments start immediately and end on the day of the replacement and the sinking fund earns interest of 12% p.a. compounded monthly.

6 A printing press currently costs R850 000. The value of the machine is expected to drop at a rate of 7% per annum simple interest, whilst the cost of a replacement machine escalates at a rate of 14% p.a. compounded annually. The press is expected to have a useful lifetime of8 years.

6.1. Calculate the scrap value of the old machine.

6.2. Calculate the cost of the replacement machine.

6.3. Calculate the amount needed to replace the old machine, if the scrap value is used as part of the payment for the new machine.

6.4. A sinking fund is set up to provide for this balance, paying interest at 15%p.a. compounded monthly. Determine the monthly amount that should be paid into the sinking fund to realize this. Payments start immediately and end 6 months before replacement.

7. How much money must Sandile invest monthly into an ordinary annuity to realise R1 000 000 in 20 years' time if the current rate of investment is calculated at an effective 9% per annum compounded annually? (The payments stretch over the 20 year period)

8. How long should an investor continue to make monthly investments of R1 200 at a rate of 12% per annum compounded monthly if he wishes to have at least R200 000 in order to buy a car cash? Assume that his first payment is immediately and that his last payment is made on the day the investment matures.

9. Harry wishes to start saving for his retirement. He estimates that he will need R4 000 000 in order to retire comfortably. He invests R5 000 a month at an interest rate of 13% per annum, compounded monthly. How many years will he have to wait to retire?

10. Ndumiso starts a 5 year savings plan. He deposits R5 000 into a savings account at the end of every month. The interest rate on the account is 6% per annum compounded quarterly.

NB: interest has to be compounded at the same rate as the payments.

(a) Determine the nominal monthly interest rate which Ndumiso receives.

(b) Determine the amount Ndumiso will receive at the end of the 5 year period.

11. Ahmed deposits R2 000 into a savings account at the end of every quarter for 16 years. The account offers an interest rate of 8,5% p.a. compounded quarterly. Due to

financial difficulty he is unable to make his last two payments. How much will he have in his account after 16 years?

12. At the beginning of January a factory purchases a new delivery truck which they expect to replace in 5 years' time. In order to replace the truck the company will need R200 000 after trading in the current vehicle. What monthly payments must be made into a sinking fund which provides an interest rate of 6,25% p.a. compounded monthly, if payments start 4 months after the original vehicle is initially purchased?

13. Bridgette started saving for a car in Grade 8. She deposited R1500 into a saving account at the end of each term until the end of Matric. If there are 4 terms per year.

   (a) How much will Bridgette have saved for her car ,if she earns 4,3% interest p.a. compounded quarterly?

   (b) Bridgette's parents would like her to have R95 000 to spend on a car at the end of Matric. To help her achieve this, they make equal monthly deposits into an account that earns 3,8% interest p.a. compounded monthly. How much do Bridgette's parents need to deposit each month?

14. At the end of January 2012 Brian deposits R2 500 into a savings account that promises an interest rate of 4,5% compounded quarterly. How many months will Brian have to wait before he has R 372 533,37 in his account, assuming that he continues making the same payment every 3months?

15. Siphiwe wants to save R100 000 for a deposit on a house. He can afford to save R4000 per month and invests in unit trusts that provide an average return of 8,5% p.a. compounded monthly. How long will Siphiwe have to wait to buy his house? Give your answer to the nearest year.

16. Suppose that at the beginning of the month, R1000 is deposited into a bank. At the end of the month, a further R1000 is deposited and a further R1000 at the end of the next month. This continues for eight years. If the interest rate is 6% per annum compounded monthly, how much will have been saved after eight year period?

17. Patrick decided start saving money for a period of eight years starting on 31 December 2009. At the end of January 2010 (in one month's time), he deposited R2300 into the savings plan. Thereafter, he continued making deposits of R2300 at the end of each month for the planned eight year period. The interest rate remained fixed at 10% per annum compounded monthly.

   (a) How much will he have saved at the end of his eight year plan which started on the 31 December 2009?

   (b) If Patrick leaves the accumulated amount in the bank for a further three months, what will the investment then be worth?

18. Anna wants to save R300 000 by paying monthly amounts of R4000, starting in one month's time, into a savings account paying 15% p.a. compounded monthly. How long will it take Anna to accumulate the R300 000?

six months thereafter,R2000 is deposited into the account. The interest rate is 16% p.a. compounded half-yearly. How long will it take to accumulate R100 000?

20. Anisha and Lindiwe each received R12 000 to invest for a period of 5 years. They invested the money at the same time according to the following options:

- **Anisha:** 8,5% p.a. simple interest. At the end of 5 years she will receive an additional bonus pay-out of exactly 7,5% of the original amount.

- **Lindiwe:** 8,5% compounded quarterly.

Who will have the larger final amount after 5 years? Justify your answer with appropriate calculations.

21. NHS bought office furniture that cost R120 000. After how many years will the furniture depreciate to a value of R41 611,57 according to the diminishing-balance method, if the rate of depreciation is 12,4% p.a.?

22. Tebogo opened a savings account with a single deposit of R5000 at the beginning of June 2015.He then made 24 monthly deposits of R800 at the end of every month, starting at the end of June 2015. The account earned interest at 15% p.a. compounded monthly. Calculate the amount that should be in his savings account immediately after he makes the last deposit.

23.Mary invests R2 000 annually, starting on her 21st birthday. The investment earns 12% interest compounded annually. Calculate the value of Mary's investment directly after her deposit on her 26th birthday.

24.An amount of R500 is deposited at the beginning of every month starting on the 1st of January and ending on the 1st of December. Determine the amount that will be in the account at the end of the year if the account earns an interest of 10% per annum compounded monthly.

25. Arshad's birthday is on the 1st of January. On the day he turns 20 he starts to save for his 21st birthday party by placing R200 into a savings account every month with his last payment made on his 21st birthday. How much money will Arshad have for his party, if the account promises an interest rate of 4,5% per annum compounded monthly and his party is to be held on the 1st of February?

PRESENT VALUE ANNUITIES

Present value annuity – Regular payments are made to repay a loan from a bank with interest.

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

26. A loan of R100 000 is repaid by means of 10 semi-annual payments of Rx each. If interest on the loan is charged at 16% per annum compounded semi-annually,

26.1. Determine x if the first payment is made at the end of the first half year. (R14 902 , 95)
26.2. Determine the semi-annual payment if the fist payment is in six months time and if a deposit of R15 000 was given. (R12 667, 51)

27. A loan of R1 000 is paid off by equal monthly payments of R88,85 per month at a rate of 12% p.a. compounded monthly. How long does it take to amortise the loan, if the first payment is made at the end of the first period. (12 months)

28. A loan is amortised by 24 monthly payments of R2 000 each made into an ordinary annuity with interest charged at 14% per annum compounded monthly. Determine the value of the loan. (R41 655,49)

29. A man plans to buy a house on a 24 year mortgage and can only afford to pay R2 700 per month. If the interest rate is currently 22% per annum compounded monthly, determine the size of the mortgage he can take, if he starts paying one month after the mortgage was approved. (R146 486,02)

30. A loan of R150 000 is amortized by equal quarterly payments for a period of 10 years at a rate of 14% p.a. compounded quarterly. Determine the size of each quarterly payment. (R7 024,09)

31. Determine the amount that must be invested now to realise equal monthly withdrawals of R3 500 for the next 15 years if interest is at 16% p.a. compounded monthly. The first withdrawal will be in one months' time. (R238 305,86)

32. Sandra buys a slimming machine, and pays a deposit of R2 000 on the purchase. The balance is paid off by 36 equal monthly instalments of R 1800 each. Interest is calculated at 23% p.a. compounded monthly.

32.1. Calculate the purchase price of the slimming machine if the first payment is made at the end of the first month. (R48 499,95)

32.2. What amount would she have saved if she made the purchase cash? (R18 300,05.)

32.3. If she took a loan for the balance and paid this loan back at 23% p.a. compounded quarterly with equal quarterly payments, would she have saved on the loan repayments? (she pays R847,32 more.)

33. A competition makes a startling claim that you can win a prize of 1 million rand. The small print informs us that the prize will be paid out in equal annual instalments of R50 000 over the next 20 years, starting now. Assuming an average inflation rate of 14% p.a. over the next twenty years, show that the present value of the prize is significantly less than the claimed 1 million rand.
(It is R668 843,47 short of R1 million!!!!)

34. When considering the purchase of a house, Mr Pillay has to take the following into account:

- The house is on the market for R570 000
- He has R80 000 available as a deposit
- The Bank's condition for granting a mortgage bond (loan) for the balance is that his monthly repayments may not exceed $\frac{1}{3}$ of his monthly salary.
- His salary is R17 500 per month
- The bank is offering mortgage bonds at 16,25% p.a. compounded monthly, repayable in equal monthly instalments over 20 years.

34.1. Show that Mr Pillay does not meet the 'third requirement' above. Set your argument out clearly.

34.2. As he is determined to purchase the house, he decides on a two-pronged strategy :

- to put in an offer to purchase which is R50 000 less than the asking price;
- to ask the bank to let him repay the bond over a longer period

Calculate how many years he will need to pay off the loan. (23 years and 10 months)

PRESENT VALUE AND OUTSTANDING BALANCE

35. A loan is paid off by 3 equal payments of R3 000 over a period of 3 years. Determine the original amount of the loan if interest is charged at a rate of 10% per annum compounded annually.

36. David takes out a bank loan to pay for his car. He repays the loan by means of monthly payments of R4000 for a period of five years starting one month after granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the purchase price of his new car.

37. Peter inherits R400 000 from his father. He invests the money at an interest rate of 12% per annum compounded monthly. He wishes to earn a monthly salary from the investment for a period of twenty years starting in one month's time. How much will he receive each month?

38. Pat borrows R500 000 from a bank and repays the loan by means of monthly payments of R8000, starting one month after granting of the loan. Interest is fixed at 18% per annum compounded monthly. How many payments of R8000 will be made?

39. Simphiwe takes out a twenty year loan of R100 000. She repays the loan by means of equal monthly payments starting three months after granting of the loan. The interest rate is 18% per annum compounded monthly. Calculate the monthly payments.

40. How long will it take to repay a loan of R400 000 if the first quarterly payment of R17 000 is made three months after the granting of the loan and interest rate is 16% per annum quarterly?

41. Simon wishes to purchase a car for R80 000. If he borrows the money at an interest rate of 8,5% per annum compounded monthly, calculate:

41.1 the monthly instalments if the loan is to be paid back over 5 years.

41.2 The outstanding balance on the loan after 3 years.

41.3 The amount that would be saved by settling the loan after 3 years.

42. A loan of R50 000 is to be paid off by equal quarterly payments over 10 years. If the interest is charged at rate of 10% per annum compounded quarterly, determine:

42.1 The value of the repayments.

42.2 The outstanding balance of the loan after 7 years.

43. A loan of R60 000 is paid off over a period of 5 years by equal monthly payments at an interest rate of 9,5% per annum compounded monthly. Determine the balance outstanding on the loan after 3 years.

44. A loan of R200 000 is paid off over a period of 8 years by equal monthly payments at an interest rate of 8% per annum compounded monthly. Determine the balance outstanding on the loan after 6 years.

45. Donna wishes to purchase a car that costs R192 000. She takes out a 5 year loan at an interest rate of 12% p.a. compounded monthly.

45.1 What are the monthly instalments that Donna will have to pay on her loan?

45.2 What is the balance outstanding on the loan after Donna makes her 45th payment?

FINAL PAYMENTS WITH REDUCED VALUES

48. Candice borrows R100 000 at an interest rate of 10% p.a. compounded monthly. She repays the loan by means of equal monthly payments of R5 500 and final payment which is less than R5 500.

    48.1 How many months will it take for the loan to be repaid?

    48.2 What will the outstanding balance on the loan be after the 19th payment?

    48.3 What is the final payment?

49. Thabo borrows R5 000 from a micro lender at an interest rate of 28% p.a. compounded monthly. He repays the loan by means equal monthly payments of R800 and a final payment less than R800.

    49.1. Determine the number of payments at R800.

    49.2. What will the value of the final payment be?

50. Siphiwe borrows R10 000 on the 1st of February 2011 at an interest rate of 9,5% p.a. compounded monthly. He pays the loan back via monthly instalments of R450. The loan agreement allows Siphiwe to make his first payment on the 1st of August 2011.

    50.1 Determine how many months it will take Siphiwe to pay back the loan once payments start.

    50.2 What is the balance of the loan immediately after the 25th payment?

    50.3 What is the value of the final payment?

51. Mr Mzinyathi takes out a home loan for 1,2 million rand to be paid back over 30 years at an interest rate of 10% p.a. compounded monthly.

51.1 Determine Mr Mzinyathi's monthly repayments.

51.2 Determine the balance outstanding on the loan after 20^{th} payment.

51.3 Mr Mzinyathi has financial difficulties and is unable to make the 21^{st},22^{nd} and 23^{rd} payment. The bank agrees to restructure the loan so that it is paid off in the same amount of time. Determine the new monthly repayments.

52. A loan of R500 000 is taken out at an interest rate of 9%p.a. compounded monthly. It is repaid by equal monthly payments of R6 000 and a final payment less than R6000.

52.1 How many payments of R6000 are needed to pay off the loan?

52.2 What is the outstanding balance on the loan after the last payment of R6 000?

52.3 What is the value of the last payment?

53. A loan of R750 000 is taken out at an interest rate of 12,5% p.a. compounded biannually. It is repaid by equal biannual payments of R55 000 and a final payment less than R55 000.

53.1 How many payments are required to settle the loan.

53.2 What is the outstanding balance on the loan after the final payment of R55 000?

53.3. What is the value of the final payment?

54. Lerato takes out a loan of R100 000 to buy a car. The loan agreement stipulates that it has to be repaid by means of 238 equal payments at an interest rate of 18% p.a. compounded monthly starting 3 months after the loan is granted. Determine Lerato's monthly payments.

55. A loan of R50 000 is repaid over a period of 5 years, from the time that the loan is taken, by equal monthly payments at an interest rate of 5,8% p.a. compounded monthly. If payments start 6 months after the loan is granted, what monthly payments are required to repay the loan?

56. A loan of R150 000 is taken out at an interest rate of 7% p.a. compounded monthly. Payments start 6 months after the loan is taken out and is repaid by means of 60 equal men

56.1 Calculate the outstanding balance 5 months after the loan is taken out.

56.2 Determine the monthly repayments required to pay back the loan.

57. Jessica takes out a loan of R180 000 which is to be repaid over a 5 year period. The loan agreement allows for payments to start 3 months after the granting of the loan at an interest rate of 10,5% per annum compounded monthly.

57.1 Determine the outstanding balance of the loan 2 months after the loan is granted.

57.2 What are the monthly repayments that Jessica has to make in order to pay the loan off in 5 years?

58. A loan of R350 000 is taken out at an interest rate of 5,8% p.a. compounded quarterly. The loan is taken on the 1st March 2011. The first payment is made on the 1st December 2011 and is repaid by 72 equal payments.

58.1 What is the outstanding balance of the loan on the 1st of September 2011?

58.2 Determine the monthly repayments required to pay back the loan.

EXERCISES

4.1: Simple and Compound interest

4.1.1 The value of a car depreciates over a period of 5 years. Five years ago, the value of the car was R119 500, but it is now worth R85 670. Calculate annual rate of depreciation, if it is calculated using the reducing method. (4)

4.1.2 Sam deposited R100 000 into a fixed deposit account that pays interest at 6,2% p.a. compounded annually. After how many years will the initial deposit accumulate to R190 000? (3)

DBE May/June 2019

4.2.1 Sandile bought a car for R 180 000. The value of the car depreciated at 15% per annum according to the reducing- balance method. The book valueof Sandile'scar is currently R79 866,96.

4.2.1.1 How many years ago did Sandile buy the car? (3)

4.2.1.2 At exactly the same time Sandile bought the car, Anil deposited R49 000 into a savings account at an interest rate of 10% p.a., compounded quarterly. (3)

4.2.1.3 Has Anil accumulated enough money in his savings account to buy Sandile's car now? (3)

4.2.2 Mbali invested RIO 000 for 3 years at an interest rate of r % p.a., compounded monthly. At the end of this period, she received R12 146,72. Calculate r, correct to ONE decimal place. (5)

Exercise 4.3: Annuities

4.3.1 Margaret decides to start saving money for an overseas trip in 10 years' time. She immediately deposits R1 200 000 into a savings account every 3 months for 10 years. The interest rate is 10,5% p.a. compounded quarterly.

4.3.1.1 How many payments of R1 200 does she make? (1)

4.3.1.2 How much money is in her account after her final payment at the end of the 10^{th} year? (3)

4.3.2 John wishes to buy a house for R1 300 000. He obtains a bank loan at a rate of 10% p.a. compounded monthly. The loan must be paid over 20 years.

4.3.2.1 Calculate his monthly payment. (3)

4.3.2.2What is the outstanding balance after the 215^{th} payment. (4)

4.3.2.3 If he had decided to pay R13 000 each month instead, how many payments would he have had to make in order to pay off the loan? (4)

4.3.3 Carla borrowed R600 000 from a bank to start a business. The bank charges her interest at 9% p.a. compounded monthly. The loan is to be repaid by monthly repayments over a period of 10 years from the date it was granted. Calculate the monthly repayment if Carla is only able to start repaying the loan from exactly 4 months after it was granted. (4)

4.3.4 Zanele opened a savings account on 30 June 2012. The account paid interest at the rate of 6% p.a. compounded monthly. She made her first deposit of R1 500 on 31 July 2012 and continued to deposit the same amount at the end of every month thereafter until 30 June 2016.

4.3.4.1 Calculate how much Zanele had in this account on 30 June 2016. (3)

4.3.4.2 Calculate how much interest Zanele earned from 1 July 2016 to 30 June 2017 if she continues to save R1 500 a month for a further year. (4)

DBE/Feb.- Mar. 2018

4.3.5On 30 June 2013 and at the end of each month thereafter, Asif deposited R2 500 into a bank account that pays interest at 6% per annum, compounded monthly. He wants to continue to deposit this amount until 31 May 2018.

Calculate how much money Asif will have in this account immediately after depositing this amount until 31 May 2018. (3)

4.3.6 On 1 February 2018, Genevieve took a loan of R82 000 from the bank to pay for her studies. She will make her first payment of R3 200 on 1 February 2019 and continue to make payments of R3 200 on the first of each month thereafter until she settles the loan. The bank charges interest at 15% per annum, compounded monthly.

4.3.6.1 Calculate how much Genevieve will owe the bank on 1 January 2019. (3)

4.3.6.2 How many instalments of R3 200 must she pay? (5)

4.3.6.3 Calculate the final payment, to the nearest rand, Genevieve has to pay to settle the loan. (5)

DBE May/June 2019

4.3.7 Exactly 10 months ago, a bank granted Jane a loan of R800 000 at an interest rate of 10,25% p.a. compounded monthly.
The bank stipulated the loan:

- Must be repaid over 20 years
- Must be repaid by means of monthly repayments of

R7 853,15, starting one month after the loan was granted

4.3.7.1 How much did Jane owe immediately after makingher 6^{th}repayment? (4)

4.3.7.2 Due to financial difficulties, Jane missed the 7^{th}, 8^{th} and 9^{th} payments. She was able to make payments from the end of the 10^{th} month onwards. Calculate Jane's increased monthly payment in order to settle the loan in the original 20 years. (5)

4.3.8.1 Piet takes a loan from a bank to buy a car for R235 000. He agreesão repay the loan over a period of 54 months. The first instalment will be paid one month after the loan is granted. The bank charges interest at p.a., compounded monthly.

4.3.8.1.1Calculate Piet's monthly instalment. (4)

4.3.8.1.2 Calculate the total amount of interest that Piet will pay during the first year of the repayment of the loan. (6)

4.3.9.1 Selby decided today that he will save RI 5 000 per quarter over the next four years. He will make the first deposit into a savings account in three months' time and he will make his last deposit at the end of four years from now.

4.3.9.1.1 How much will Selby have at the end of four years if interest is earned at 8,8% per annum, compounded quarterly? (3)

4.3.9.1.2 If Selby decides to withdraw R100 000 from the account at the end of th•ee years from now, how much will he have in the account at the end of four years from now? (3)

4.3.10 Tshepo takes out a home loan over 20 years to buy a house that costs RI 500 000.

4.3.10.1 Calculate the monthly instalment if interest is charged at 10,5% p.a., compoundemonthly. (4)

4.3.10.2 Calculate the outstanding balance immediately after the 144^{th} payment

7. DATA HANDLING

QUESTION 2

In an experiment, a group of 23 girls were presented with a page containing 30 coloured rectangles. They were asked to name the colours of the rectangles correctly as quickly as possible. The time, in seconds, taken by each of the girls is given in the table below.

12	13	13	14	14	16	17	18	18	18	19	20
21	21	22	22	23	24	25	27	29	30	36	

2.1 Calculate:

2.1.1 The mean of the data (2)

2.1.2 The interquartile range of the data (3)

2.2 The standard deviation of the times taken by the girls is 5,94. How many girls took longer than ONE standard deviation from the mean to name the colours? (2)

2.3 Draw a box and whisker diagram to represent the data on the number line provided in the ANSWER BOOK. (3)

2.4 The five-number summary of the times taken by a group of 23 boys in naming the colours of the rectangles correctly is (15 ; 21 ; 23,5 ; 26 ; 38).

2.4.1 Which of the two groups, girls or boys, had the lower median time to correctly name the colours of the rectangles? (1)

2.4.2 The first three learners who named the colours of all 30 rectangles correctly in the shortest time will receive a prize. How many boys will be among these three prizewinners? Motivate your answer. (2)
[13]

QUESTION 3

3.1 A company has 132 employees working in their Cape Town branch. The distance (x), in kilometres, the travel to work each day is summarised in the frequency table below:

Interval	Frequency	Cumulative frequency
$0 < x \leq 5$	12	12
$5 < x \leq 10$	29	41
$10 < x \leq 15$	13	54
$15 < x \leq 20$	63	117
$20 < x \leq 25$	12	129
$25 < x \leq 30$	3	132

3.1.1 Determine the median interval for this data. (1)
3.1.2 Determine the estimated mean distance covered. (2)
3.1.3 Determine the standard deviation for the data. (2)
3.1.4 Draw an ogive (cumulative frequency curve) for this data on the axes on the answer sheet. (3)

3.2 The number of points scored by two Formula 1 racing drivers during the course of a season are given in the table below. Note that Driver B does only 12 races.

A	1	1	1	2	6	6	8	8	8	8	10	10	10
B	1	2	6	8	8	8	8	8	8	10	10	10	

3.2.1 Using your calculator, determine the mean and standard deviation for each driver. (4)

3.2.2 Discuss the performance of each driver by referring to your answers to Question 1.2.1 above. (2)

QUESTION 4

4.1 Consider the table below, where the age of the car is given in years, and the selling price is given in Rands.

Age of car	1	2	3	4	5	6	7
Selling price	132 000	129 900	121 000	110 000	80 000	74 900	69 900

4.1.1 Determine the value of the correlation coefficient, ***r***, and use it to describe the relationship between the age of the car and its selling price. (4)

4.2 The following table shows a comparison between the number of goals scored and the number of red cards issuedat each World Cup Tournament held since 1970:
(www.planetworldcup.com)

Year	Goals scored per tournament (x)	Red cards per tournament (y)
1970	95	0
1974	97	5
1978	102	3
1982	146	5
1986	132	8
1990	115	16
1994	141	15
1998	171	22
2002	161	17
2006	147	28
2010	145	17
2014	171	10

4.2.1 Draw a scatter plot for the data . (3)

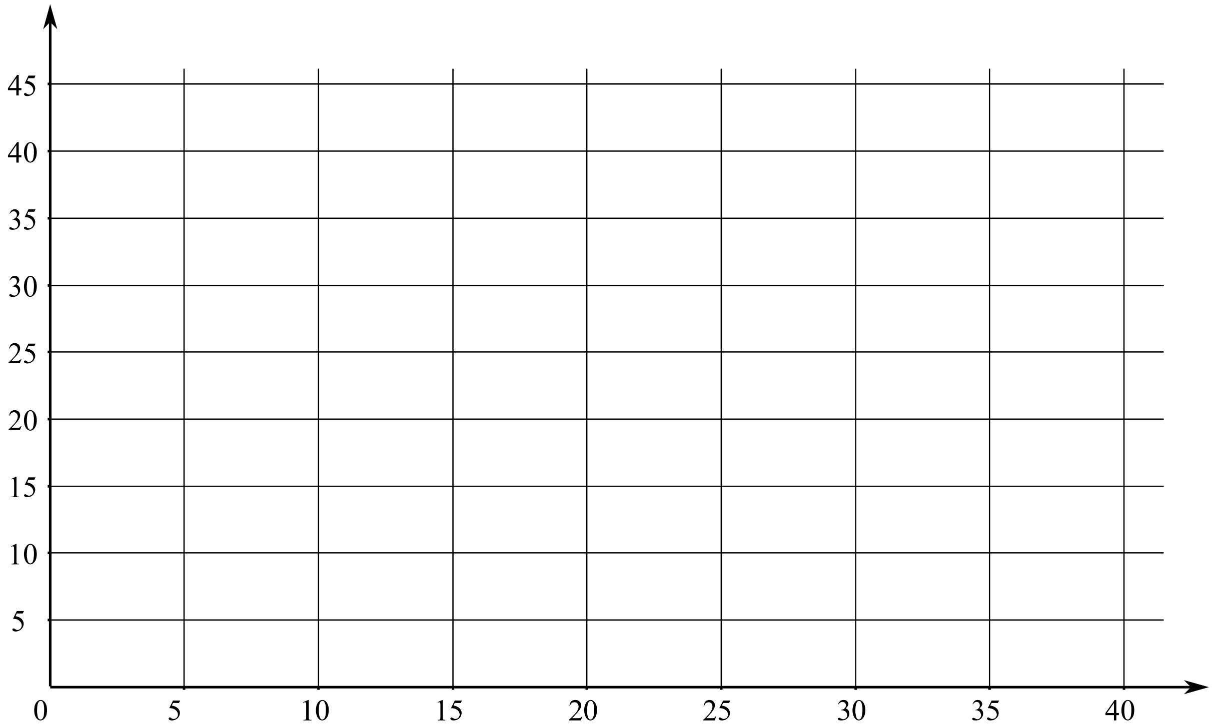
4.2.2 Determine the equation of the least squares line. (4)

QUESTION 5

5.1 The following table summarises delivery data gathered from a pizza company on a Monday night. x represents the number of minutes taken to deliver a pizza from the time the order was taken.

Interval	Frequency	Cumulative Frequency
$15 \le x < 20$	6	6
$20 \le x < 25$	13	19
$25 \le x < 30$	12	31
$30 \le x < 35$	10	41
$35 \le x < 40$	2	43

5.1.1 Draw an ogive curve that represents the number of deliveries over time.

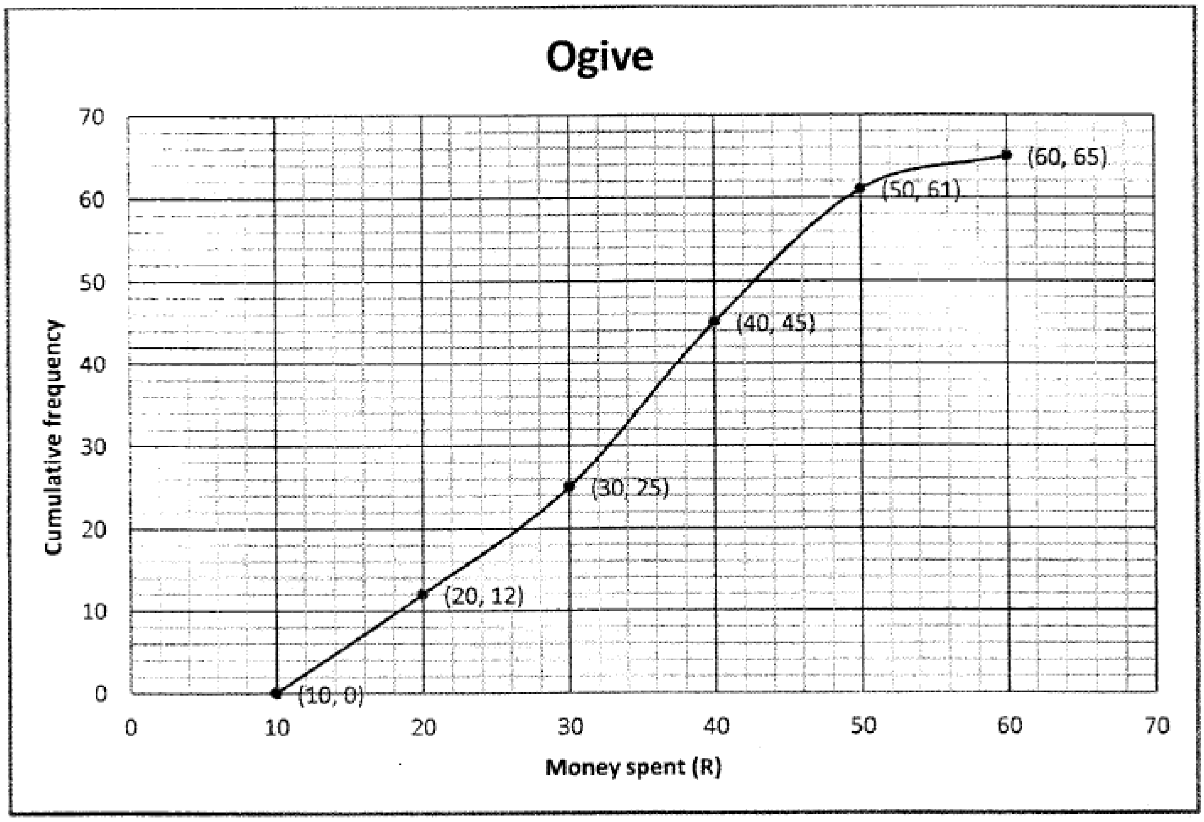


(3)

5.1.2 Use the ogive to determine the percentage of customers who will receive their pizza after 32 minutes. (2)

5.1.3 Use your graph to find the lower and upper quartiles and the median. Draw them onto the graph and label the values clearly. (2)

5.2 The amount of money, in rands, that learners spent while visiting a tuck shop at school on a specific day was recorded. The data is represented in the ogive below



An incomplete frequency table is also given for the data

Amount of money (in R)	$10 \leq x < 20$	$20 \leq x < 30$	$30 \leq x < 40$	$40 \leq x < 50$	$50 \leq x < 60$
Frequency	a	13	20	b	4

5.2.1 How many learners visited the tuck shop on that day?
5.2.3 Write down the modal class of this data. (1)
5.2.3 Determine the values of a and b in the frequency table.
(2)
5.2.4 Use the ogive to estimate the number of learners who spent at least R45 on the day the data was recorded at the tuck shop.
(2)

QUESTION 6

6.1 The ages of 25 residents of Sunny day security village are recorded below.

25 35 42 49 53 54 60 64 65 67 68 69 70
72 72 75 77 78 82 85 89 90 91 97

6.1.1 Represent the above data using a box and whisker diagram.

6.1.2 Could the distribution of the ages of the residents be described as symmetrical?Justify your answer.

6.1.3 Determine the mean age of the residents at Sunnydale, and calculate how many are within one standard deviation of the mean (5)

QUESTION 7

The heights of several plants (in cm) were measured at a certain stage after planting, and the following data was recorded:

x = days after planting	14	20	8	15	18	11	14
h = height (cm)	6	11	3	8	10	4	

The record of the last height has been lost, but we do know that the regression line had equation $h = 0{,}72x - 3{,}31$.

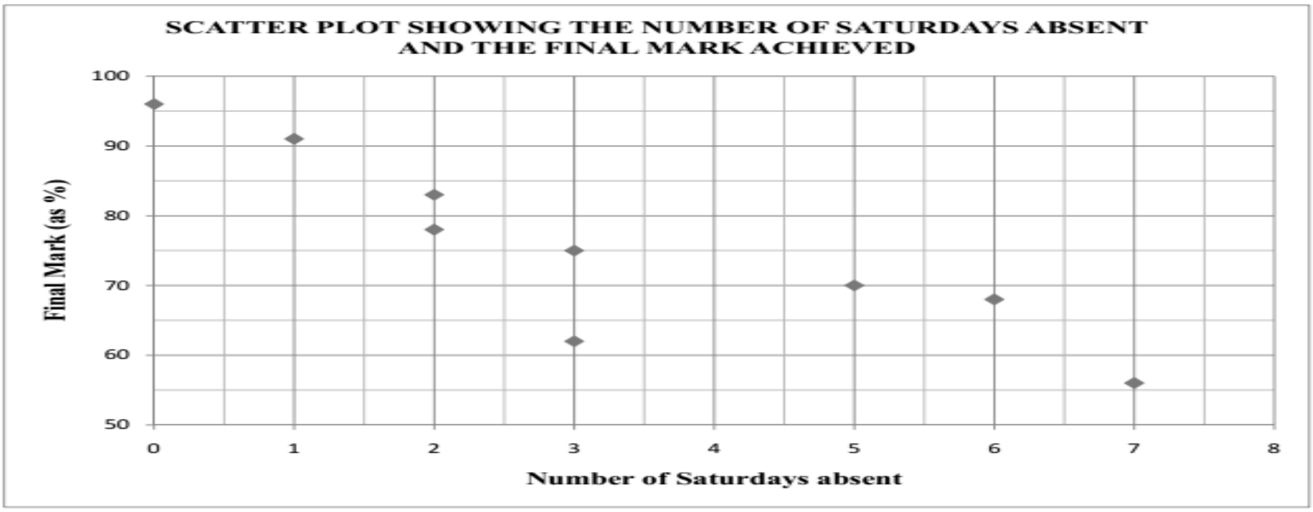
7.1.1 Estimate, to the nearest centimeter, what the last recorded height was. (2)

7.1.2 Calculate the correlation coefficient for the data relating to the first 6 plants (i.e. ignoring the last column). (2)

7.1.3 Sometime later another plant's height 25 days after planting was found to be 20 cm. Comment on how surprising (or not) this is in the light of your previous results. (2)

7.2 A group of students attended a course in statistics on Saturdays over a period of 10 months. The number of Saturdays on which a student was absent was recorded against the final mark of the student. The information is shown in the table below and the scatter plot is drawn for the data.

Number of Saturdays absent	0	1	2	2	3	3	5	6	7
Final mark (as %)	96	91	78	83	75	62	70	68	56



7.2.1 Calculate the equation of the least squares regression line. (3)

7.2.2 Draw the least squares regression line on the scatter plot on the diagram sheet. (2)

7.2.3 Calculate the correlation coefficient. (2)

7.2.4 Comment on the trend of the data. (2)

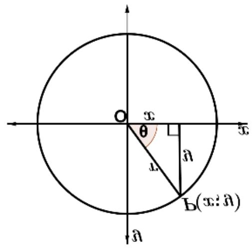
7.2.5 Predict the mark for a student who was absent for 4 Saturdays. (2)

6 **TRIGONOMETRY**

DEFINITIONS AND SIMPLE RELATIONSHIPS

1. DEFINITIONS

If given a point P(x; y) and radius = r, then



$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

(a) Note that the radius is always positive
(b) The signs of the coordinates of P change in the different quadrants
(c)If OP moves anticlockwise from the positive x −axes, then θ is negative.
(c) Integral multiples of 360° may be added to θ without changing the values of the function ∴ **200°**; **−160°**; **560° and 920°** are the same.

1. SIGNS OF RATIOS IN THE FOUR QUADRANTS

The diagram shows those ratios which are positive in each quadrant.
We call this diagram the CAST diagram

Any function of $(180° \pm \theta)$ or $(360 \pm \theta)$ is numerically equal to the same function of θ
We have to decide whether the ratio will be positive or negative in that specific quadrant.

EXAMPLES /ACTIVITIES

(Basic Trig, Simplification, Identities, Equations)

1 Given: $\sin\alpha = \frac{3}{5}$ and $90^{\circ} < \alpha < 270^{\circ}$

With the aid of a sketch and **without using a calculator**, determine:

1.1 $\tan\alpha$ (3)

1.2 $\cos(90+\alpha)$ (2)

1.3 $\cos 2\alpha$ (3)

2 Given: $\sin 38^{\circ} = p$, determine the following in terms of p.
(Without using a calculator.)

2.1 $\cos(-38^{\circ})$ (3)

2.2 $\sin 76^{\circ}$ (2)

3 Simplify the following without using a calculator:

$$\frac{\sin 150^\circ . \tan 225^\circ}{\sin\left(-30^\circ\right).\sin 420^\circ}$$ (6)

4 Prove:

$$\frac{2\sin^2 x}{2\tan x - \sin 2x} = \frac{1}{\tan x}$$ (6)

5. Prove the following identity:

$$\frac{\cos A - \cos 2A + 2}{3\sin A - \sin A} = \frac{1 + \cos A}{\sin A}$$ (6)

6 Given: $\sin\alpha = \frac{3}{5}$ and $90^\circ < \alpha < 270^\circ$

With the aid of a sketch and **without using a calculator**, determine:

$\tan\alpha - \cos\alpha$ (2)

7 Given: $\sin 38^\circ = p$, determine the following in terms of p.

(Without using a calculator.)

7.1 $\cos\left(-38^\circ\right)$ (3)

7.2 $\sin 76^\circ$ (2)

8 Simplify the following without using a calculator:

$$\frac{\sin 150^\circ . \tan 225^\circ}{\sin\left(-30^\circ\right).\sin 420^\circ}$$ (6)

9. Prove:

$$\frac{2\sin^2 x}{2\tan x - \sin 2x} = \frac{1}{\tan x}$$ (6)

10. If $4\tan\theta = 3$ and $0^\circ \le \theta \le 360^\circ$, determine with the aid of a diagram:

10.1 $\sin\theta + \cos\theta$ (3)

10.2 $\tan 2\theta$ (3)

11. If $\sin 36^\circ \cos 12^\circ = p$ and $\cos 36^\circ \sin 12^\circ = q$, determine in terms of p and q the value of:.

11.1 $\sin 48^\circ$ (3)

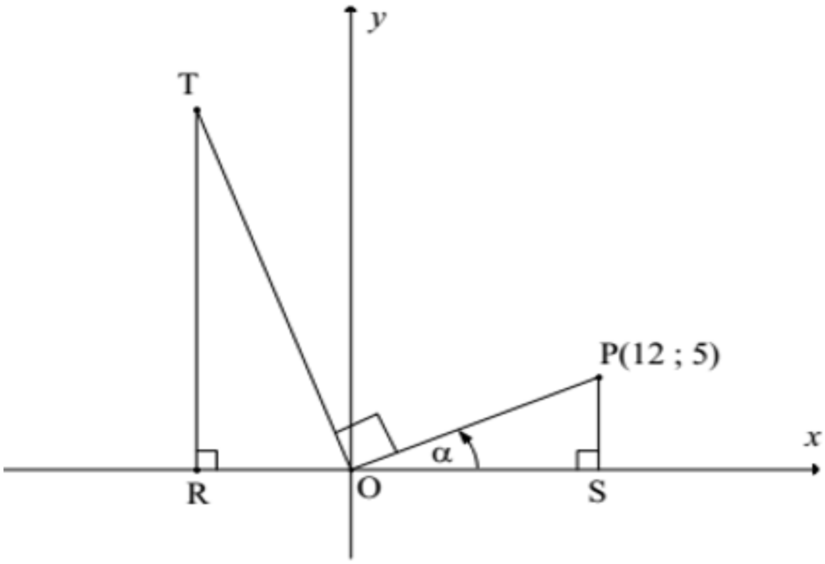
11.2 $\sin 24^\circ$ (2)

11.3 $\cos 24^\circ$ (2)

12. If $\sin 28° = a$ and $\cos 32° = b,$ determine the following in terms of *a* and *b*:
12.1.1 $\cos 28°$
12.1.2 $\cos 64°$
12.1.3 $\sin 4°$

12.2 Prove without the use of a calculator, that if $\sin 28° = a$ and $\cos\ 32° = b,$ then:
$b\sqrt{1-a^2}\ -a\sqrt{1-b^2} = \frac{1}{2}$

13. In the diagram, P is the point (12;5). OT $\perp$ OP. PS and TR are perpendicular to the *x*-axis.
$P\hat{O}A = \alpha$ and
OR=7,5 units.



Determine:

13.1	$\cos\alpha$	(3)
13.2	$T\hat{O}R$ in terms of α	(3)
13.3	The length of OT.	(2)
		[08]

14. **Without using a calculator**, Determine the following in terms of $\sin 36°$:
14.1.1 $\sin 324°$ (1)
14.1.2 $\cos 72°$ (2)

14.2 $1-\frac{\tan^2\theta}{1+\tan^2\theta} = \cos^2\theta$ (4)

14.3 $1-\frac{\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} = \frac{1}{4}$ (6)

14.4 Given: $\cos(A-B) = \cos A\cos B + \sin A\sin B$

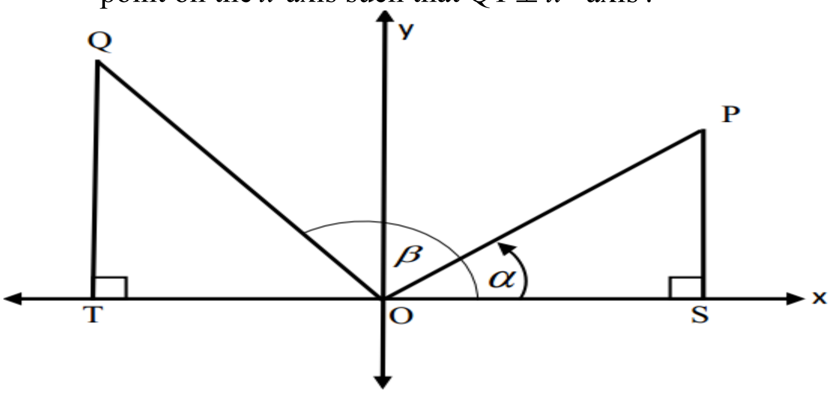
14.4.1 Use the formula for $\cos(A-B)$ to derive the formula for $\sin(A-B)$ (4)

14.4.2 **Without using a calculator**, show that:

$$\sin(x+64°)\cos(x+379°)+\sin(x+19°)\cos(x+244°)=\frac{1}{\sqrt{2}}$$ (6)

[23]

15. In the diagram below, the equation of OP is given by $3y-2x=0$. S is a point on the x-axis such that $PS \perp x$ - axis. $S\hat{O}P = a$. The line segment OQ is drawn such that $S\hat{O}Q = b$. T is a
point on the x-axis such that $QT \perp x$ - axis.



15.1 Show that $\tan\alpha = \frac{2}{3}$ (4)

15.2 Calculate the value of $\sin\alpha$ (2)

15.3 Write down $Q\hat{O}P$ in terms of and α and β. (2)

15.4 If it is given that $\sin\beta = \frac{3}{5}$, find the value of $Q\hat{O}P$ (3)

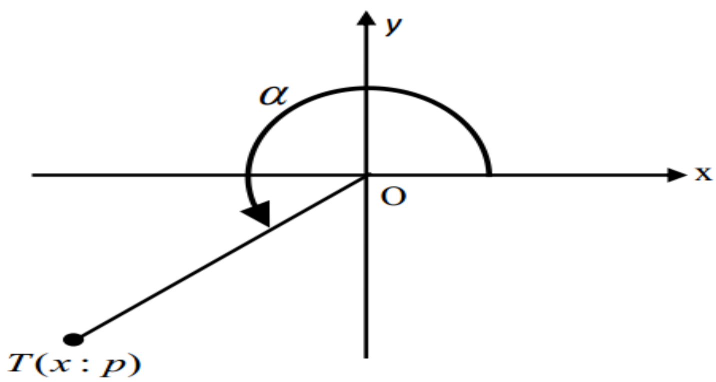
[15]

16.

16.1 If $x = 3\sin\theta$ and $y = 3\cos\theta$, determine the value of x^2+y^2

16.2 Simplify to a single term:
$\sin(540°-x).\sin(-x) - \cos(180°-x).\sin(90°+x)$

16.3 In the diagram below, T($x;p$) is a point in the third quadrant and it is given that
$\sin\alpha = \frac{p}{\sqrt{1-p^2}}$



16.3.1 Show that $x = -1$

16.3.2 Write $\cos(180° + \alpha)$ in terms of p in its simplest form.

16.3.3 Show that $\cos 2\alpha$ can be written as $\frac{1-p^2}{1+p^2}$

17 If $\sin\alpha = -\frac{5}{13}$; $\tan \alpha < 0$ and $\cos\beta = \frac{3}{5}$; $180° < \beta < 360°$, evaluate:

i sin 2α

ii $\cos(180° - 2\beta)$

TRIGONOMETRICAL IDENTITIES AND EQUATIONS

The Prior Knowledge The Background Knowledge
The Assumed Knowledge The Previous Knowledge
The Perceived Knowledge

1. The Difference between an Equation and an Identity

(a) $\boldsymbol{Sin\theta = 0,5}$ is a trigonometrical equation which is only true for certain values of $\boldsymbol{\theta}$, e.g. $\mathbf{30}°$ and
150°; in the interval $[\mathbf{0}°; \mathbf{360}°]$

(b) $\boldsymbol{tan\theta = \frac{sin\theta}{cos\theta}}$ is a trigonometrical identity which is true for all values of $\boldsymbol{\theta}$, for which both sides are meaningful, i.e. for which both sides are defined.

(c) An identity consists of two sides, namely, a left hand side (LHS) and a right hand side (RHS).

(d) If we have to prove an identity, we have to simplify one side until it is equal to the other side. (Sometimes it is necessary and sufficient to simplify both sides separately)

$(\boldsymbol{e})$ $\boldsymbol{tan\theta = \frac{sin\theta}{cos\theta}}$ (f) $\boldsymbol{sin^2\theta + cos^2\theta = 1}$

(g) Should squares occur, try to rewrite them using the squares identity

(h) Replace function values of $(\mathbf{90}° \pm \boldsymbol{\theta})$; $(\mathbf{180}° \pm \boldsymbol{\theta})$; and $(\mathbf{360}° \pm \boldsymbol{\theta})$, for example, with function
values of $\boldsymbol{\theta}$.

(i) Express all the remaining functions in terms of $\boldsymbol{Sin\theta\ andCos\theta}$

(j) Now try to simplify by means of division or the addition of fractions (Using the LCM)

(k) Sometimes it is necessary to multiply the expression (LHS or RHS) by one, e.g. by $\boldsymbol{\frac{1+sin\theta}{1+sin\theta}}$

2. Function values

(a) Sometimes you often have to determine the values of $\boldsymbol{\theta}$ for which an expression is invalid

(b) Division by zero (0) is invalid (undefined), meaningless.

(c) Certain trigonometrical function values of $\boldsymbol{\theta}$ if $\boldsymbol{\theta} \in [-\mathbf{360}°; \mathbf{360}°]$ like $\boldsymbol{tan\theta}$ are undefined if $\boldsymbol{\theta} = \pm\mathbf{90}°$; $\pm\mathbf{270}°$. i.e. where the denominator is equal to zero.

3. General Solution of Trigonometrical Equations

(a). When the interval in which a solution is found, is not restricted, trigonometric equations can have
infinitely many solutions.

(b) The "general solution" is a way of writing down "all" these solutions.

(c) In order to determine the value of $\boldsymbol{\theta}$ if $\boldsymbol{sin\theta = 0,766}$ we do the following:

$\boldsymbol{sin\theta = 0,766}$

(Remember $\boldsymbol{sin\theta > 0}$ in the first and second quadrant)

$\boldsymbol{\theta = 50}°$ or 130°

(d) Since the period of $\boldsymbol{sin\theta}$ is 360°, we may add integral multiples of 360°

$\therefore \boldsymbol{\theta = 50° + k.360°}$ or $\boldsymbol{\theta = 130° + k.360°, k \in \mathbb{Z}}$

$\therefore \boldsymbol{\theta = 50°, -310°; 130°;\ -230°; where\ \theta \in [-360°; 360°]}$

4. Hints for the solution of trigonometric equations

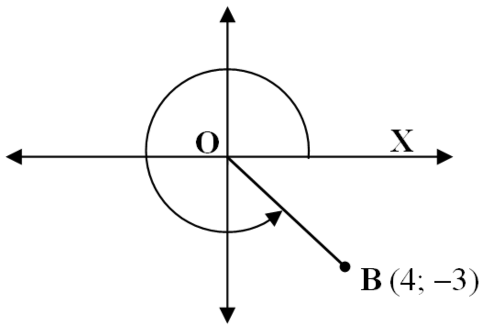
(a) Normally the equation is simplified to one trigonometrical ratio with one angle.
(b) Simplify the equation by using identities where possible.
(c) Compound angles and double angles must be simplified to single angles.
(d) Factorization, as in the solution of algebraic equations, is often used.
Look out for:
(i) Common factor
(ii) Difference of squares
(iii) Quadratic trinomials
(iv) Grouping of polynomials
(e) When squaring a solution on both sides, additional solutions are usually obtained. Do not forget to check your answers.
(f) Take care not to divide by an unknown variable. Make absolutely sure that the value is not equal
to zero.
(g) Use the rules to determine in which quadrants the angle will be
(h) Remember the general solution

Examples /Activities

QUESTION 1 (Basic Trig, Simplification, Identities, Equations)

(a) Refer to the sketch below.

(b) θ is the reflex angle $X\hat{O}B$ with B (4; –3) given.



Determine, without the use of a calculator, the value of $\dfrac{\sin 2\theta}{\cos\theta}$. (3)

(b) (i) Prove: $\dfrac{\sin 5x + \sin x}{2\cos 2x} = \sin 3x$

(Hint: $5x = 3x + 2x$ and $x = 3x - 2x$) (5)

(ii) For which values of x is this identity, above, undefined? Give the general solution. Round off your answer to 1 decimal place.
(3)

[11]

QUESTION 2

2.1 Given: $\sin 2A = \frac{\sqrt{15}}{8}$ for $0° \leq 2A \leq 90°$

2.1.1 Determine the value of $\cos 2A$ without the use of a calculator. (3)

2.1.2 Hence, find the value of $\cos A$ without the use of a calculator.
Leave your answer in surd form. (4)

2.2 Given $\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

2.2.1 Derive the formula for $\cos(A+B)$. (3)

2.2.2 If $A + B = 90°$, show that: $\cos^2(A-B) = 4\cos^2 B \sin^2 B$ (4)

[14]

QUESTION 3 (Reduction and identities)

3.1 Simplify, without the use of a calculator: $\frac{\sin 140°.\sin 120°}{\sin 110°.\sin 340°}$ (6)

3.2 Given: $\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \tan x$

3.2.1 Prove the identity. (4)

3.2.2 Determine for which value(s) of x the identity is invalid if $x \in [0°; 360°]$. (3)

[13]

Question 4 (Identities)

Prove the fundamental Identity

$\frac{1}{cosA-\sin A} + \frac{1}{cosA+\sin A} = \frac{\tan 2A}{sinA}$ (6)

QUESTION 5 (Identities)

a) Prove that $\frac{(1+\tan^2\theta)\sin(90°+\theta)}{1-\tan\theta} = \frac{1}{\cos\theta - \sin\theta}$ (5)

b) For which value(s) of θ is the above identity undefined (4)

[09]

<u>QUESTION 6</u> **(compound angles)**

7.1 Simplify cos5β cos 3β + sin5β sin 3β to a trigonometric expression involving only cosβ. (2)

7.2 Hence solve for β if cos5β cos 3β +sin5β sin 3β + cosβ = 0 (5)

7.3 Show that sin(x − 30°) = 2cos$x$ can be rewritten as $\sqrt{3}\sin x = 5\cos x$. (3)

7.4 Hence solve for x ∈ [−270° ; 90°] if sin(x −30°) = 2cosx (4)

[14]

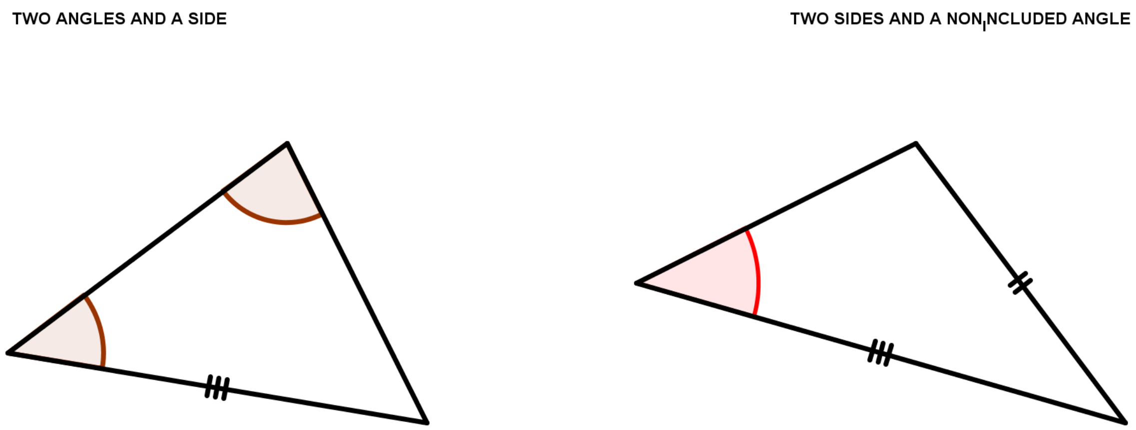
TRIGONOMETRIC GRAPHS

FORMULAE AND THE SOLUTION OF TRIANGLES (HIGHTS AND DISTANCES)

2. THE SINE FORMULA

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \quad \textbf{or} \quad \frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

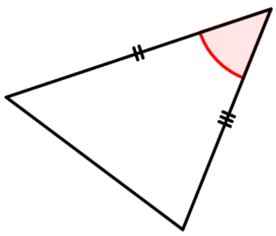
Remember: The sine formula is used if **two angles and a side** are given in a triangle, or if **two sides and a non-included angle** are given.



Remember: Opposite the longest side is the largest angle.
∴ When the triangle is obtuse-angled, the longest side is opposite the obtuse angle.

3. THE COSINE FORMULA

$$a^2 = b^2 + c^2 - 2bcCosA$$
$$b^2 = a^2 + c^2 - 2acCosB$$
$$c^2 = a^2 + b^2 - 2abCosC$$



Remember: We use the cosine formula in this form to determine the third side of a triangle when two
sides and the included angle are given.

$$CosA = \frac{b^2+c^2-a^2}{2bc} \quad CosB = \frac{a^2+c^2-b^2}{2ac} \quad CosC = \frac{a^2+b^2-c^2}{2ab}$$

Remember: We use this formula in this form if **three sides** of a triangle are given and an angle must be
calculated.
NB:
The Cosine-formula is used when the information in the triangle entails: S, S, S or S, A, S.

4. THE AREA FORMULA

Area of a Δ**ABC**$= \frac{1}{2}\boldsymbol{abSin\hat{C}}$

Area of a Δ**ABC** $= \frac{1}{2}\boldsymbol{bcSin\hat{A}}$

Area of a Δ**ABC** $= \frac{1}{2}\boldsymbol{acSin\hat{B}}$

Remember: The area formula is used to determine the area of a triangle
An unknown side can also be determined if the area, a side and an angle are given
Note that this formula actually means: The area of a triangle= $\frac{1}{2}$(product of two adjacent sides) multiplied by the sine of an included angle.
∴ In order to apply this formula, you only need: S, A, S in the triangle.

5. SOLVING PROBLEMS IN TWO DIMENSIONS

Always in problems involving two or more triangles, the same method as for a single triangle is used.

HINTS, CLUES

- The problem usually involves two triangles with a common side
- Often, one of the triangles is right-angled
- Draw a large neat sketch showing all the given information
- Use Geometry to obtain additional information, e.g. exterior angle of a triangle, corresponding and
  alternate angles
- Decide in which triangle the required side occurs. Start with the other triangle and calculate the
  common side using the sine or cosine formula
- Then use the sine formula or the cosine formula or trigonometrical ratios to solve the problem.

6. PROVING A FORMULA

- Sometimes we are required to prove some sort of a formula, before calculating a side or an

  angle.
- We use the same procedure as in solving a problem
- Trigonometrical identities such as: $\boldsymbol{Sin}(\mathbf{90}° - \boldsymbol{\theta}) = \boldsymbol{Cos\theta}$, $\boldsymbol{Cos}(\mathbf{90}° - \boldsymbol{\theta}) = \boldsymbol{Sin\theta}$

$\boldsymbol{Sin}[\mathbf{180}° - (\boldsymbol{x} + \boldsymbol{y})] = \boldsymbol{Sin}(\boldsymbol{x} + \boldsymbol{y})$, etc. are used.

PROBLEMS IN THREE DIMENSIONS

- In three-dimensional problems right angles often don't look like right angles.
- Draw all vertical lines, vertical, so that a right angle may look like this:
- Always shade the horizontal plane roughly.
- Where you encounter problems with three triangles, you must work from the one with the most
  information via the second to the third.
- The cosine formula is used more often than in problems in two dimensions.

Examples/Activities on:
3D PROBLEMS

Question 1.

In the sketch, KL is a vertical tower and L, M and N are all points in the horizontal plane.
$M\hat{L}N = (90° + \alpha)$; $\hat{N} = 2\alpha$ and $\hat{K} = \alpha$. KM = 2 units.

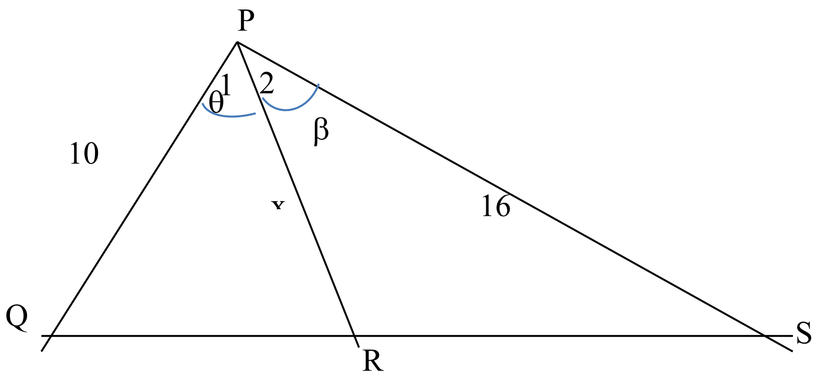
a) Express ML in terms of a trigonometric ratio of α and hence show that
MN = 1 unit. (4)

b) Prove, using the sine rule, that
LN = $1 - 4\sin^2\alpha$ (HINT: $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$) (5)

[09]

Question 2. (Area Rule)

ΔPQS has PQ = 10 units; PR = x units; PS = 16 units. Point R divides QS in the ratio 3 : 5. $\hat{P}_1 = \theta$ and $\hat{P}_2 = \beta$. (See the sketch below.)



a) Write down an expression for the area of ΔPQR in terms of x and θ. (2)

b) (i) Complete the statement:

"Area of ΔPQR : Area of ΔPRS = ____________________________ (2)

(ii) Hence, determine the ratio $\frac{\sin\theta}{\sin\beta}$. (2)

c) If $Q\hat{P}S = 90°$, calculate θ correct to one decimal place. (4)

[10]

Question 3.

In the diagram alongside, MT is a vertical structure.

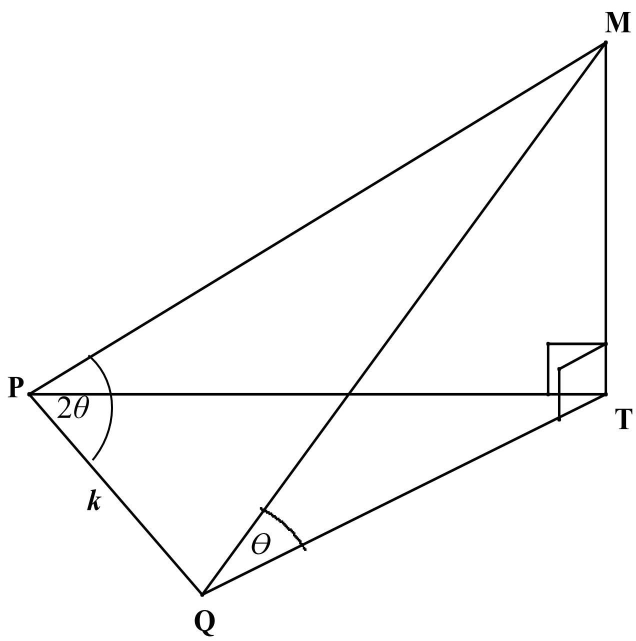
P, Q and T are three points in the same horizontal plane.

The angle of elevation of M from Q is θ.

PQ = k metres.

PM = 2 PQ , $M\hat{P}Q = 2\theta$

$\frac{1}{2}$area Δ MPQ $= k^2 \cdot \sin\theta \cdot \cos\theta$



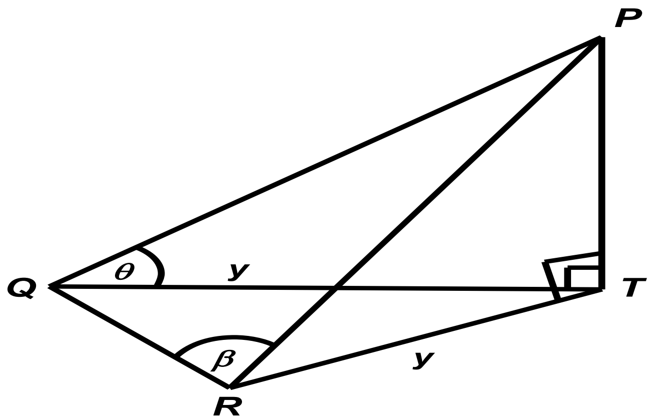
3.1 Hence, or otherwise, prove that $MQ = k.\sqrt{1+8\sin^2\theta}$ (5)

3.2 Find the value of MQ, rounded off to the nearest metre if $k = 139,5$ m and $\theta = 42^\circ$. (2)

[7]

Question 4

A vertical post, PT, has its foot in the same horizontal plane as points Q and R. $TQ = TR = y$ and $PQ = PR$. The angle of elevation from Q to P is θ and $P\hat{R}Q = \beta$.



4.1 Express PQ in terms of y and θ. (2)

4.2 Prove that $QR = \frac{2y.\cos\beta}{\cos\theta}$ (4)

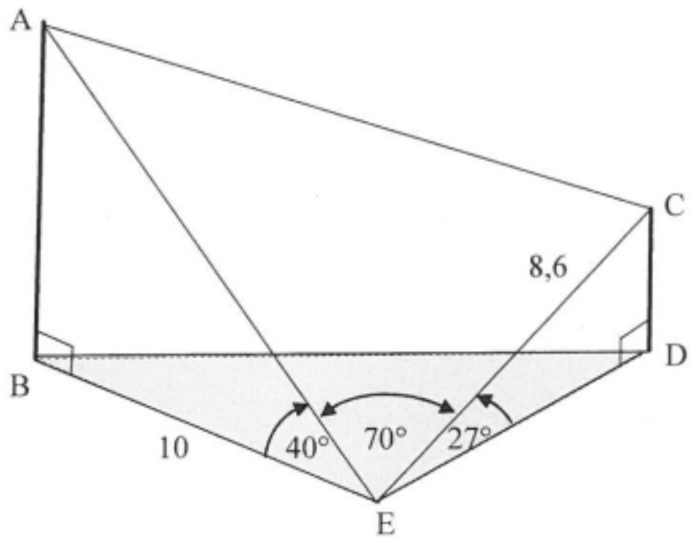
[06]

Question 5.

In the diagram, B, E and D are points in the same horizontal plane. AB and CD are vertical poles.

Steel cables AE and CE anchor the poles at E. Another steel cable connects A and C. CE = 8,6 m, BE 10 m,

$A\hat{E}B = 40^\circ$, and $C\hat{E}D = 27^\circ$.



Calculate the:

5.1 Height of the pole CD. (2)

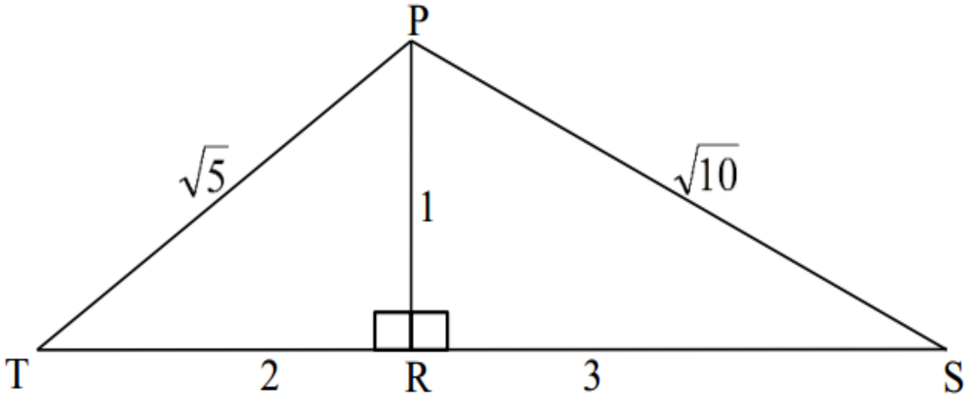
5.2 Length of the cable AE. (2)

5.3 Length of the cable AC. (4)

[08]

QUESTION 3

In the diagram PR ⊥ TS in obtuse triangle PTS. PT = $\sqrt{5}$; TR = 2; PR = 1;
PS = and RS = 3



3.1 Write down the value of:

3.1.1 $\sin \hat{T}$ (1)

3.1.2 $\sin \hat{T}$ (1)

3.2 Calculate without using a calculator, the value of $\sin(\hat{T} + \hat{S})$ (5)

3.3 Determine the value of:

3.3.1 $\dfrac{1}{\cos(360° - \theta).\sin(90° - \theta)} - \tan^2(180° + \theta)$ (6)

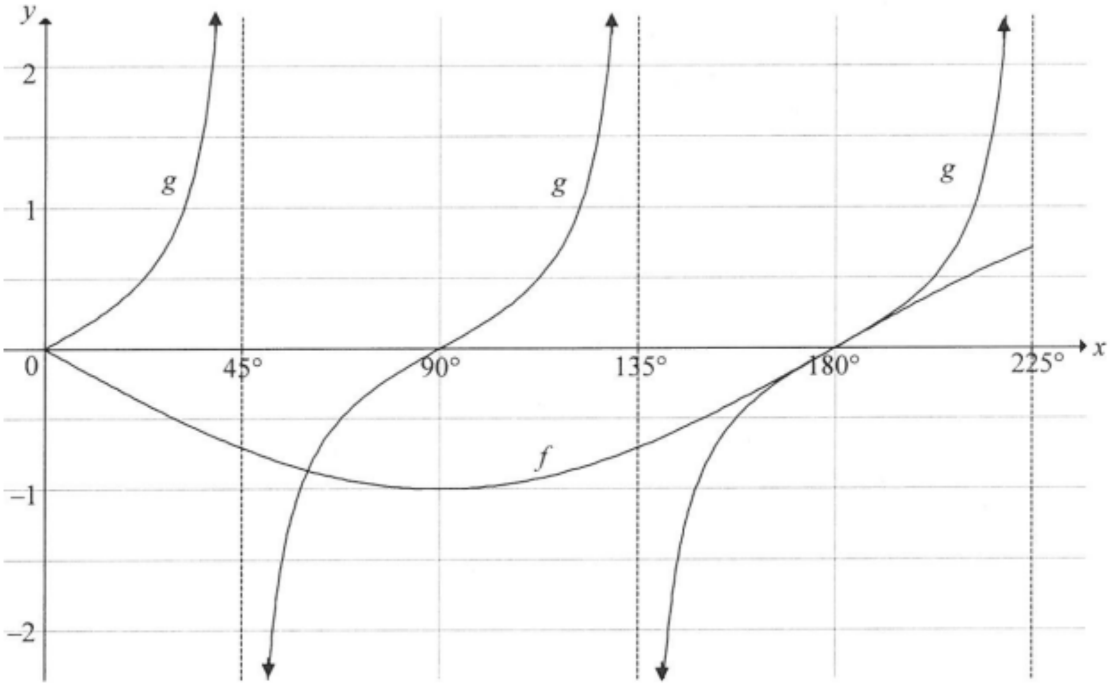
3.3.2 If sin$x$-cos$x$ = $\dfrac{3}{4}$, calculatethe value of sin2x without using a calculator (5)

TRIGONOMETRIC GRAPHS

Question 1.

In the diagram, the graphs of functions $f(x) = a\sin x$ and $g(x) = \tan bx$ care drawn on the same system of axes

for the interval $0° \leq x \leq 225°$.

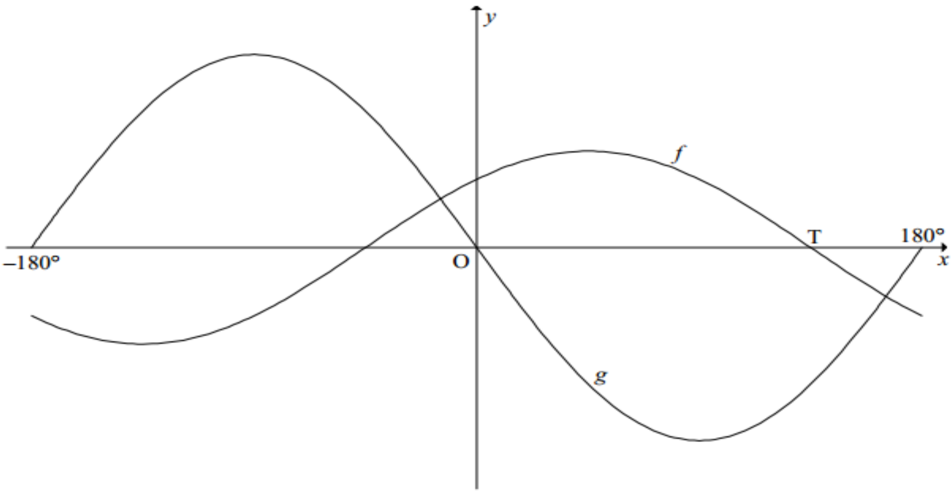


1.1	Write down the values of *a* and *b*.	(2)
1.2	Write down the period f(3x).	(2)
1.3	Determine the values of x in the interval $90° \leq x \leq 225°$.	(3)
		[07]

Question 2.

The graphs of $f(x) = \cos(x - 45°)$ and $g(x) = -2\sin x$ are drawn below for x Ɛ [-180° 180°].

The pointy T is an *x*-intercept of f as indicated in the diagram:



2.1	Show that $\cos(x - 45°) = -2\sin x$ can be written as $\tan x = -0{,}2612$.	(4)
2.2	Solve the equation: $\cos(x - 45°) = -2\sin x$ for $x \in [-180° ;180°]$.	(3)
2.3	Write down the coordinates of point T.	(2)

2.4	Write down the interval for which $f(x) \geq g(x)$.	(2)
2.5	Write down the interval for which both *f* and *g* are strictly increasing.	(3)
2.6	The graph *h* is obtained when the graph *f* is shifted 45° to the right. Write down the equation of *h* in its simplest form.	(2) **[16]**

QUESTION 3

Draw the graphs of the following equations on the same systems of axes if $x \in [-360; 360]$:

3.1	f(x) = sin$x$ and g(x) = cos$x$	(6)
3.2	h(x) = cos2x and t(x) = $\sin(x - 45°)$	(6)
3.3	f(x) = $\tan\frac{1}{2}x$ and $\cos(x + 30°)$	(6)

7 ANALYTICAL GEOMETRY

Represent geometric figures on a Cartesian co-ordinate system, and derive and apply, for any two points $(x_1;y_1)$and $(x_2;y_2)$, a formula for calculating:

(a) the distance between the two points

(b) the gradient of the line segment joining the points (including collinear points).

(c) the co-ordinates of the mid-point of the line segment joining the points

Use a Cartesian co-ordinate system to derive and apply:

(a) the equation of a line through two given points

(b) the equation of a line through one point and parallel or perpendicular to a given line

(c) the inclination of a line.

Use a two-dimensional Cartesian co-ordinate system to derive and apply:

(a) the equation of a circle (any centre);

(b) the equation of a tangent to a circle given a point on the circle.

NOTE: Learners are expected to know and use as an axiom: "the tangent to a circle is perpendicular to the radius drawn to the point of contact."

<u>Formulae for Co-ordinate Geometry</u>

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad \mathrm{M}\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$$

$$y=mx+c \qquad y-y_1=m(x-x_1) \qquad m=\frac{y_2-y_1}{x_2-x_1} \qquad m=\tan\theta$$

$$(x-a)^2+(y-b)^2=r^2$$

<u>LEARNING HINTS</u>

N.B. Mathematical language and terminology must be learnt in more detail

NB "Only Analytical methods must be used:"

1. *Learners must learn which formula is to used to prove the most basic aspects of Analytical Geometry.*

   Eg. Bisect is to cut into two equal parts
   If two lines are parallel then their gradients are equal
   The lines joining collinear points have the same gradient
   If two lines are perpendicular to each other then the product of their gradients = - 1

2. Learners need to remember that the product of gradients equals -1, is accepted to prove $\perp$. If asked to prove $\perp$, every effort must be made to show that this product = -1. Repetition of exam type questions such as this must be practised with learners.

3. *Learners should then follow the method laid out below:*

   - Select the correct formula from the data sheet
   - Label the ordered pairs using the correct two points, eg A and C.
   - Substitute correctly and accurately into your chosen formula

4. Often Analytical Geometry questions *follow on*, (scaffolding). Look out for that, as you might have already proven an aspect above, that you will require for the next sub-question

   - Even if you failed to show or prove in the previous question, accept that as true in the follow up questions.

5. *Use the diagram more effectively.*

Eg Highlight the sides you are going to use for proving perpendicular, so you can see clearly which points you are going to use for the substitution.

6. You must answer the question, and remember to conclude, exactly what you were asked to show / prove / conclude. Use wording to do this.

7. **Learners need to know the properties of all geometric figures e.g. triangles and quadrilaterals**

8. Learners need to know the 2 forms of the equations of a circle, by alluding to several different types of examples and exercises. Furthermore this formula needs to be shown to the candidates by training them to use the Data Sheets. Completing the square in terms of x and y need to be emphasised.

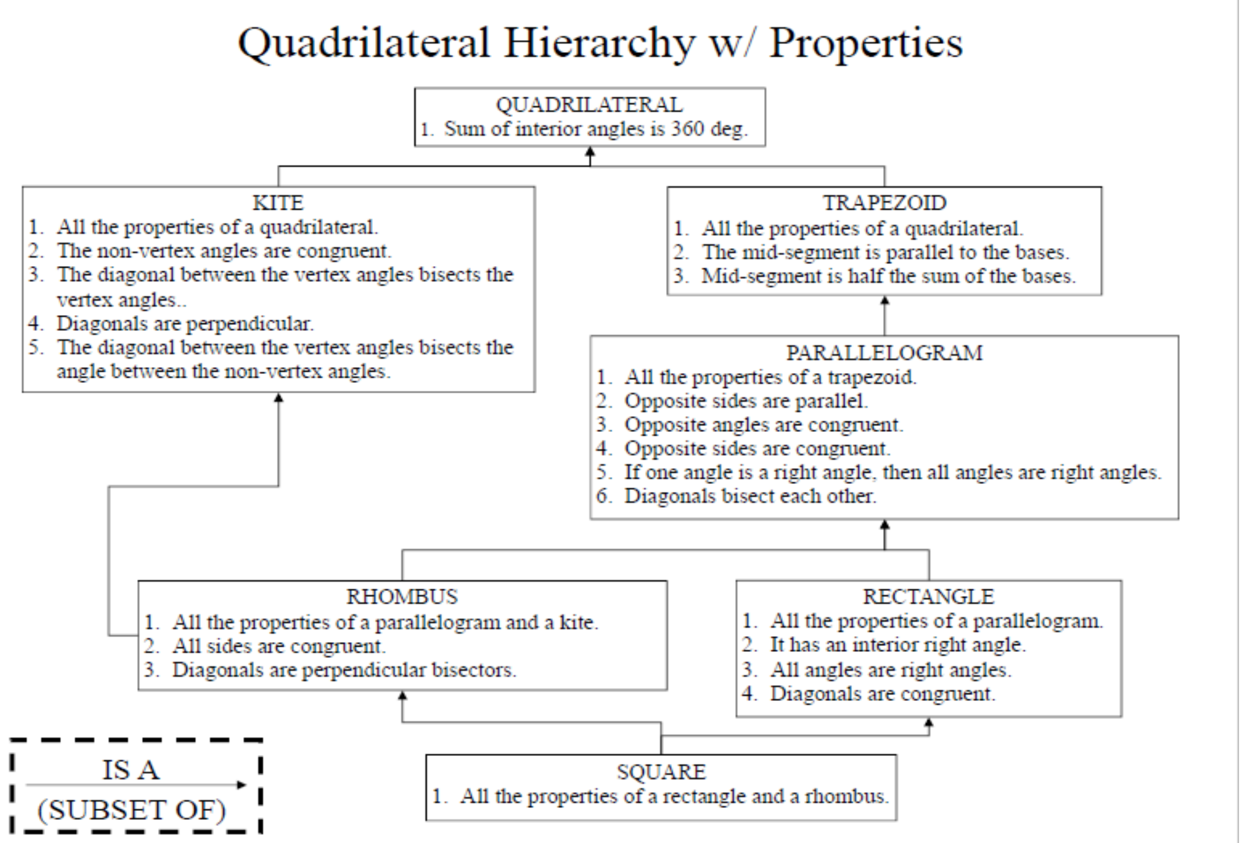
9. Learners need to be able to determine whether a particular point is inside, outside or on the circle by comparing that distance and the radius.

10. With regards to determining the equation of a line, educators must cover all aspects of the equation of a line including those passing through the origin, as well as the equations of horizontal and vertical lines in their teaching.

11. Practice exercises are often required to teach the above points.

12. Grade 11 work must ***NOT*** be ignored. Circle centre (0 ; 0) was no problem in Grade 11, but can be completely forgotten in Grade 12.

You may find the following summary of the properties of quadrilaterals useful in solving some of the Analytical Geometry problems.



NOTE:

Always draw a diagram when doing problems involving Analytical Geometry. A diagram helps you to visualise the problem accurately.

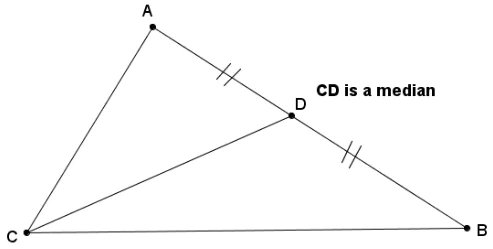
Distance Formula.

The distance between any two points A $(x_1; y_1)$ and B $(x_2; y_2)$ is given by: AB$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ OR $AB=\sqrt{(x_A-x_B)^2+(y_A-y_B)^2}$

The mid-point of a line segment

The mid-point M$(x; y)$ of a line segment joining the points A$(x_1; y_1)$ and B$(x_2; y_2)$ is given by: M$(x; y) =$ M$\left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right)$ = M$\left(\frac{x_2+x_1}{2}; \frac{y_2+y_1}{2}\right)$

Median: The median of a triangle bisect the opposite side of a triangle.
The median of a triangle bisect the area of a triangle.



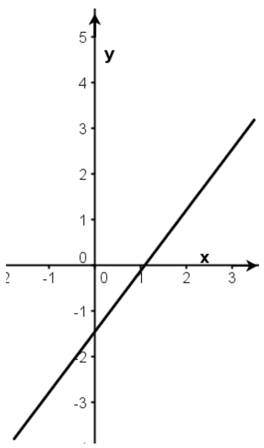
To find the equation of the median:

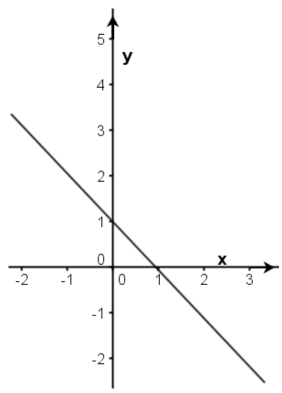
- Determine the coordinates of D using the formula for the midpoint
- Use the coordinates of C and D to find the equation.

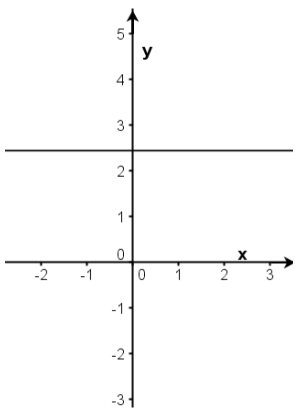
The gradient or slope of a straight line

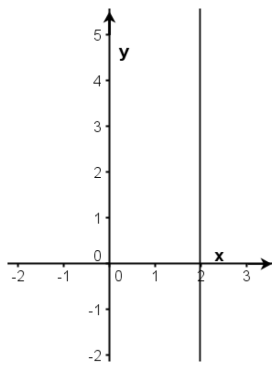
The gradient or slope of a straight line through the points A$(x_1; y_1)$ and B$(x_2; y_2)$ in which $x_1 \neq x_2$ is given by: GradAB$= m_{AB} = \frac{y_1-y_2}{x_1-x_2} = \frac{y_2-y_1}{x_2-x_1} = \frac{increase\ in\ y}{increase\ in\ x}$

$\boldsymbol{m} > 0$	$\boldsymbol{m} < 0$	$\boldsymbol{m = 0}$	$\boldsymbol{m}$**- undefined**
$\boldsymbol{tan\theta} > 0$	$\boldsymbol{tan\theta} < 0$	$\boldsymbol{tan\theta = 0}$	$\boldsymbol{tan\theta}$ **is undefined**
		Horizontal line	**Vertical line**









Perpendicular and Parallel Lines

- The product of the gradients of perpendicular lines is **-1,** i.e. $m_1 \times m_2 = -1$
- $m_1 \times m_2 = -1$ cannot be applied when one of the lines is parallel to the y −axis

- When two lines are parallel, then $m_1 = m_2$

If AB∥PQ, then $m_{AB} = m_{PQ}$ (AB and PQ not parallel to the$y - axis$)

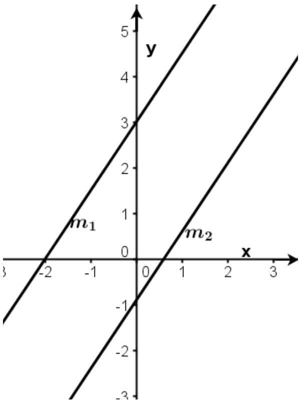
If $m_{AB} = m_{PQ}$,then AB∥PQ

If AB⊥AQ, then $m_{AB} \times m_{AQ} = -1$ (AB and PQ not parallel to the $y - axis$)

If $m_{AB} \times m_{AQ} = -1$,then AB⊥AQ

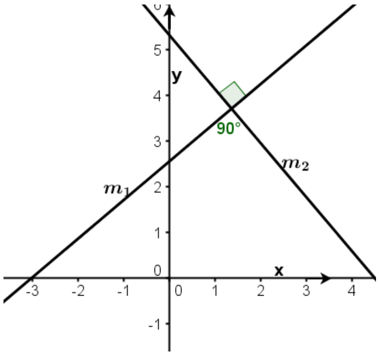
Parallel Lines

$\boldsymbol{m_1 = m_2}$



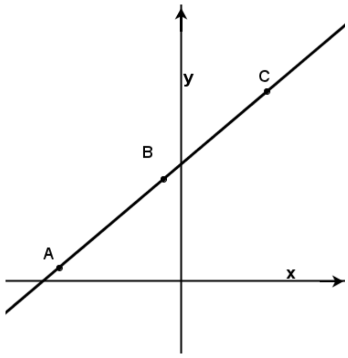
Perpendicular Lines

$\boldsymbol{m_1 \times m_2 = -1}$

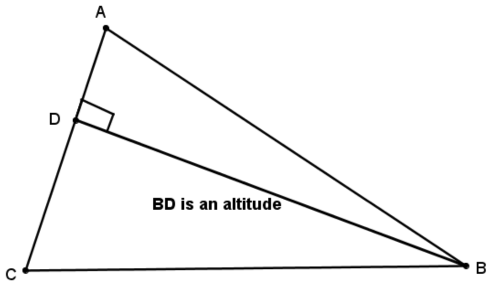


Collinear Points

$\boldsymbol{m_{AB} = m_{BC}}$



Altitude: The altitude of a triangle is perpendicular to the opposite side of a triangle.



To find the equation of an altitude:

- Determine the gradient AC= m_{AC}
- Determine the m_{BD} using the fact that $m_{BD} \times m_{AC} = -1$
- Use the coordinates of B and m_{BD} to find the equation.

Perpendicular bisector: The perpendicular bisector of a line segment is also found in a triangle.

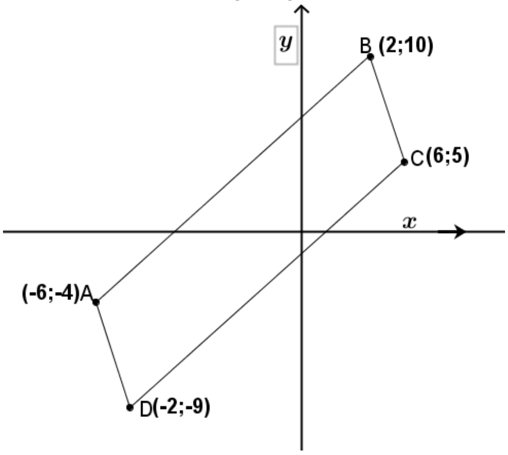
To find the equation of the perpendicular bisector:

- Determine the coordinates of M using the midpoint formula.
- Find the gradient AB i.e. m_{AB}
- Determine m_{DP} using the fact that DP⊥AB ∴ $m_{DP} \times m_{AB} = -1$
- Use the coordinates of M and m_{DP} to find the equation.

Activities:

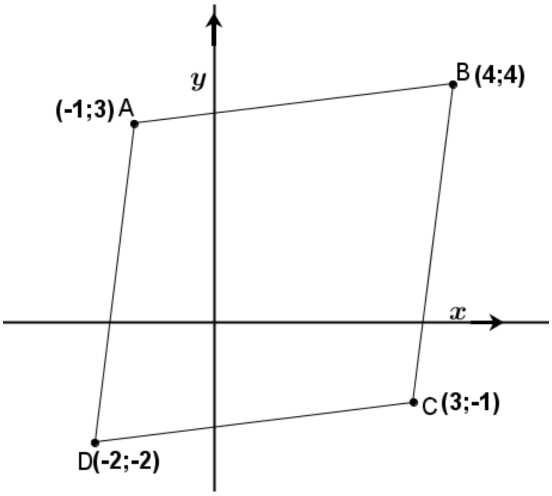
1. The distance between A $(x; 15)$ and B $(-7; 3)$ is 13 units. Calculate the possible values of x.

2. Given the following diagram:



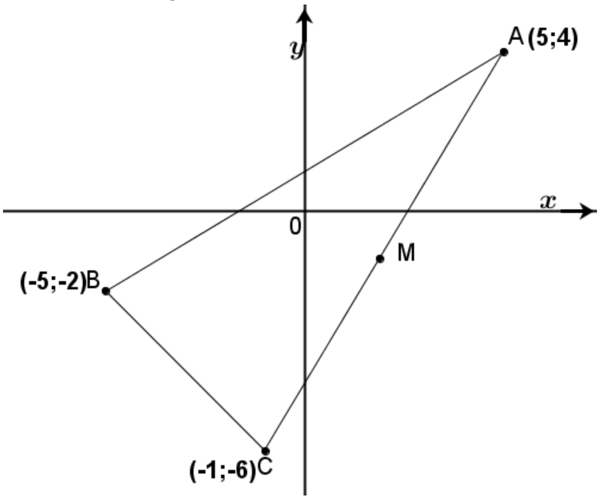
(a) Prove that ABCD is a parallelogram using the lengths of the sides.
(b) Prove that ABCD is a parallelogram using the diagonals.

3. Prove that ABCD is a rhombus.



4. Find the 4th vertex of a parallelogram PQRT, if the three given vertices are P$(6; -3)$, Q$(3; 3)$, and C$(-2; 1)$

5. From the diagram below:



(a) Calculate the coordinates of M the midpoint of AC.
(b) Determine the gradient BC.
(c) Determine the equation of the line parallel to BC that passes through M.
(d) Give the coordinates of P, the midpoint of AB.
(e) Calculate the length of BC.

(f) Prove that BC= 2PM.

NOTE:

1. If $l_1 \parallel l_2 \parallel x$ −axis then $m_1 = m_2 = 0$

2. If $l_1 \parallel l_2 \parallel y$ −axis then m_1 *and* m_2 are not defined but the lines are clearly parallel.

3. If $l_1 \parallel x$ −axis and $l_2 \parallel y$ −axis then $m_1 = 0$ and m_2 is not defined but the lines are clearly perpendicular.

Collinear points

Collinear points means points in the same straight line.

The distinct points A, B, and C will be collinear if $m_{AB} = m_{AC}$ *or* $m_{AB} = m_{BC}$
Or if $x_1 = x_2 = x_3$ or if $y_1 = y_2 = y_3$

Activities:

1. The points A $(-5; 9)$, B $(-3; y)$ and C $(2; -5)$ are given.
(a). Determine the value of y if A;B and C are collinear.
(b). Determine the value of y if BC⊥ AC

2. Calculate the value of t, if AB is parallel to the line which passes through the points C$(2; 3)$ and D$(-2; -5)$ if A$(-3; t)$ and B$(0; -2)$

3. Prove that the following points are on the same line.
(a) N$(-7; -8)$, A$(-1; 2)$ and G$(2; 7)$ (b) M$(-1; 9)$, P$(2; 3)$ and Q$(5; -3)$

4. Find the missing coordinates if these points are collinear.
(a) P$(1; 10)$, E$(x; 2)$ and T$(-5; -2)$ (b) H$(-3; 3)$, O$(-1; y)$, B$(2; -7)$

The Equation of a straight line

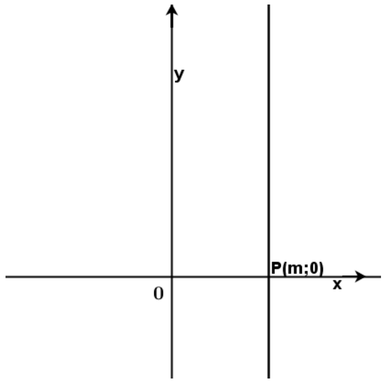
To determine the equation of a straight line:

- Determine the gradient
- Use the coordinates of a point on the straight line and simplify.

Use this Formula	**(When:)Condition**	**Graph**
$y = mx + c$	Given the gradient and the y −intercept	
$y - y_A = m(x - x_A)$	Given the gradient and one point	
$y - y_A = (\frac{y_B - y_A}{x_B - x_A})(x - x_A)$	Given two points	
$y = k$	Given a horizontal line	

$x = m$

Given a vertical line



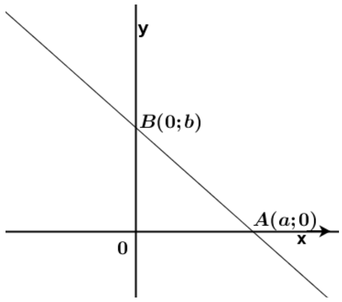
$\frac{x}{a} + \frac{y}{b} = 1$

Dual-intercept form

Given the x and y- intercepts

$(a; 0)$ is the x −intercept

$(0; b)$ is the y −intercept



Equation of a straight line through any point given a gradient

The equation of a straight line with gradient m through any point $(x_1; y_1)$on the line is given by $y - y_1 = m(x - x_1)$ or $\frac{y-y_1}{x-x_1} = m$

This can be simplified to $y = mx + c$ where m is the gradient and c is the y −intercept.

- To find the equation of a straight line, we need the gradient $\boldsymbol{m}$ and the coordinates of one point on the line.

Activities:

1. Determine the equation of a straight line passing through the points:

(a) $(3; 7)$ and $(-6; 1)$ (b) $(1; -\frac{11}{4})$ and $(\frac{2}{3}; -\frac{7}{4})$

(c) $(8; t)$ and $(t; 8)$ (d) $(2p; q)$ and $(0; -q)$

2. Determine the equation of the straight line:

(a) passing through the point $(-1; \frac{10}{3})$ and with gradient $m = \frac{2}{3}$

(b) parallel to the x −axis and passing through the point $(0; 11)$

(c) perpendicular to the x −axis and passing through the point $(-\frac{3}{2}; 0)$

(d) with undefined gradient and passing through the point $(4; 0)$

(e) with $m = 3a$ and passing through the point $(-2; -6a + b)$

Equation of a straight line through a given point and parallel or perpendicular to a given line

The gradient of a straight line through $(x_1; y_1)$ and parallel to $y = mx + c$ is m
Therefore the required equation is: $y - y_1 = m(x - x_1)$

The gradient of a straight line through $(x_1; y_1)$ and perpendicular to $y = mx + c$ where $m \neq 0$ is$-\frac{1}{m}$

Therefore the required equation is:$y - y_1 = -\frac{1}{m}(x - x_1)$

Activities:

A

1. Determine whether or not the following two lines are parallel:

(a) $y + 2x = 1$ and $-2x + 3 = y$ (b) $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$

(c) $y = 2x - 7$ and the line passing through $(1; -2)$ and $(\frac{1}{2}; -1)$

(d) $y + 1 = x$ and $x + y = 3$

2. Determine the equation of the straight line that passes through the point $(1; -5)$ and is parallel to the line $y + 2x - 1 = 0$

3. Determine the equation of the straight line that passes through the point $(-2; \frac{2}{5})$ and is parallel to the line with the angle of inclination $\theta = 145°$

B

1. Determine whether or not the following two lines are perpendicular.

(a) $y - 1 = 4x$ and $4y + x + 2 = 0$ (b) $10x = 5y - 1$ and $5y - x - 10 = 0$

(c) $x = y - 5$ and the line passing through $(-1; \frac{5}{4})$ and $(3; -\frac{11}{4})$

(d) $y = 2$ and $x = 1$ (e) $\frac{y}{3} = x$ and $3y + x = 9$

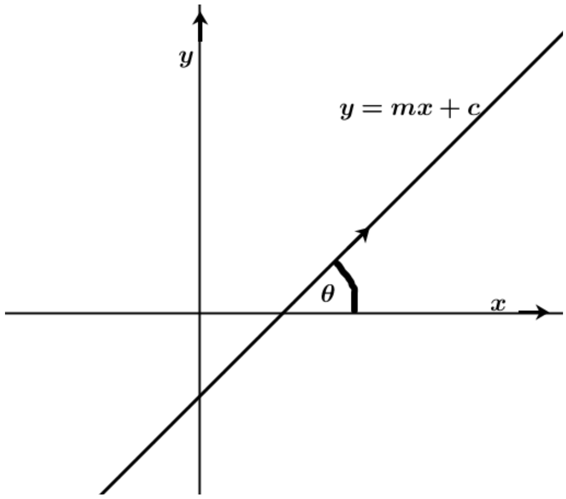
2. Determine the equation of a straight line that passes through the point $(-2; -4)$ and is perpendicular to the line $y + 2x = 1$

3. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with an angle of inclination $\theta = 135°$.

The angle of inclination of a straight line

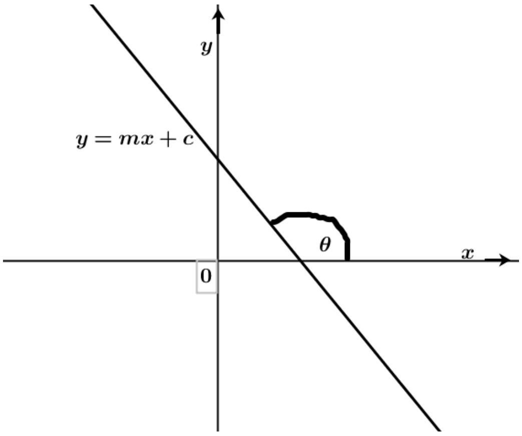
The angle which a straight line makes with the positive x −axis (inclination of the line)

The angle of inclination of a straight line $y = mx + c$ is the angle $\theta, 0° \leq \theta < 180°$, made by the straight line with the positive direction of the x −axis, measured in the anti-clockwise direction.



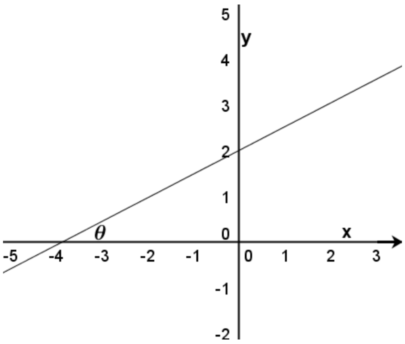
For tan$\theta = m$ where $m \geq 0$

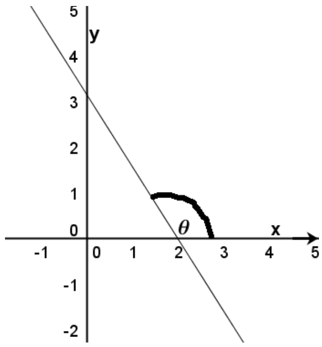
$\theta = tan^{-1}m$

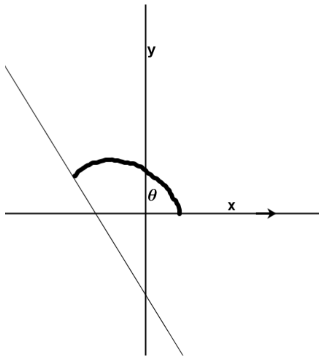


For $\tan\theta = m$ where $m < 0$

$\theta = tan^{-1}m + 180°$







$tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = m$

If $tan\theta > 0$, then θ is an acute angle

If $tan\theta < 0$, then θ is an obtuse angle

Activities:

A.

1. Determine the gradient correct to one decimal place of each of the following straight lines, given that the angle of inclination is equal to:

(a) 60° (b) 135° (c) 0° (d) 54° (e) 90°

2. Determine the angle of inclination correct to one decimal place for each of the following:

(a) a line with $m = \frac{3}{4}$ (b) $2y - x = 6$

(c) the line passes through the points A$(-4; -1)$ and B$(2; 5)$

(d) $y = 4$ (e) $x = 3y + \frac{1}{2}$ (f) $x = -0{,}25$

(g) the line passes through the points P$(2; 5)$ and Q$(\frac{2}{3}; 1)$

(h) a line with a gradient equal to 0,577

B.

1. Determine the equation of a straight line passing through the point $(3; 1)$ and with an angle of inclination equal to 135°

2. Determine the acute angle correct to one decimal place between the line passing through the points M $(-1; 1\frac{3}{4})$ and N $(4; 3)$ and the straight line $y = -\frac{3}{2}x + 4$

3. Determine the acute angle between the line passing through the points A $(-2; \frac{1}{5})$ and B $(0; 1)$ and the line passing through the points C $(1; 0)$ and D $(-2; 6)$

4. Determine the angle between the line $y + x = 3$ and the line $x = y + \frac{1}{2}$

5. Find the angle between the line $y = 2x$ and the line passing through the points P $(-1; \frac{7}{3})$ and Q $(0; 2)$

Points of intersection of the straight lines

To determine the points of intersection of two straight lines, we have to solve the two equations of the lines simultaneously.

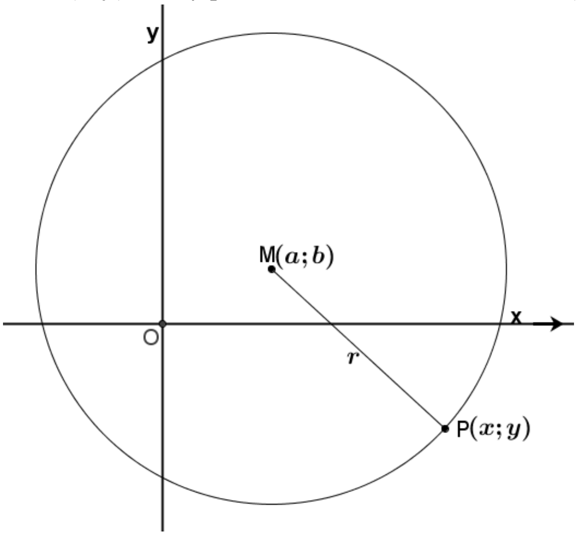
- Write the equations of both the lines $y = mx + c$
- Let the y −values equal one another
- Determine the x −value of the point of intersection
- Calculate the corresponding y −value by means of substitution.

Circles:

Definition of a Circle: The set of all points equidistant from a fixed point is called a circle.

The equation of a circle with a centre $(a; b)$ and radius r

The equation of a circle with centre $(a; b)$ and radius **r** can be found as follows:
Let P $(x; y)$ be any point on the circle with centre M $(a; b)$ and radius **r**



$MP = r$
$MP^2 = r^2$
$(x - a)^2 + (y - b)^2 = r^2$ Distance formula
This equation $(x - a)^2 + (y - b)^2 = r^2$ defines a circle with radius **r** and centre $(a; b)$

You will also be given the equation in the form: $x^2 + y^2 + px + qy = k$ and a sked to find the centre and the radius of the circle.
It is necessary to write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$
Arrange the equation in the form $x^2 + px + \cdots + y^2 + qy + \cdots = k$
Complete the square $x^2 + px + (\frac{p}{2})^2 \quad + \quad y^2 + qy + (\frac{q}{2})^2$
$(x + \frac{p}{2})^2 \quad + \quad (y + \frac{q}{2})^2$

Activities:

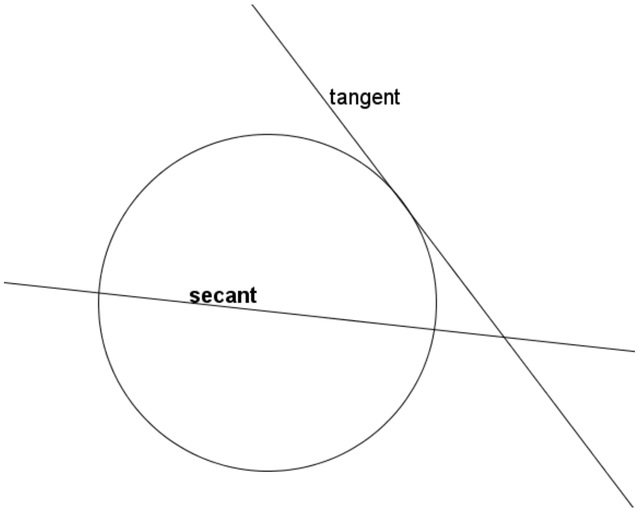
1. Determine the coordinates of the centre of the circle and the radius for each of the following circles:
(a) $x^2 + y^2 - x - 2y - 5 = 0$
(b) $x^2 + y^2 + 2x - 6y + 9 = 0$
(c) $x^2 + y^2 - 4x - 6y + 9 = 0$

2. Find the centre and the radius of each of the following circles.
(a) $x^2 + y^2 = 36$ (b) $(x - 2)^2 + y^2 = 12$ (c) $x^2 + (y - 5)^2 = 18$
(d) $x^2 + y^2 - 6x + 10y = 18$

3. Calculate the distance from the centre of the following circles to the origin.
(a) $(x + 3)^2 + (y + 6)^2 = 20$ (b) $(x + 1)^2 + (y - 7)^2 = 32$

The equation of the tangent to a circle

- A secant of a circle is a straight line that cuts the circle at two distinct points.
- A tangent to a circle is a straight line that touches the circle at one point only.



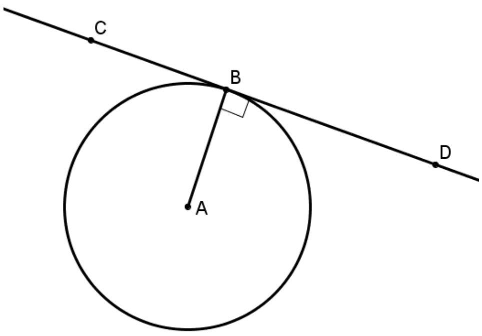
NOTE:

From Euclidean Geometry, the angle between the tangent and the diameter at the point of contact is 90° (A radius is perpendicular to a diameter at the point of contact)

NOTE:

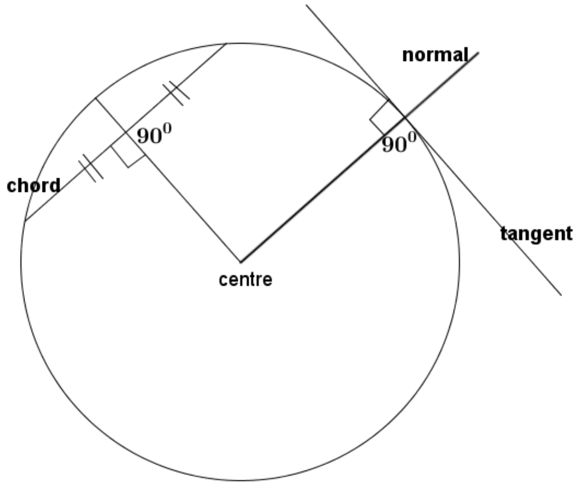
- Two tangents drawn to a circle from the same point outside the circle are equal in length.
- The quadrilateral formed by two tangents from the same point outside the circle and the two radii at the points of contact is often a kite.
- The diagonals of a kite are perpendicular to one another and the diagonal between the equal tangents bisects the other diagonal

The equation of the tangent to a circle



- Determine the gradient AB, m_{AB}
- Determine the gradient CD, using AB⊥CD
- Use the gradient CD and the point B to find the equation CD

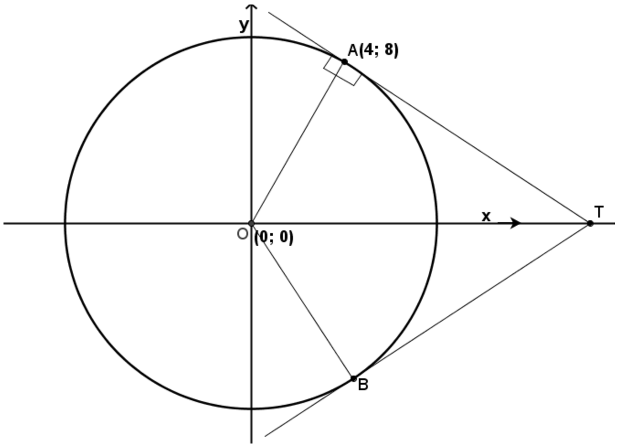
The following need to be revised and be remembered for the purposes of understanding and answering questions in Analytical Geometry.



- The line from the centre of the circle perpendicular to a chord bisects the chord.
- The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- A tangent to a circle is perpendicular to the radius at the point of contact.

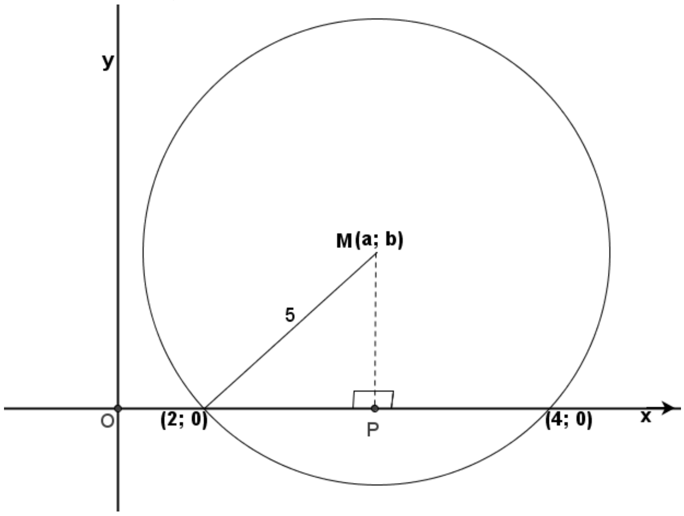
Activities:

1. In the diagram below TA and TB are tangents to the circle centre O.

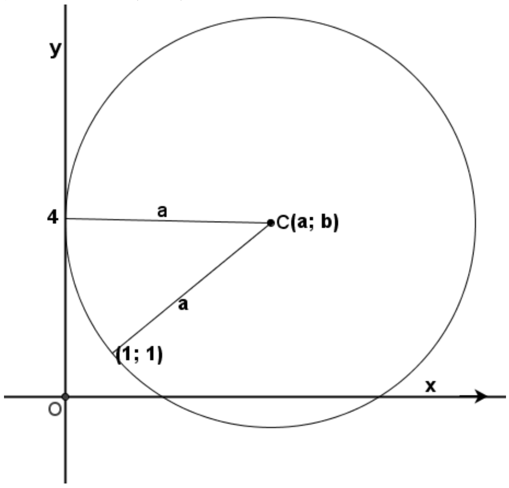


(a) Find the equation of the circle.
(b) Find the equation of TA.
(c) Find the equation of TB.

2. From the diagram below, find the equation of the circle if the radius is 5 units and the centre is at M $(a; b)$.



3. Using the diagram below, find the equation of the circle if the centre is at C$(a; b)$ and the circle touches the $y-$ axis at $(0; 4)$



4. The circle with the equation $x^2 + y^2 = 169$ cuts the $y-$axis at M and N, and the $x-$ axis at P and Q. A line with the equation $y = x - 7$ cuts the circle at A and B.
(a) Write down the coordinates of M; N; P and Q.
(b) What are the equations of the tangents at M and N.?
(c) Find the coordinates of A and B.
(d) Find the length of AB.

5. A circle with centre $(0; 0)$ passes through A $(3; -4)$ and B $(-5; 0)$
(a) Find the equation of the circle.
(b) Find the equation of the tangent at A.
(c) Find the equation of the radius that passes through the mid-point of AB.

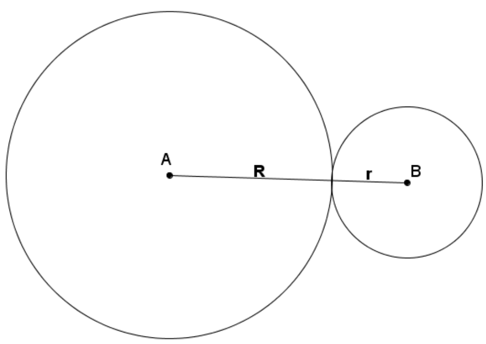
6. Find the equation of the tangent to the circle $x^2 - 2x + y^2 + 4y = 5$ at the point $(-2; -1)$

7. Find the length of the tangent to $x^2 + y^2 + 2x - 4y = 8$ from the point $(2; 9)$

8. Determine the length of the tangent from the point H $(6; -2)$ to the point of intersection with the circle $x^2 + y^2 - 6x + 2y + 8 = 0$

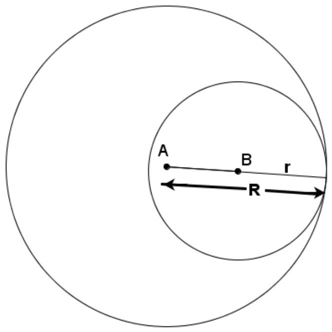
Circles with centres A and B

The two circles touch externally



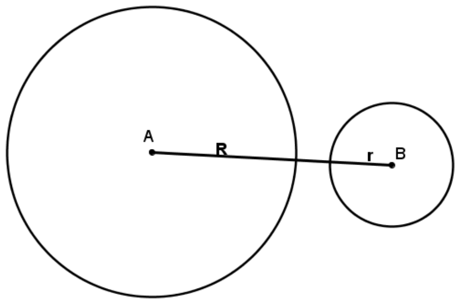
AB= R+r

The two circles touch internally



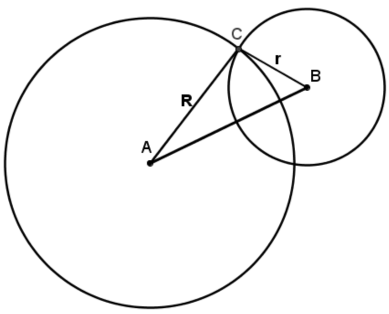
AB= R− r

The two circles Never touch



AB>R+r

The two circles Intersect



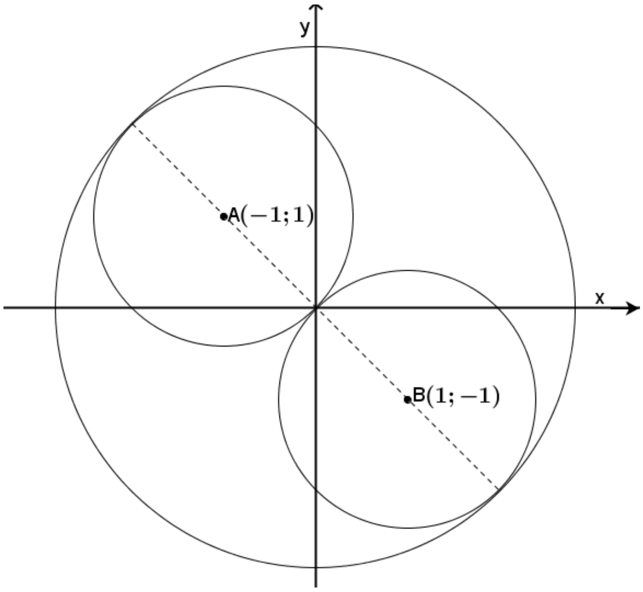
AB<R+r

Activities:

1. Show that the circles with equations $x^2 + y^2 - 16x - 8y + 35 = 0$ and $x^2 + y^2 = 5$ touch each other.

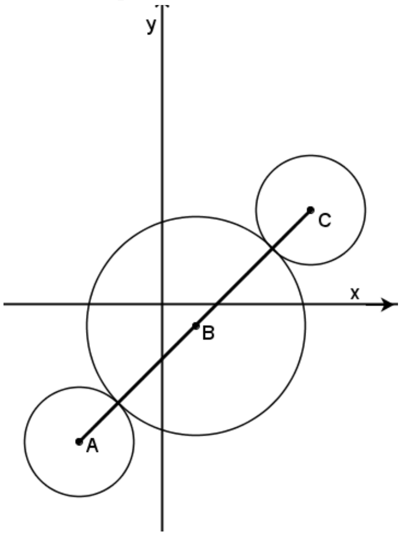
2. Given the circles: $x^2 + y^2 + 2x - 6y + 9 = 0$ and $x^2 + y^2 - 4x - 6y + 9 = 0$
Show that the two circles touch externally.

3. The equation of a bigger circle is $x^2 + y^2 = 8$ and the smaller circles are centred at A and B respectively.



(a) Write down the equations of the circles centred at A and B respectively.
(b) Prove that the circle centred at A and the bigger circle touch internally.
(c) Give the equation of the bigger circle if it is translated 2 units up and 3 units left.

4. When newspapers were printed by lithograph, the newsprint had to run over three rollers, illustrated in the diagram by three circles. The centres A, B and C of the three circles are collinear. The equations of the circumferences of the outer circles are: $(x+12)^2+(y+15)^2=25$ and $(x-24)^2+(y-12)^2=100$.
Find the equation of the central circle.



The division of a line segment (Optional)

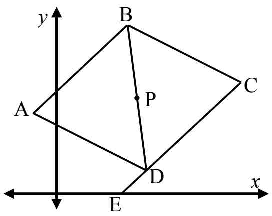
I P $(x;y)$ divides the line AB with A $(x_a;y_a)$ and B $(x_b;y_b)$ in the ratio of $m:n$, the coordinates of P will be: $(\frac{nx_a+mx_b}{m+n};\frac{ny_a+my_b}{m+n})$

ACTIVITIES

1. Find the distance between the points given below. Leave answers in surd form.
   (a) $(3;0)$ and $(0;3)$ (b) $(4;0)$ and $(2;3)$
2. The vertices of a triangle are given as $O(0;0)$, $B(\sqrt{2};\sqrt{2})$ and $C(x;-\sqrt{2})$.
   (a) Determine the value of x for which $B\hat{O}C=90°$.
   (b) Hence determine the perimeter of the triangle.

3. ABCD is a parallelogram with $A(-1;4)$, $B(3;6)$, $C(x;y)$ and $D(4;1)$.

Determine:



(a) the gradient of AB. (b) the midpoint P of BD.

(c) the coordinates of C. (d) the equation of CD.

(e) the coordinates of E if E is the x-intercept of the line CD.

(f) the inclination of line AE (g) the size of $A\hat{E}D$.

(h) the length of BC

4. $A(-7;3)$ and $B(5;7)$ are two given points. Determine:

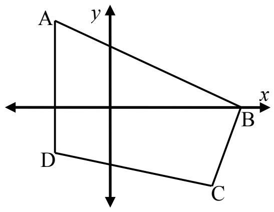
(a) the coordinates of M, the midpoint of AB.
(b) the distance from $C(-2;-5)$ to M.
(c) the equation of the line AB.
(d) the equation of the line perpendicular to AB which passes through M.

5. ΔPQR is right-angled at P. The points $P(2;6)$, $Q(-1;2)$, and $R(6;k)$ are given.

Determine:
(a) k
(b) the coordinates of S, such that PQRS is a rectangle.

6. The points $A(-2;\frac{7}{2})$, $B(5;0)$, $C(3;-4)$ and $D(-2;-\frac{3}{2})$

form a quadrilateral. Show that:



(a) $AB//DC$

(b) $AB \perp BC$

7. PQSR is a quadrilateral with $P(-3;2)$, $Q(1;4)$ and $S(2;-1)$. Determine the:

(a) inclination of QS (b) midpoint A of PS
(c) coordinates of $R(x;y)$ such that PQSR is a parallelogram.
(d) equation of RS (e) coordinates of M, the x-intercept of RS
(f) length of QR

8. Give the equations of the circles, centre $(0;0)$,

(a) with radius 7 (b) with radius $3\sqrt{2}$
(c) through $(0;5)$ (d) through $(-2;0)$

(e) through $(2;7)$ (f) through $(-4;\sqrt{2})$

9. Give the radius of the circle whose centre is at the origin, in the simplest surd form if necessary:

(a) $x^2+y^2=36$ (b) $x^2+y^2=20$
(c) through $(-5;-12)$ (d) through $(-8;4)$

10. The line $x=8$ cuts the circle $x^2+y^2=100$ at two points.

(a) Calculate the y-coordinates of the two points.
(b) Find the co-ordinates of the points that lie on the line $x=-8$ using symmetry.

11. The circle with equation $x^2 + y^2 = 169$ cuts the y-axis at M and N and the x-axis at P and Q.
A line with equation $y = x - 7$ cuts the circle at A and B.

(a) Write down the co-ordinates of M, N, P and Q.
(b) Calculate the co-ordinates of A and B.
(c) Calculate the length of the chord AB.

12. Determine the radius and the coordinates of the centre of each of the circles:

(a) $(x-4)^2 + (y+3)^2 = 25$

(b) $x^2 + (y-2)^2 = 0$

(c) $x^2 + 2x + y^2 = 0$

(d) $x^2 - 8x + y^2 + 6y = 2$

(e) $x^2 - 5x + y^2 + 6y = 9$

(f) $x^2 + x + y^2 - 4y - 4 = 0$

(g) $x^2 + y^2 + 10x - 3y + 1 = 0$

(h) $x^2 + y^2 - 2x - y - 12 = 0$

(i) $2x^2 + 2y^2 - 4x - 12y = 3$

13. Determine the equation of the circle:
(a) with centre $(-2;-1)$ and radius 4
(b) with centre $(3;-3)$ and radius $3\sqrt{5}$
(c) with centre $(-3;1)$ and passing through $(2;-2)$
(d) with centre $(1;-1)$ and passing through $(-3;4)$

14. The equation of a circle with radius $3\sqrt{2}$ is $x^2 - 6x + y^2 + 2y - m = 0.$

(a) Determine the coordinates of the centre of the circle.
(b) Find the value of m.

15. Determine the equation of the circle:

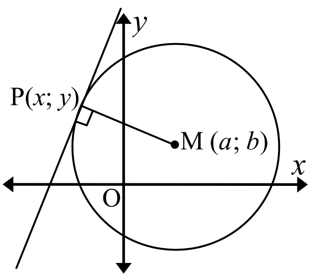
(a) with BC as diameter if B and C are the points $(2;-3)$ and $(6;-1)$ respectively.
(b) with centre $(b;4)$, $b > 0$, which cuts the x-axis at $(1;0)$ and $(7;0)$.

16. Find the equation of the circle with centre $(-5;2)$ and touching the y-axis.

17. Find the equation of the tangent to the given circle at the given point:

1. $x^2 + y^2 = 25$ at $(-3;4)$
2. $x^2 + y^2 = 5$ at $(-1;-2)$
3. $(x-3)^2 + (y+1)^2 = 20$ at $(5;3)$
4. $(x+3)^2 + (y-4)^2 = 25$ at the origin

18. The drawing on the right shows a tangent drawn to a circle at the point P(x; y). Write down the gradient of:
(a) MP in terms of a, b, x and y.
(b) the tangent in terms of a, b, x and y.

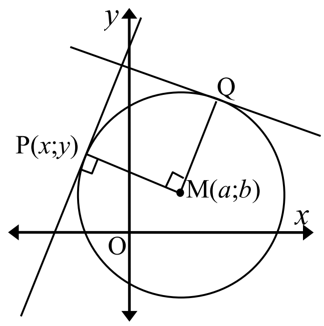


19. Let the centre of the circle in question 18 be M(3; 2) and P the point (-3; 4).

(a) Calculate the length of the radius MP.
(b) Write down the equation of the circle.
(c) Calculate the gradient of the radius MP.
(d) Find the equation of the tangent through the point P.

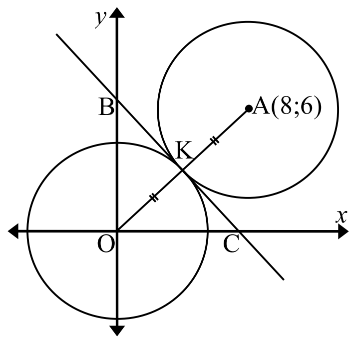
20. The radius MP and the tangent as described in question 19

through an angle of 90° so that the new point of contact is Q.



(a) Write down the gradient and the equation of the new radius MQ.
(b) Calculate the co-ordinates of the new point(s) of contact.
(c) Write down the gradient of the new tangent.
(d) Find the equation of the new tangent through Q.

21. O and A are the centres of two circles with equal radii. The two circles touch each other at point K and OKA is a straight line.
(a) Calculate the co-ordinates of K.
(b) Write down the equation of circle O.
(c) Write down the equation of circle A.
(d) Find the equation of BKC, a common tangent to both circles.
(e) Does OC = OB? Explain.
(f) What kind of shape will OCAB be? Explain.



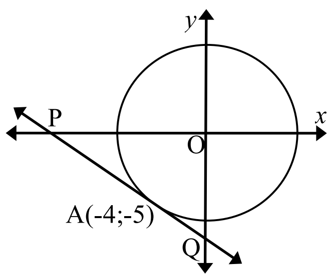
22. A tangent and the circle $x^2 + y^2 = 25$ touch at (3;–4).

(a) Find the equation of the tangent at (3;–4).
(b) Find the point on the circle where another tangent will be parallel to the tangent in question 5(a).

23.The drawing on the right shows a circle with centre at the

origin and a tangent PQ that touches the circle at $A(-4;-5)$

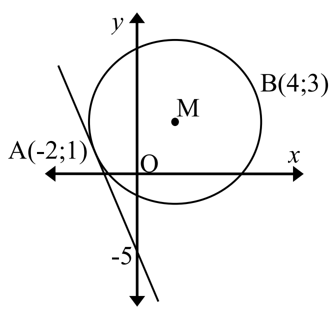
with P and Q on the x- and y-axes respectively.



(a) Determine the equation of the circle.
(b) Determine the gradient of OA.
(c) Determine the equation of the tangent PQ.
(d) Write down the co-ordinates of P and Q.

24.The circle $x^2 + y^2 = 10$ and the point $P(-4;-2)$ are given.
(a) Show by calculation that P lies outside the circle.
(b) PT is a tangent to the circle at T. T lies in the second quadrant and PT is parallel to the radius OA of the circle with A(I; 3). Show that the equation of the tangent PT is $y - 3x - 10 = 0$.
(c) Calculate the length of PT.

25. $A(-2;1)$ and $B(4;3)$ are two points on a circle with centre M. The equation of the tangent to the circle at A is $3x + y + 5 = 0$.
Find the:
(a) equation of the perpendicular bisector of AB, and deduce that AB is a diameter of the circle
(b) equation of MA
(c) co-ordinates of M equation of the circle.



26. Find the equations of the tangents:
(a) to $x^2 + y^2 = 5$ with a gradient of -2.
(b) to $x^2 + y^2 = 18$ with an inclination of 135°.

(c) to $(x+1)^2+y^2=20$ which is parallel to $2y-x=0$.

(d) to $(x-2)^2+(y+3)^2=16$ which is parallel to the $y-$axis.

27. (a) Find the equation of the circle, with centre the origin, through $A(-2;4)$.

(b) Find the equation of the tangent to the circle at A.

(c) If the tangent cuts the $x-$axis at B, find the length of AB.

(d) Find the equation of the other tangent to the circle from B, if C is the point of contact of the tangent to the circle.

(e) Show that AB = BC.

28. Find the equation of a circle with centre $C(2;3)$ and which touches the $x-$axis.

29. Find the equation of the circle touching the $x-$axis at $(3;0)$, passing through $(1;2)$.

30. The straight line $y=x+2$ cuts the circle $x^2+y^2=20$ at A and B.

(a) Determine the coordinates of A and B.

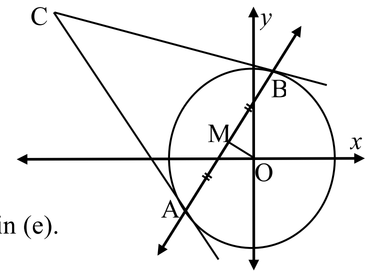
(b) Determine the length of chord AB.

(c) Determine the coordinates of M, the midpoint of chord AB.

(d) Show that $OM \perp AB$ if O is the origin.

(e) Determine the equations of the tangents to the circle at A and B.

(f) Determine the coordinates of C, the point of intersection of the tangents in (e).



31. The straight line $y=-2x+c$ cuts the circle $(x+1)^2+y^2=20$ at $A(x;y)$.

(a) Determine the equation of the radius through A.

(b) Determine the coordinates of A and hence the value of C.

32. BE and EL are tangents to a circle, centre M, at D and E respectively. The equation of the circle is $x^2-8x+y^2-4y+15=0$ and the equation of the tangent BE is $y=\frac{1}{2}x+c$.

(a) Determine the centre M and the radius of the circle.

(b) Determine the equation of the radius MD and hence show that the coordinates of D are $(3;4)$.

(c) If the coordinates of L are $(9;0)$ calculate the length of LN.

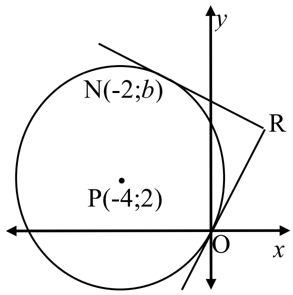
33. A circle with centre $P(-4;2)$ has the points $O(0;0)$ and $N(-2;b)$ on the circumference. The tangents at O and N meet at R. Determine:

(a) the equation of the circle

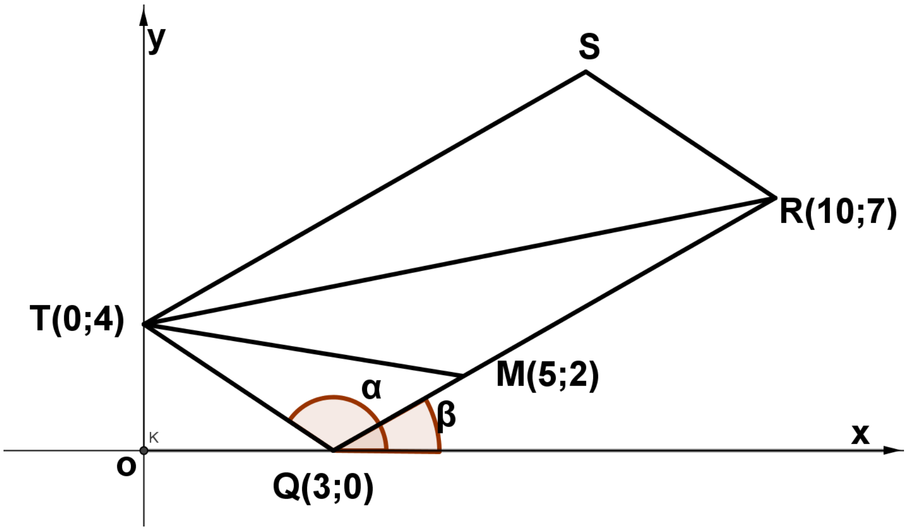
(b) the value of *b*.

(c) the equation of OR.

(d) the coordinates of R.



34. In the diagram, Q(3;0), R(10;7), S and T(0;4) are vertices of a parallelogram QRST. From T a straight line is drawn to meet QR at M(5;2). The angles of inclination of TQ and RQ are α and β respectively.

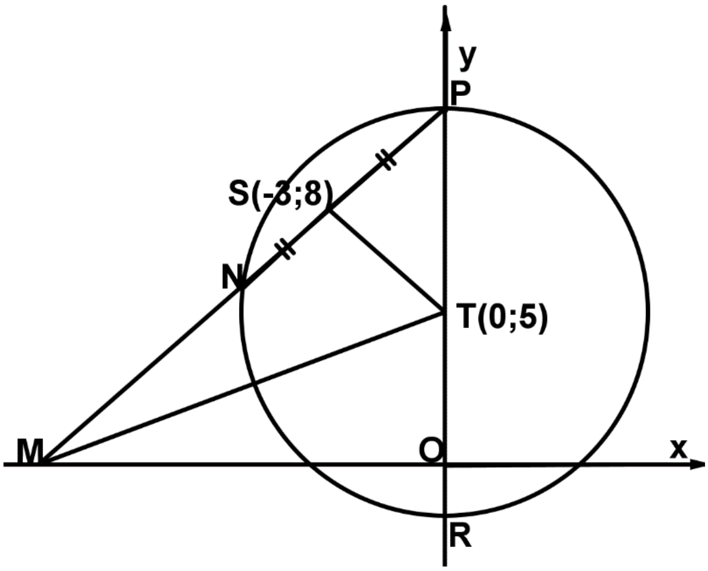


34.1 Calculate the gradient of TQ (1)
34.2 Calculate the length of RQ. Leave your answer in a surd form. (2)
34.3 F(k;-8) is a point in a Cartesian plane such that T, Q and F are collinear. Calculate the value of k. (4)
34.4 Calculate the coordinates of S. (4)
34.5 Calculate the size of $T\hat{S}R$ (6)
34.6 Calculate, in the simplest form, the ratio of:

34.6.1 $\frac{MQ}{RQ}$ (3)

34.6.2 $\frac{area\ of\ \Delta TQM}{area\ of\ parm\ RQTS}$ (3)

35. In the diagram, the circle, having the centre T(0;5), cuts the y-axis at P and R. the line through P and S(-3;8) intersects the circle at N and the x-axis at M . NS = PS. MT is drawn.



35.1 Give a reason why TS ⊥ NP.
(1)
35.2 Determine the equation of a line passing through N and and P in the form $y = mx + c$. (5)
35.3 Determine the equations of the tangents to the circle that are parallel to the x-axis. (4)
35.4 Determine the length of MT (4)
35.5 Another circle is drawn through the points S, T and M. Determine, with reasons, the equation of this circle in the form $(x - a)^2 + (y - b)^2 = r^2$. (5)

8 EUCLIDEAN GEOMETRY

In Grades 10 to 12 Geometry ceases to deal mainly with finding the size of angles and line segments but focuses more on proof. A Geometry proof is different in that you start from scratch unlike proofs in Trigonometry, Analytical Geometry and Algebra where you start, either on the left hand side (LHS) or right hand side (RHS).
It is therefore important for all the learners to understand a basic deductive proof in Euclidean geometry and be able to write/construct a simple deductive proof before proceeding to complex proofs found in Grade 11 and 12.

Important points about proof in Geometry

1. Read the problem carefully for understanding. You may need to underline important points and make sure you understand each term in the given and conclusion.
2. Draw the sketch if it is not already drawn. The sketch need not be accurately drawn but must as close as possible to what is given i.e. lines and angles which are equal must look equal or must appear parallel etc. Also indicate further observations based on previous theorems.
3. Indicate on the figure drawn or **given** all the equal lines and angles, lines which are parallel, drawing in circles, measures of angles given if not already indicated in the question. It might be more helpful to have a variety of colour pens or highlighters for this purpose.
4. Usually you can see the conclusion before you actually start your **formal proof** of a rider. Always to write the reason for each important statement you make, quoting in brief the theorem or another result as you proceed.
5. Sometimes you may need to work backwards, asking yourself what I need to show to prove this **conclusion** (required to be proved) and then see if you can prove that as you reverse.
6. **WRITE GEOMETRY REASONS CORRECTLY. Refer to acceptable reasons as reflected in the Examination Guidelines.**

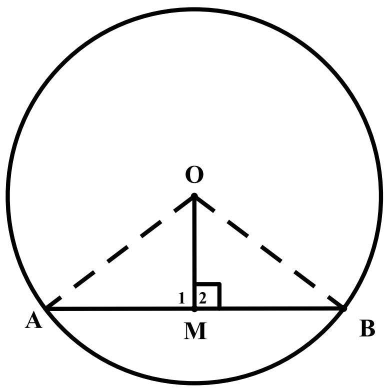
Note:
There are six (6) theorems which learners must know how to prove/examinable. The rest are corollaries (results flowing directly from theorems which can be proved as riders or axioms (self-evident truths which require no proofs) and the 'common notions'. It is very important to know how to prove these 6 theorems as 2 or more are sure 'spots'. Acceptable short (abbreviated) forms of theorems must be known. There are some meaningless or wrong/incorrect short forms of theorems which are not acceptable.

Examinable Theorems
Note the following when proving the theorems

- The construction, if applicable should be clearly indicated in the diagram
- Naming of angles after the construction should be in lie with the statements

A) The line drawn from the centre of a circle perpendicular to a chord bisects the chord

R.T.P : AM = MB
Proof : Draw radii OA and OB



In Δ OAM and Δ OBM

1. OA = OB (radii)
2. OM is common
3. $\hat{M}_1 = \hat{M}_2 = 90°$ (adjacent supp.)

$\therefore \Delta OAM \equiv \Delta OBM$ (90°HS)

$\Rightarrow AM = MB$

B) The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

Given : Circle with centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{C}B$ at the circle

R.T.P : $A\hat{O}B = 2 \times A\hat{C}B$

Construction : Draw CO and produce

Diagram 1

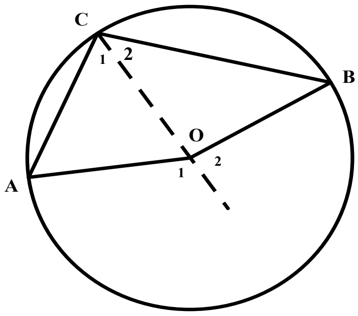


Diagram 2

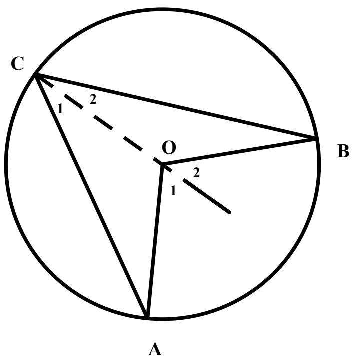
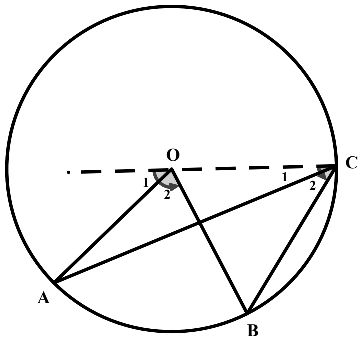


Diagram 3



Proof:

$\hat{O}_1 = \hat{C}_1 + \hat{A}$ (Ext $\angle$of Δ = sum of int opp $\angle$'s)

but $\hat{C}_1 = \hat{A}$ ($\angle$'s opp = sides OA and OC radii)

$\therefore\ \hat{O}_1 = 2\hat{C}_1$

similarly $\hat{O}_2 = 2\hat{C}_2$

In Diagram 1 & 2 : $\hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore\ A\hat{O}B = 2 \times A\hat{C}B$

In Diagram 3 : $\hat{O}_2 - \hat{O}_1 = 2(\hat{C}_2 - \hat{C}_1)$

$\therefore\ A\hat{O}B = 2 \times A\hat{C}B$

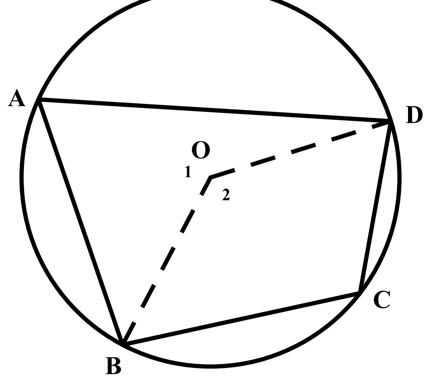
C) The opposite angles of a cyclic quadrilateral are supplementary

Given any circle with centre O, passing through the vertices of cyclic quadrilateral ABCD

R.T.P.: $\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$

Construction : Draw BO and OD

Proof:



$\hat{O}_2 = 2\hat{A}$ ($\angle$ at the centre $= 2 \times \angle$ at circle)

$\hat{O}_1 = 2\hat{C}$ ($\angle$ at the centre $= 2 \times \angle$ at circle)

$\hat{O}_1 + \hat{O}_2 = 2(\hat{A} + \hat{C})$

but $\hat{O}_1 + \hat{O}_2 = 360°$ ($\angle$'s around a point)

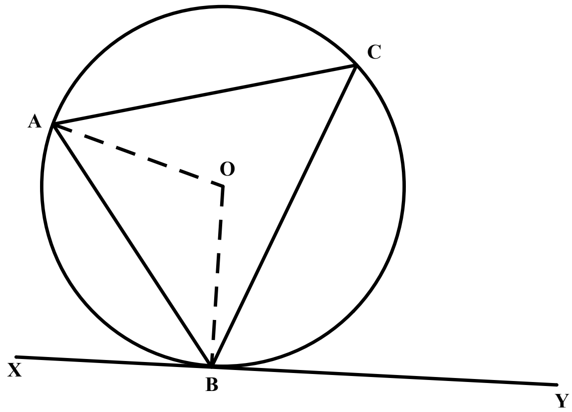
hence $\hat{A} + \hat{C} = 180°$

also $\hat{B} + \hat{D} = 180°$ (sum of int $\angle$'s of quad)

D) Angle between a tangent and the chord is equal to the angle in the alternate segment.

RTP: $A\hat{B}X = \hat{C}$

Construction:Construct radii AO and OB

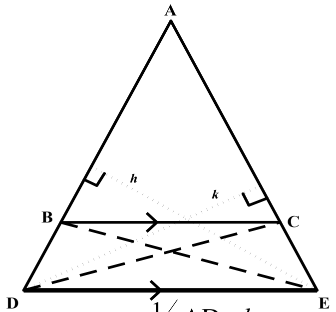


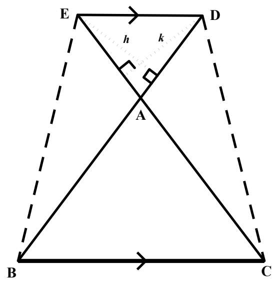
$$X\hat{B}O = 90° \text{ (tangent} \perp \text{radius)}$$
$$O\hat{B}A = 90° - A\hat{B}X$$
$$B\hat{A}O = 90° - A\hat{B}X \text{ } (\angle\text{'s opp} = \text{sides BO \& OA radii})$$
$$A\hat{O}B = 180° - (90° - A\hat{B}X + 90° - A\hat{B}X) \text{ (sum of } \angle\text{'s in } \Delta)$$
$$= 2\,A\hat{B}X$$
$$\text{but } A\hat{O}B = 2\hat{C} \quad (\angle \text{ at centre} = 2 \times \angle \text{ at circle})$$
$$\therefore \quad 2A\hat{B}X = 2\hat{C}$$
$$\Rightarrow \quad A\hat{B}X = \hat{C}$$

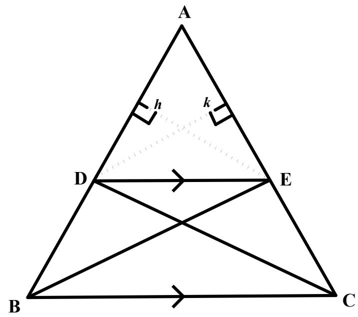
E) The line drawn parallel to one side of a triangle divides the other two sides proportionally

Given : Δ ABC, D lies on AB and E lies on AC. And DE // BC.

R.T.P : $\dfrac{AD}{DB} = \dfrac{AE}{EC}$







$$\frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ BDE}} = \frac{\frac{1}{2}\text{AD} \times h}{\frac{1}{2}\text{DB} \times h} = \frac{\text{AD}}{\text{DB}} \text{ (same height)}$$

$$\frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ CED}} = \frac{\frac{1}{2}\text{AE} \times k}{\frac{1}{2}\text{EC} \times k} = \frac{\text{AE}}{\text{EC}} \text{ (same height)}$$

$$\text{but Area } \Delta \text{ BDE} = \text{Area } \Delta \text{ CED} \text{ (same base \& between // lines)}$$

$$\therefore \quad \frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ BDE}} = \frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ CED}}$$

$$\therefore \quad \frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}}$$

F. Equiangular triangles are similar

In Δ AXY and ΔDEF

1. $AX = DE$ (construction)
2. $AY = DF$ (construction)
3. $\hat{A} = \hat{D}$ (given)

$\therefore \Delta AXY \equiv \Delta DEF$ (SAS)

now $A\hat{X}Y = \hat{E}$ but $\hat{E} = \hat{B}$ (given)

$\therefore A\hat{X}Y = \hat{B}$

$\Rightarrow XY \,//\, BC$ (correponding $\angle$'s =)

now $\frac{AB}{AX} = \frac{AC}{AY}$ (line // one side of Δ)

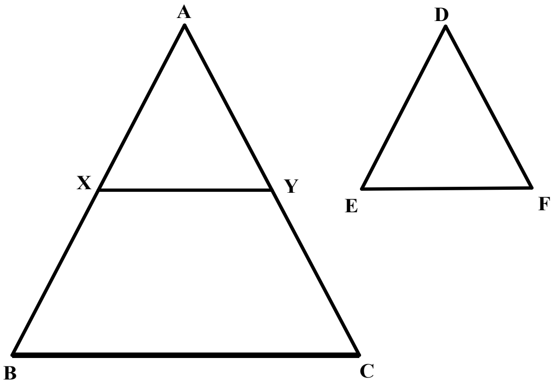
but $AX = DE$ and $AY = DF$ (construction)

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

similarly by marking off equal lengths on BA and BC

it can be shown that : $\frac{AB}{DE} = \frac{BC}{EF}$

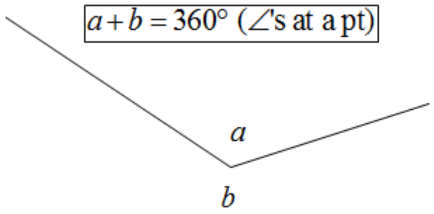
$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$



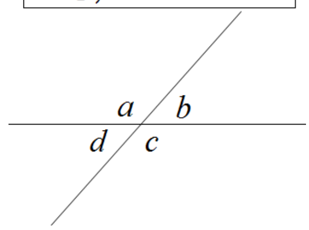
Note: Learners must be able to identify, visualise theorems , axioms to apply in every situation. When presented with a diagram they should be able to write the theorem in words.

SUMMARY FOR EUCLIDEAN GEOMETRY

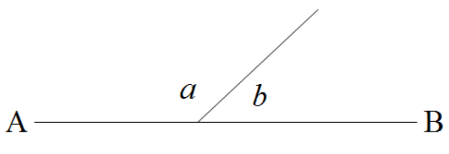
$a + b = 360°$ ($\angle$'s at a pt)



$\left.\begin{array}{l} a = c \\ b = g \end{array}\right\}$ (vert.opp.$\angle$'s)

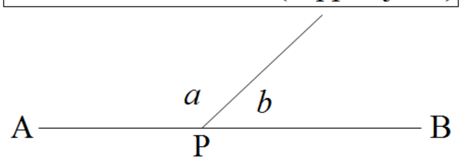


$a + b = 180°$ (adj.$\angle$'s st line)

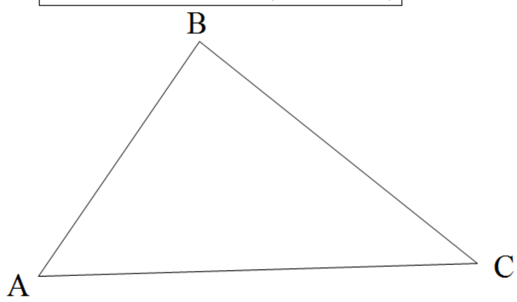


$\Leftrightarrow$

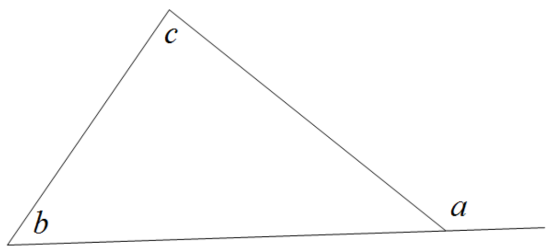
If $a + b = 180°$,
then APB is a st line (supp.adj.$\angle$'s)



$\hat{A} + \hat{B} + \hat{C} = 180°$ ($\angle$ sum Δ)

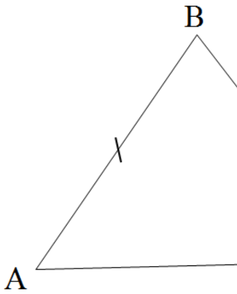


$\hat{A} = \hat{B} + \hat{C}$ (ext.$\angle$ Δ)



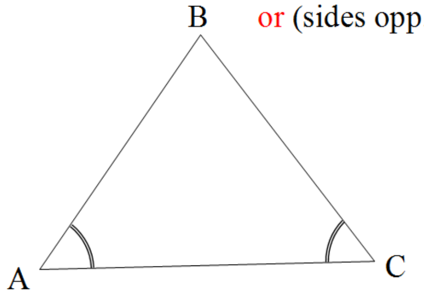
$\boxed{\hat{A} = \hat{C} \text{ (isos } \Delta)}$

or (∠'s opp. = sides)

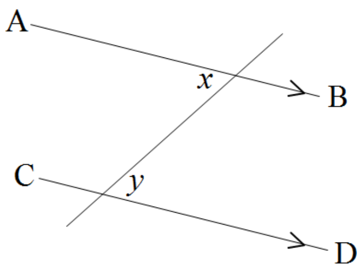


$\boxed{BA = BC \text{ (isos } \Delta)}$

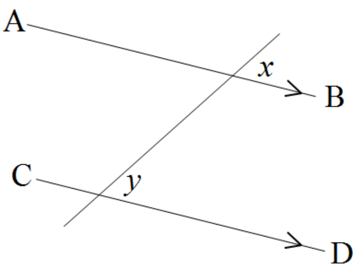
or (sides opp. = ∠'s)



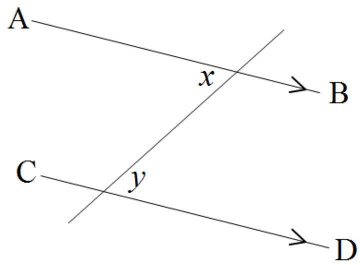
$\boxed{x = y \text{ (alt.∠'s; AB//CD)}}$

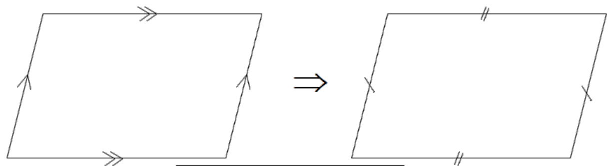


$\boxed{x = y \text{ (corres.∠'s; AB//CD)}}$

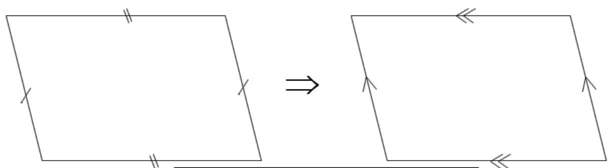


$\boxed{x + y = 180° \text{ (coint.∠'s; AB//CD)}}$

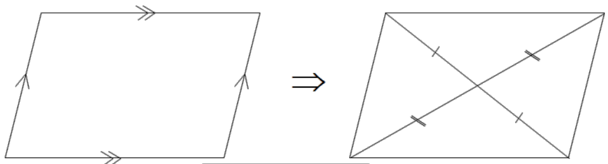




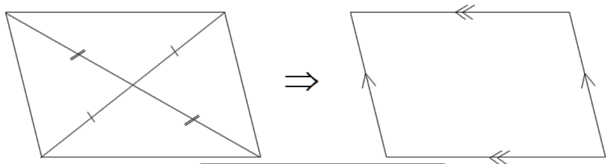
(opp. sides parm)



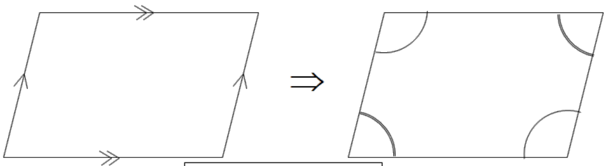
(conv. opp. sides parm)



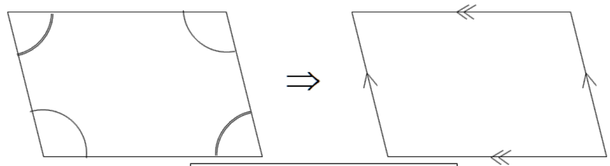
(diags parm)



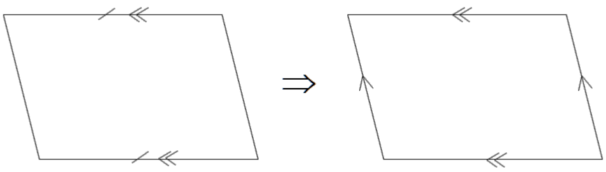
(conv. diags parm)

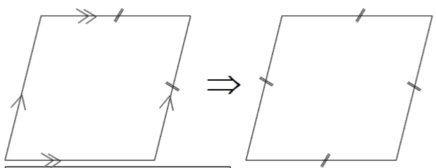


(opp.∠'s parm)

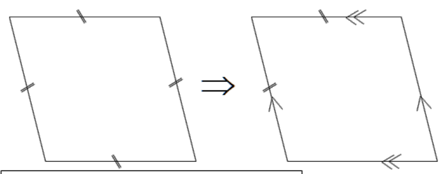


(conv.opp.∠'s parm)

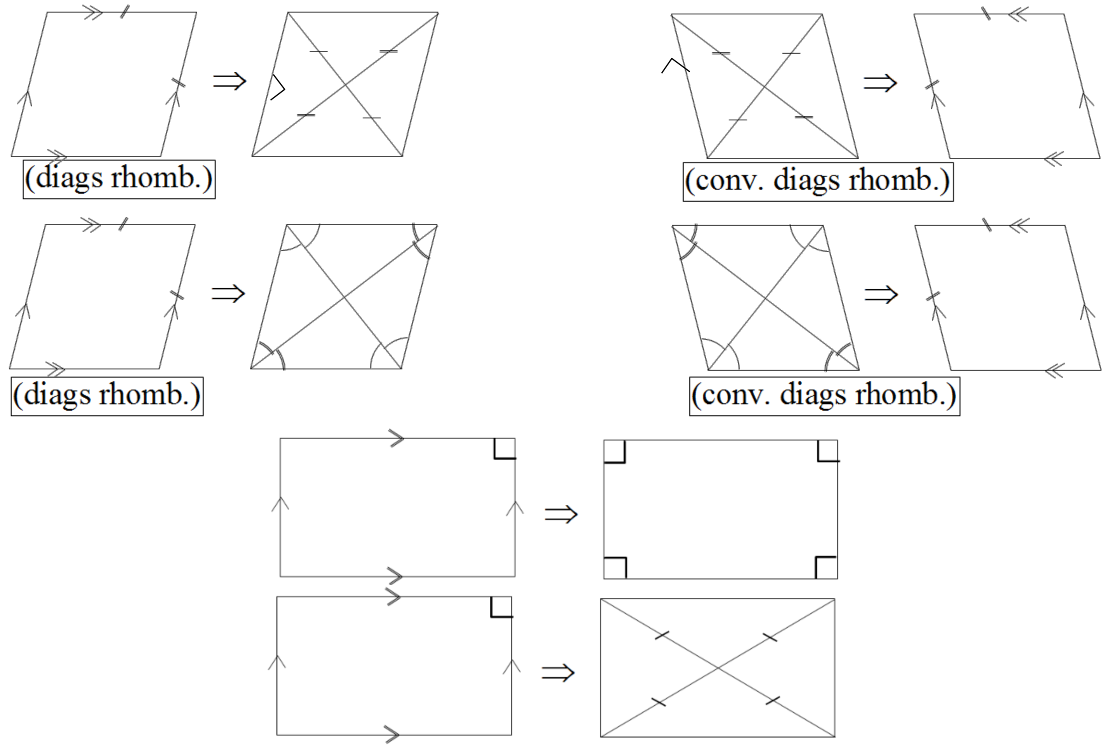


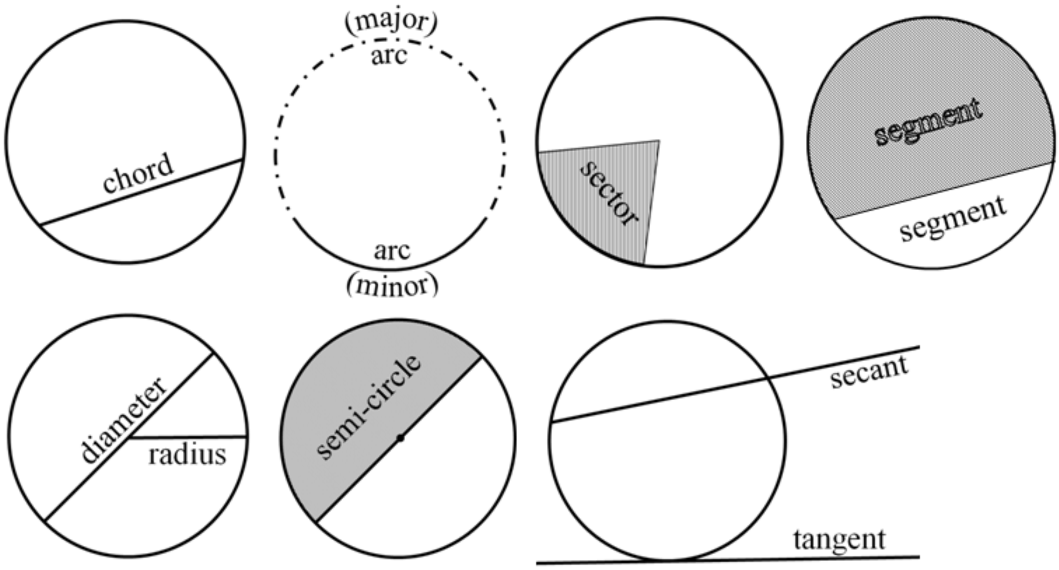


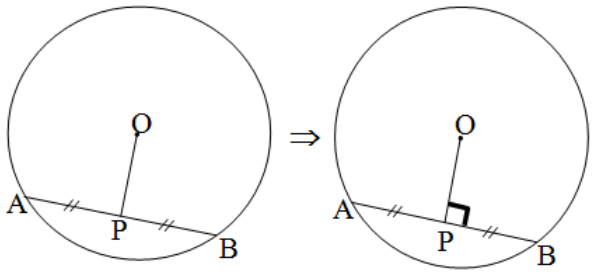
(sides of rhomb.)



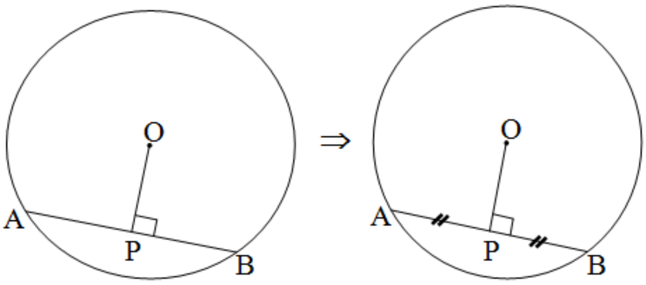
(conv. sides of rhomb.)







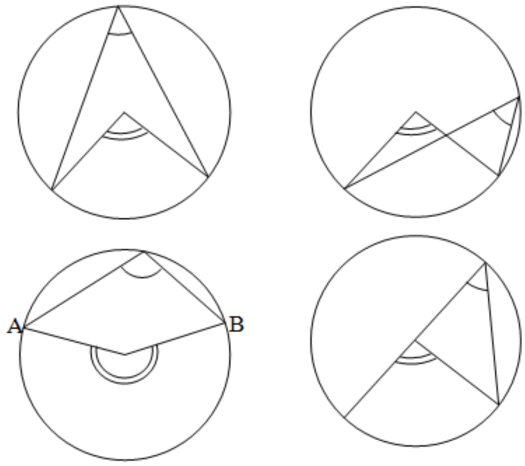
i.e. $AP = PB \Rightarrow OP \perp AB$



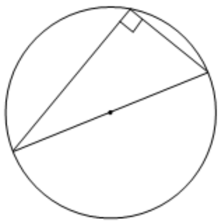
i.e. $AP = PB \Rightarrow OP \perp AB$

$\boxed{\text{Reference:}}$ $\angle$ **at centre**

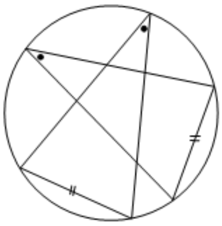
i.e. $\widehat{O} = 2\widehat{P}$, O centre



$\boxed{\text{Reference:}}$ $\angle$ **in semi-circle**

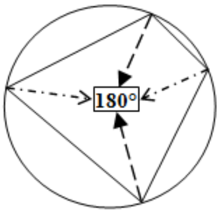


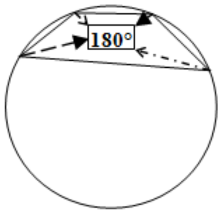
Reference: $\angle$'s **subt. by = chords**



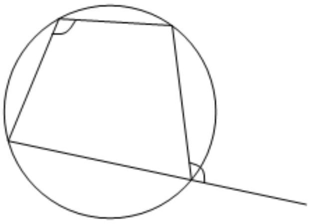
Reference: **opp.** $\angle$'s **cyclic quad.**

i.e. $\widehat{A}+\widehat{C}=180^\circ$; $\widehat{B}+\widehat{D}=180^\circ$

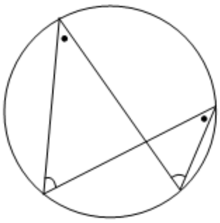




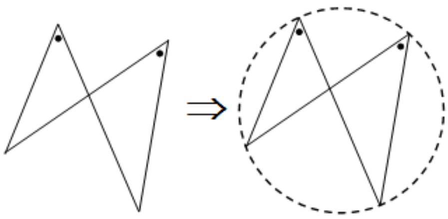
Reference: **ext.** $\angle$ **cyclic quad.**



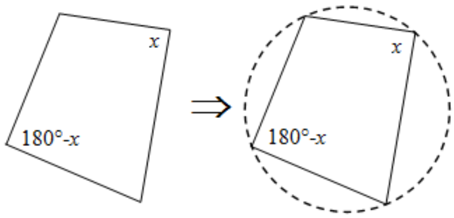
Reference: $\angle$'s **in same segment**



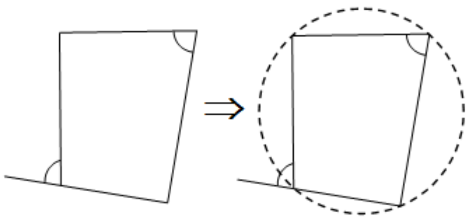
Reference: **conv.** $\angle$**'s in same segm.**



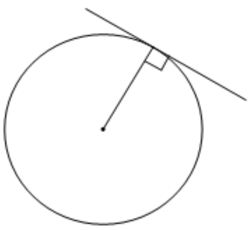
Reference: **conv. opp.** $\angle$**'s cyclic quad.**



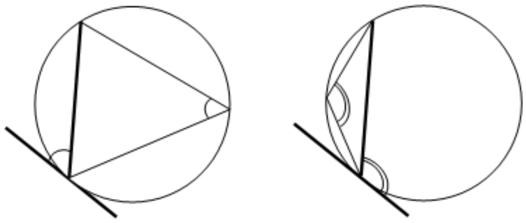
Reference: **conv. ext.** $\angle$ **cyclic quad.**



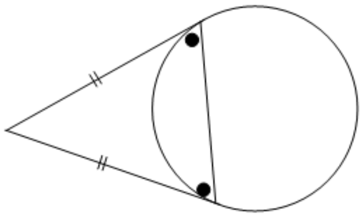
Reference: **rad. tan.**



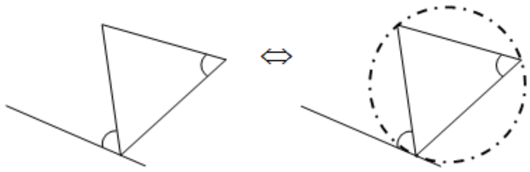
Reference: **tan. chord**



Reference: **tans from a common pt**



Reference: **conv. tan. chord**



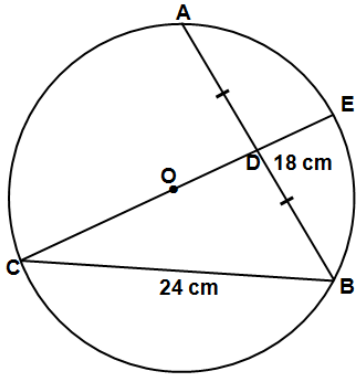
PRACTICAL EXERCISES

QUESTION 1

1.1 Complete the following so that the Euclidean Geometry statement is true:

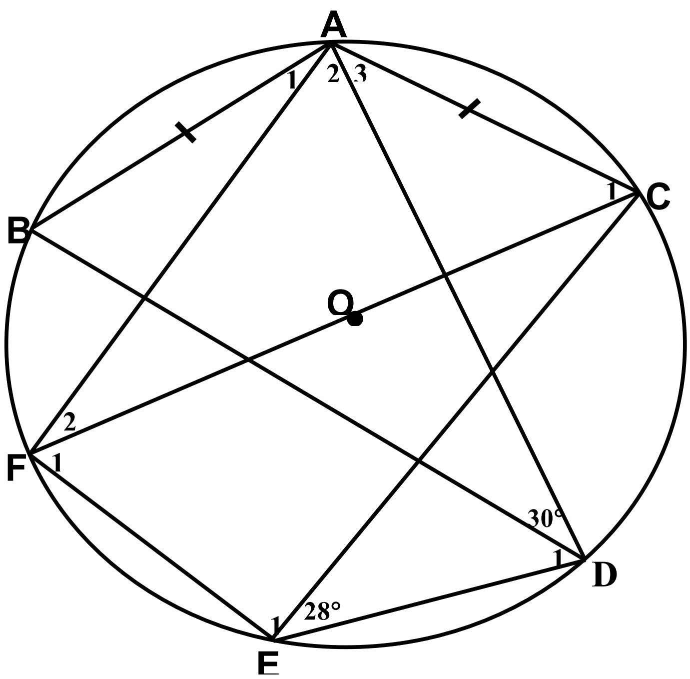
A line drawn from the centre of a circle to the midpoint of the chord is ....to the chord (1)

1.2 In the circle with centre O, chord AB = 18cm and AD = DB. Chord CB = 24 cm.



1.2.1 Calculate the length of CD. Leave the answer in simplest surd form. (4)

1.3. In the diagram, O is the centre of the circle. Chords AB = AC. $C\hat{E}D = 28^\circ$ and $A\hat{D}B = 30^\circ$



Calculate, with reasons, the sizes of the following angles:

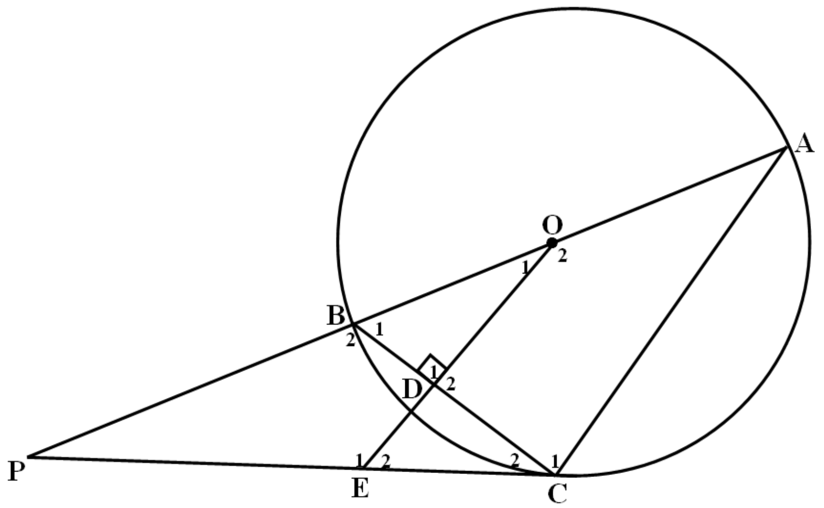
1.3.1 $\hat{E}_1$ (2)

1.3.2 $\hat{A}_2$ (3)

1.3.3 $\hat{F}_2$ (2)

QUESTION 4

4.1 In the diagram, AB is a diameter of circle, centre O. AB is produced to P. PC is a tangent to the circle at C. OE $\perp$ BC at D.



4.1.1 Prove, with reasons, that EO || CA. (4)

4.1.2 If $\hat{C}_2 = x$, name with reasons, two other angles each equal to x. (3)

4.1.3 Calculate the size of $\hat{P}$ in terms of x. (2)

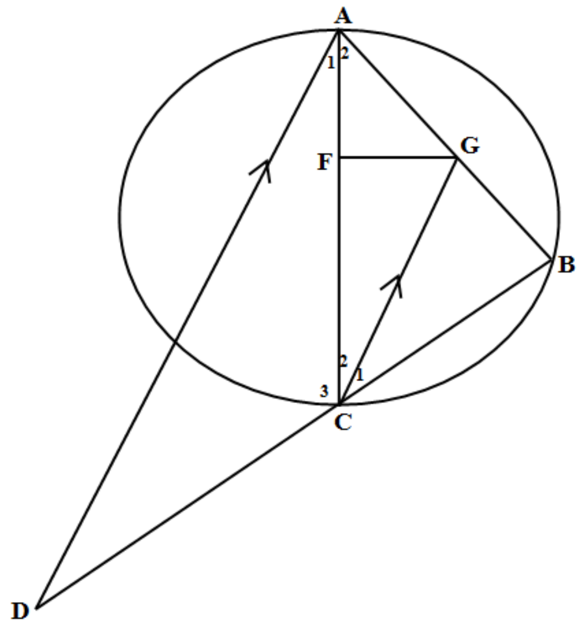
[14]

QUESTION 5

5.1 Complete the following so that the Euclidian Geometry statement is true:

A line drawn parallel to one side of a triangle divides the other two sides ……. (1)

5.2 In the diagram below, CG bisects $A\hat{C}B$. AD || GC.

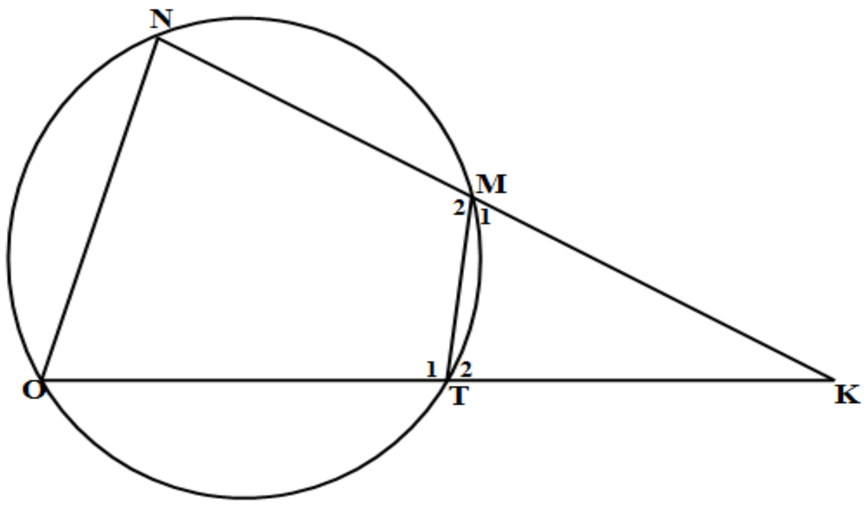


Prove, with reasons, that:

5.2.1 AC = DC (4)

5.2.2 $\frac{BC}{AC}=\frac{BG}{AG}$ (3)

5.3 In the diagram, KMN and KTO are two secants of a circle.



5.3.1 Prove that $\Delta MTK \parallel\!\mid \Delta ONK$. (3)

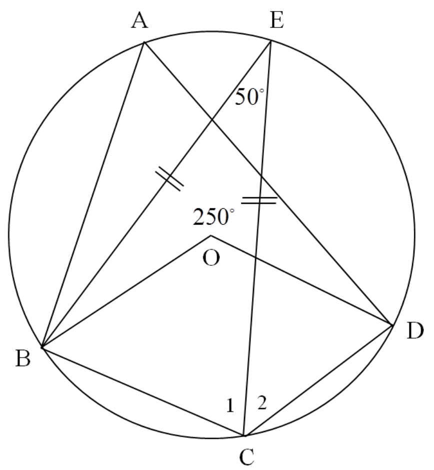
5.3.2 Hence, prove that KM.KN = KT.KO (1)

5.3.3 Calculate KT if OT = 6 units, MN = 3 units and MK = 5 units. (4)

[16]

QUESTION 6

Refer to the figure below:



The circle, centred at O, has points A, B, C, D and E on the circumference of the circle. Reflex angle $B\hat{O}D = 250°$ and $B\hat{E}C = 50°$. Chord BE = EC.

Determine the following, **stating all necessary reasons**:

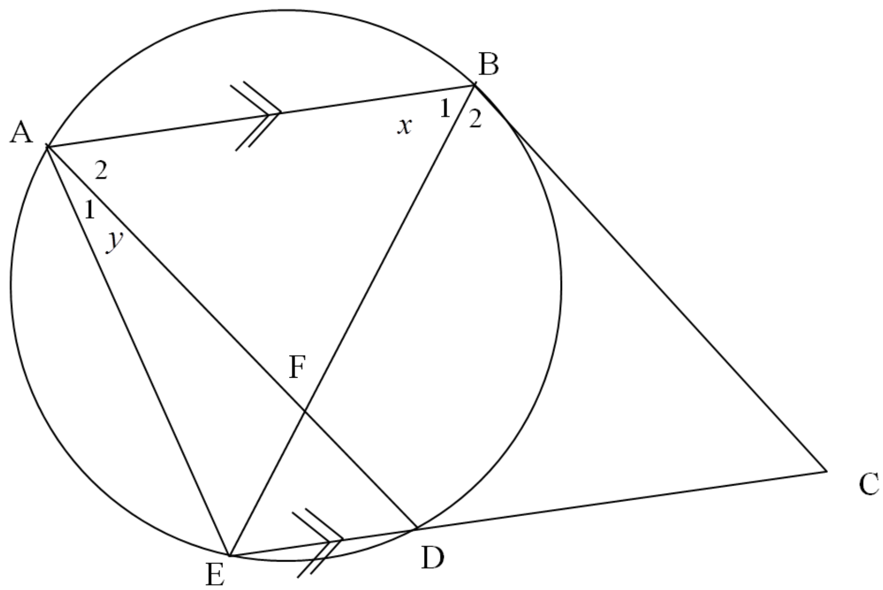
(a) $\hat{A}$ (2)

(b) $B\hat{C}D$ (1)

(c) $\hat{C}_2$ (2)

QUESTION 7

Tangent BC touches the circle ABDE at B. Chords AD and BE intersect at F. Chord ED is produced to C. AB // ED. It is further given that $\hat{B}_1 = x$ and $\hat{A}_1 = y$.



NOTE: Reasons must be given.

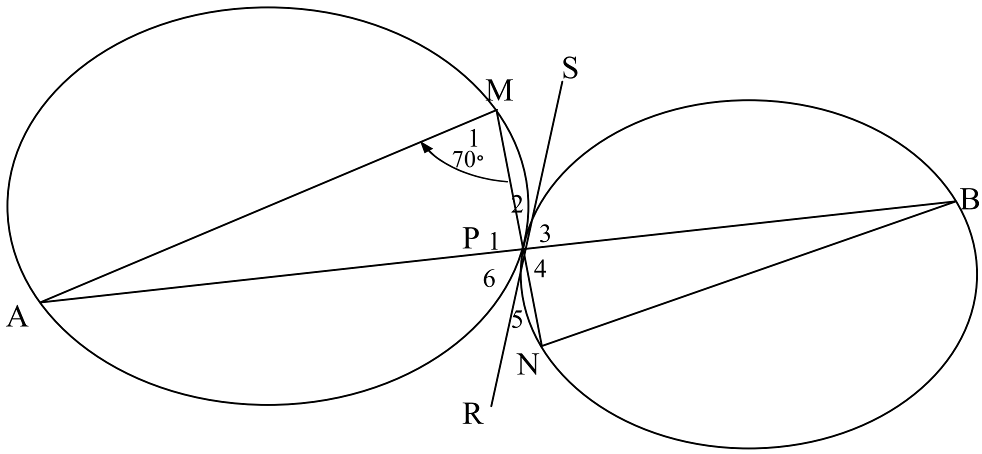
(a) Determine the magnitude of $\hat{C}$ in terms of x and y. (5)

(b) John says to Becky, "I am sure I can prove quad BCDF is cyclic".
Becky says, "This is impossible".
Who is correct?
Motivate your answer by performing all of the necessary calculations. (3)

[8]

QUESTION 8 (Feb 2012)

In the diagram below, AM is the diameter of the bigger circle AMP. RPS is a common tangent to both circles at P. APB and MPN are straight lines.



8.1 State the size of $\hat{P}_1$. (1)

8.2 Hence, show that BN is the diameter of the smaller circle. (2)

8.3 If $\hat{M}_1 = 70°$, calculate the size of each of the following angles:

8.3.1 $\hat{A}$ (1)

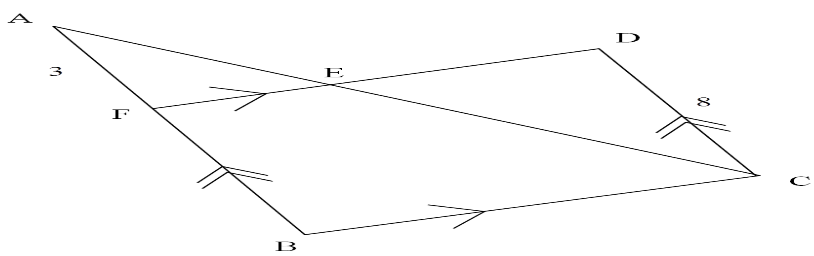
8.3.2 $\hat{P}_6$ (1)

8.3.3 $\hat{B}$ (2)

[7]

QUESTION 9

9.1.In the figure AB // CD and FD // BC.
CD : AF = 3:1



9.1.1.Determine the value of $\frac{ED}{BC}$,giving reasons. (Hint: Prove a triangle similar to ΔABC (7)

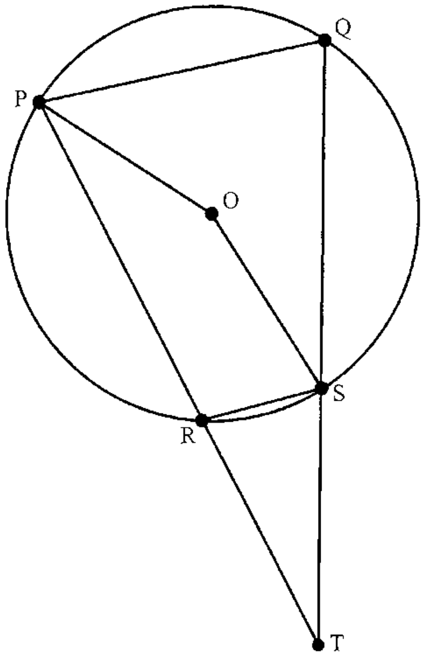
9.1.2 ΔPQR and ΔPST are right-angled triangles with RQ=SP=2RS. S is a point on RP with ST $\perp$PQ.

Write down, without proof, a pair of triangles that are similar (1)

9.1.3 Show that $ST = \frac{2}{\sqrt{13}}.QR$ (6)

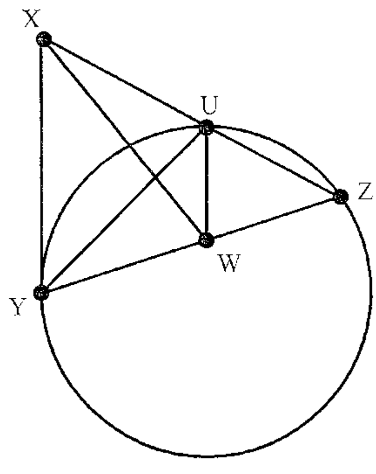
Question 10

In the figure, PQSR is a cyclic quadrilateral inscribed in circle O. QS and PR are extended to meet at T. If P, O, S and T are concyclic,



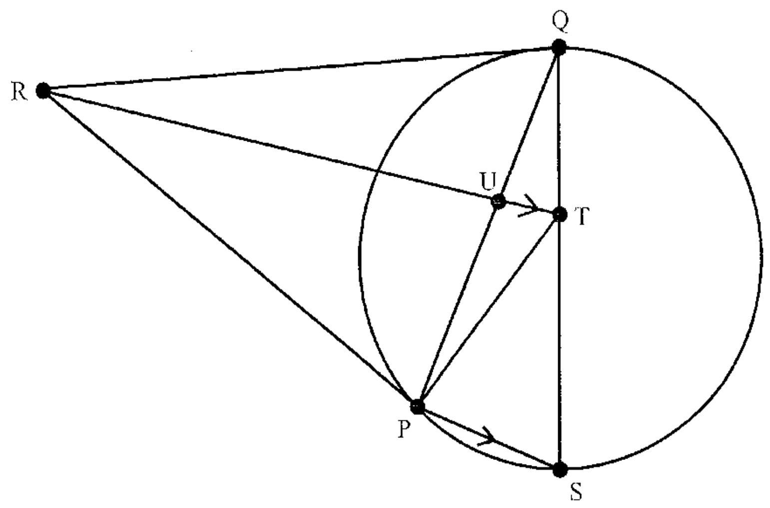
10.1 Prove that: PQ || RS (6)

10.2 In the figure, YZ and UZ and UY are chords of the circle. W is a point on Y and UW is joined. ZU is extended to X so that XY is a tangent to the circle. XY = XW.



10.2.1 Prove that XUWY is a cyclic quadrilateral. (7)
10.2.2 Prove that XW is a tangent to circle UWZ. (4)

10. 3. In the figure, RQTP is a cyclic quadrilateral and RT bisects $Q\hat{T}P$. RT || PS, while QS is a diameter for circle PQS.



Prove that:

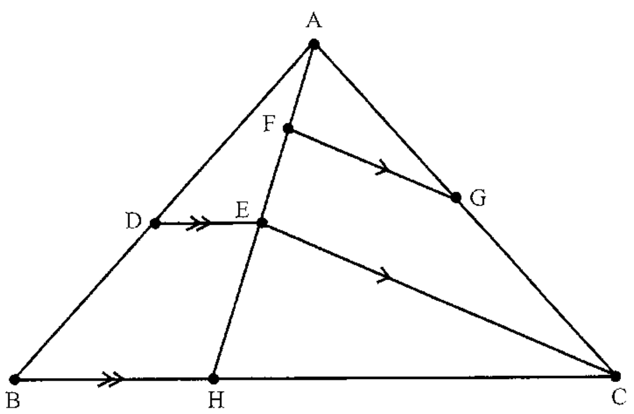
10.3.1 RP = RQ (3)

10.3.2 RQ and RP are tangents to the circle PQS. (4)

10.3.3 RT⊥QP (3)

10.3.4 QT=PT=TS (5)

10.4. In the figure, DE || BH and FG || EC, where F is the midpoint of line segment AE. It is further given that $\frac{AD}{DB} = \frac{2}{3}$. Determine:



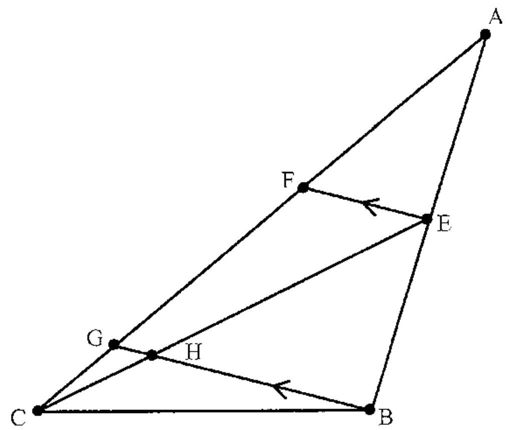
10.4.1 $\frac{DE}{BH}$ (3)

10.4.2 $\frac{AG}{AC}$ (1)

10.4.3 $\frac{FE}{EH}$ (2)

10.3.4 $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta ABH}$ (3)

10.4 In the figure, AF = 2CG and FE || GB. $\dfrac{AE}{AB} = \dfrac{2}{5}$

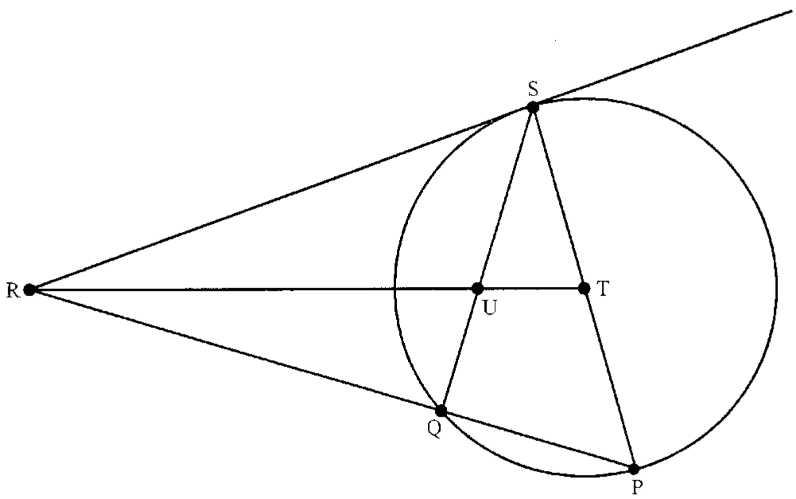


10.4.1 Determine $\dfrac{AF}{FG}$ (2)

10.4.2 Determine $\dfrac{CH}{HE}$ (4)

10.4.3 Determine $\dfrac{\text{Area } \Delta BCG}{\text{Area } \Delta AFE}$ (4)

10.5 In the figure, PS is a diameter of the circle. The tangent at S meets PQ produced at R. RT bisects $S\hat{R}P$. The radius of the circle is 2,5 units and the length of RS is 10 units.



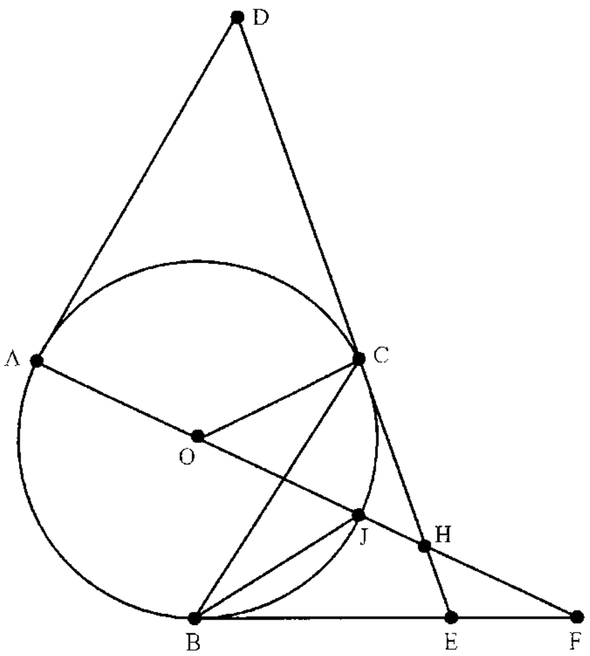
Prove:

10.5.1 $RS^2 = RQ \cdot RP$ (5)

10.5.2 Find the length of RQ (5)

10.5.3 Find the numerical value of the ratio $\dfrac{\text{Area } \Delta \text{ RSP}}{\text{Area } \Delta \text{ RQS}}$ (4)

10.6 In the figure, AD, DC and BE are tangents to the circle. CO is a radius and chord BC is drawn. Radius AO is drawn and extended to cut the circle at J and BE is extended at F.



10.6.1 Prove $\Delta DAH \parallel\!\!| \Delta OCH$ (4)

10.6.2 Prove $OH = \dfrac{AO.DH}{DC}$ (6)

10.6.3 Prove $\Delta JBF \parallel\!\!| \Delta BAF$ (4)

10.6.4 Prove $BF^2 = JF.AF$ (3)

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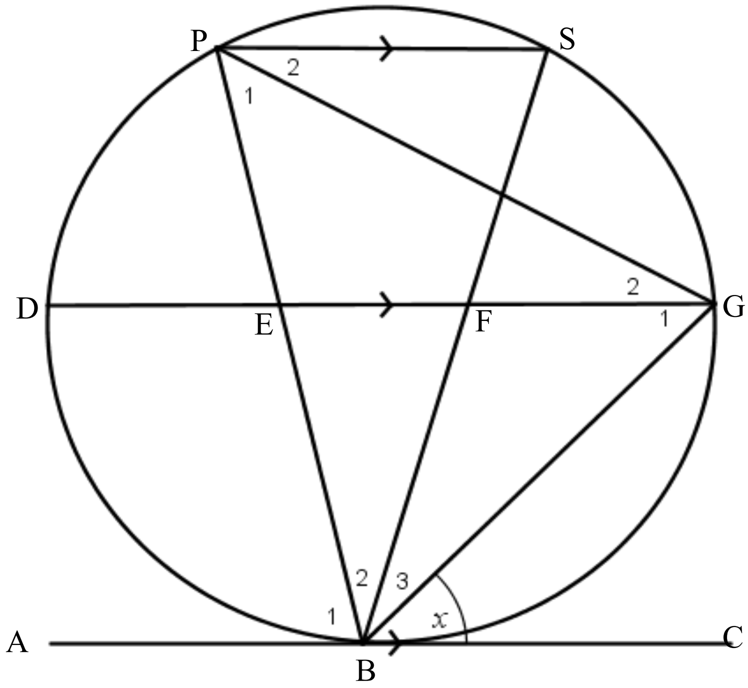
Question 11

In the diagram, P, S, G, B and D are points on the circumference of the circle such that PS || DG || AC. ABC is a tangent to the circle at B. $G\hat{B}C = x$.



11.1 Give a reason why $\hat{G}_1 = x$. (1)

11.2 Prove that:

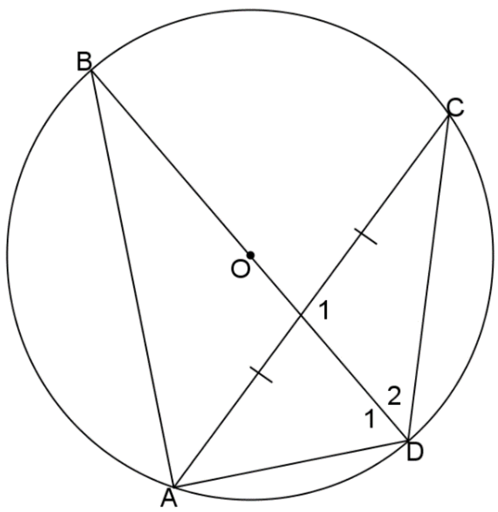
11.2.1 $BE = \dfrac{BP.BF}{BS}$ (2)

11.2.2 $\Delta BGP \parallel\!| \Delta BEG$ (4)

11.2.3 $\dfrac{BG^2}{BP^2} = \dfrac{BF}{BS}$ (3)

QUESTION 12

In the diagram, O is the centre of the circle. A, B, C and D are points on the circumference of the circle. Diameter BD bisects chord AC at E. Chords AB, CD and AD are drawn. $\hat{C} = 43°$.



12.1 Give a reason for DE $\perp$ AC. (1)

12.2 Calculate, giving reasons, the size of $\hat{B}$. (2)

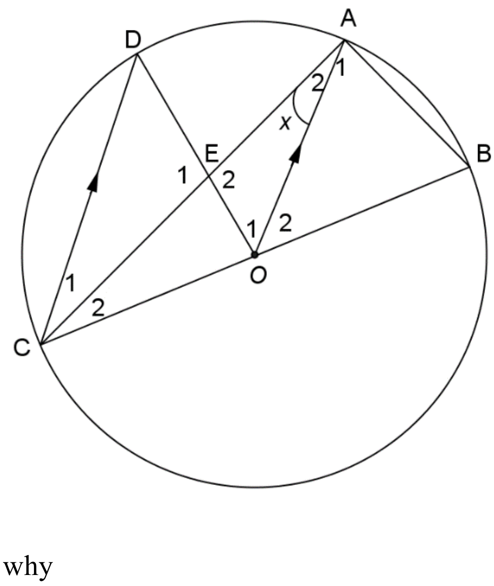
12.3 Prove that $\hat{E}_1 = B\hat{A}D$. (2)

12.4 The length of the diameter of the circle is 28 units. Calculate the length of AB. (2)

[7]

QUESTION 13

In the diagram, O is the centre of the circle. A, B, C and D are points on the circumference of the circle and CB is the diameter of the circle. Chord CA intersect radius OD at E. AB is drawn. CD | | OA and $\hat{A}_2 = x$.



13.1 Give reasons why

13.1.1 $\hat{C}_1 = x$ (1)

13.1.2 $\hat{C}_2 = x$ (1)

13.2 Determine, giving reasons, the size of the following angles in terms of x.

13.2.1 $\hat{A}_1$ (3)

13.2.2 $\hat{O}_1$ (2)

13.2.3 $\hat{O}_2$ (2)

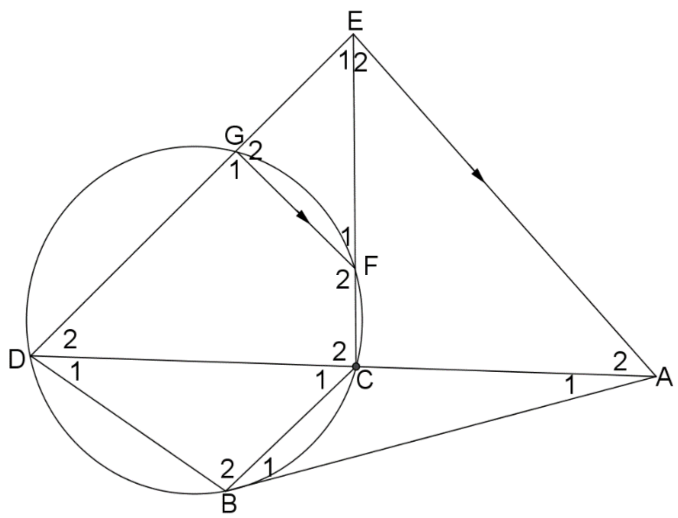
13.3 For which value of x will ABOE be a cyclic quadrilateral? (3)

[12]

QUESTION 14

14.1 Complete the following statement of the theorem in the ANSWER BOOK: (1)
If two triangles are equiangular, then the corresponding sides are ...

14.2 In the diagram, DGFC is a cyclic quadrilateral and AB is a tangent to the circle at B. Chords DB and BC are drawn. DG and CF produced meet at E and DC is produced to A. EA | | GF.



14.2.1 Give a reason why $\hat{B}_1 = \hat{D}_1$. (1)

14.2.2 Prove $\Delta ABC \,|||\, \Delta ADB$. (3)

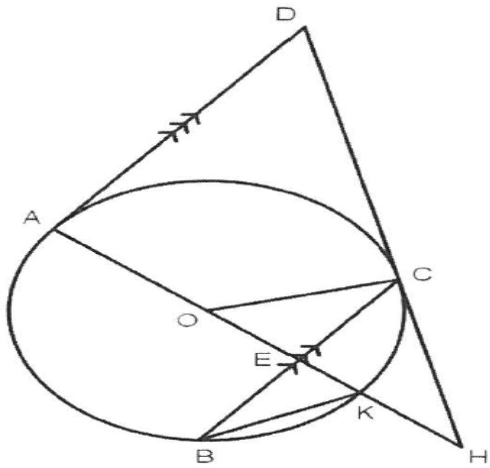
14.2.3 Prove $\hat{E}_2 = \hat{D}_2$. (4)

14.2.4 Prove $AE^2 = AD \times AC$. (4)

14.2.5 Hence deduce that AE=AB

Question 15

In the diagram below ,AD and DCH are tangents to the circle through A,C,K,and B. O is the centre of the circle. Diameter AK is produced to meet tangent DCH at H. OC,BC and BK are drawn. $AD \parallel BC$



Prove that:

15.1.1 AOCD is a kite (4)

15.1.2 ADCO is a cyclic quad (4)

15.1.3 $\hat{D} = 2\hat{B}$ (3)

15.1.4 (3)

15.1.5 OH×DC=AO×DH (5)

15.1.6 $\frac{AE}{EH} = \frac{AD}{DH}$ (3)