kzn education
Department:
Education
KWAZULU-NATAL

MATHEMATICS

LEARNER ASSISTANCE REVISION BOOKLET

GRADE 12

2020

This document has been compiled by the FET Mathematics Subject Advisors together with Lead Teachers. It seeks to unpack the contents and to give more guidance to teachers.

TABLE OF CONTENTS

PREAMBLE

This supporting document is meant to assist learners to understand basic concepts in this subject. It focuses mainly on levels 1 to 2 and a bit of level 3 questions/activities. When studied well, the document can help learners to do just about enough to get a pass mark during examinations. If a learner wishes to get even better marks, however, he or she is advised to work on levels 3 and 4 questions. Unfortunately, those challenging questions/activities are not contained herein.

TIPS ON ANSWERING MATHEMATICS QUESTION PAPER

1. It is important that you arrive at the writing venue 30 minutes before the starting time. This will give you time to relax and be ready for a writing session.
2. You are allowed 10 minutes for reading, utilize this time profitable.
3. Identify questions where you are more likely to earn more marks and start with those. Remember you can start with any question e.g. question 10 then question 4 etc.
4. In Paper 1 it is always advisable to start with question 1, when you are able to score marks automatically you develop confidence and will be able to try even challenging questions,
5. Use the calculator that you have been using when practising for writing.
6. Utilize all the time (3hrs) given for writing the paper.

HINTS TO LEARNERS TO ACHIEVE A PASS IN MTHEMATICS

Mark distribution for Mathematics CAPS papers: Grades 12				
PAPER 1 : Bookwork max 6 marks				
No	**Description**	**Gr. 12**	**Distinction**	**Pass**
1	Patterns & Sequences	**± 25**	± 22	± 10
2	Annuities & Finance	**± 15**	± 10	± 5
3	Functions & Graphs	**± 35**	± 30	± 10
4	Algebra and Equations (and inequalities)	**± 25**	± 15	± 15
5	Differential Calculus	**± 35**	± 30	± 10
6	Probability	**± 15**	± 13	± 5
TOTAL		**150**	**120**	**55 (45 needed)**
PAPER 2 : Bookwork 12 marks (Gr 11 or 12 theorems or Trig Proofs)				
No.	Description	Gr 12	**Distinction**	**Pass**
1	Analytical Geometry	**± 40**	± 35	± 15
2	Euclidean Geometry	**± 50**	± 40	± 18
3	Trigonometry	**± 40**	± 30	± 6
4	Data Handling	**± 20**	± 15	± 12
TOTAL		**150**	**120**	**51 (45 needed)**

THESE MARKS ARE JUST AN ESTIMATION BUT THEY CAN SERVE AS A GUIDE AS TO WHICH AREAS YOU NEED TO FOCUS ON:

KEY CONCEPTS TO FOCUS ON <u>TO PASS</u>

PAPER 1

Algebra and Equations (and inequalities)

This section must really be done well during the revision process.

- Learn solving for x in a quadratic equation (use formula all the time if you trouble with inspection). *Do this daily if you are not getting it right!*
- Simultaneous equations must be learnt over and over.
- Inequalities must be learnt (to the very least...find critical values)

Patterns & Sequences

- Grade 11 quadratic number patterns. Learn the easiest method to find the coefficients in the formula $T_n = an^2 + bn + c$. Learn to find the value of a particular term and also to find the number of a given term.
- Gr 12 arithmetic and geometric sequences formulae and substitution.
- Learn mixed number patterns.
- Learn derivation of sum to arithmetic and geometric formulae proofs.
- Learn sum to infinity formula and basic substitution.

Annuities & Finance

- Learn Gr 11 financial maths and how to find *r* or *n* using logs. Substitution into formula is essential.
- Learn the grade 12 future value and present value formula properly and the substitution into these formula. *Do not learn Difficult Problems now if your goal is just to pass!!!!*

Functions & Graphs

- Graph sketching is essential. Learn use of calculator to find critical points before drawing.
- Learn how to find critical points (intercepts, turning points, asymptotes etc.) from a given graph.

Calculus

- First principles to be learnt thoroughly. Don't forget $\frac{1}{x}$ or x^3 but concentrate on easier examples.
- Rules for differentiation to be learnt to get at least some of the available marks.
- Graph sketching is essential. Learn how to find critical points using calculators and how to sketch...even if points are incorrect!
- Learn how to find critical points (intercepts, turning points, inflection points etc.) from a given graph.
- In applications learn to find the derivative always and equate to zero if an equation is provided. Also to substitute answer to get maximum or minimum.

Probability

- Venn Diagrams and Contingency tables must be done very well.
- Easier counting principle examples must be done.
- Proving Independent events

PAPER 2

Coordinate Geometry

- In Gr 11 concentrate on finding distance, gradient, mid-point, equation of a line and inclination.
- In Gr 11 concentrate on questions regarding perpendicular lines.
- Properties of polygons must be summarized.

- In grade 12 using completing the square to find the coordinates of the centre of the circle must be drilled.
- Finding the equation of the tangent must be covered.

Euclidean Geometry

- Approximately 28 marks of 50 comes from level 1 & level 2 type problems
- MUST be able to prove ALL 6 Examinable proofs.
- Problems involving numerical calculation of angles or finding angles equal to "*x*" must be covered extensively.
- Basic Proving triangles similar is also often not a very challenging problem.

Trigonometry

Do Not frustrate yourself with too much trigonometry during the revision process and waste valuable time!

- Focus on Grade 11 work. Reduction formulae and drawing the angle in quadrants to solve. e.g. $\boldsymbol{sin23^\circ = a}$ type questions...
- learn trig graph sketching using a calculator.
- *DO NOT learn* identities, general solutions or 2D and 3D problems...if your focus now is just passing!!!

Data Handling.

This section must be learnt very well and use as many examples from past papers to drill learners.

- Learn calculations of measures of central tendency using calculators.
- Learn standard deviation calculations.
- Box and whisker diagrams and drawing and interpretation of ogives are essential.
- Scatter plots, Regression & Correlation Coefficients will carry about 10 marks and the questions are fairly limited to past year questions must be studied extensively.

ALGEBRA EQUATIONS AND INEQUALITIES.
This section accounts for ±25 marks in Paper 1.
If understood properly, it also increases marks in other sections, e.g. Functions

- This section must be done really well when preparing for all the next examinations, namely **JUNE, PREPARATORY AND FINAL EXAMINATIONS**
- Learn **solving for *x* in a quadratic equation**, which must be in standard form.
  (use formula all the time if you struggle with inspection). ***Do this daily if you are not getting it right!***
- Basic understanding of **LAWS OF EXPONENTS** is important when manipulating equations with exponents.
- Simultaneous equations must be learnt over and over.
- Inequalities must be learnt (to the very least...find critical values).

<table>
<tr><th colspan="2">ALGEBRA, EQUATIONS AND INEQUALITIES – CONCEPTS INVOLVED</th></tr>
<tr><th colspan="2">TYPES AND GROUPS OF QUESTIONS:</th></tr>
<tr><td>• a × b = 0 standard form
• factorization: any method
• transposing
• formula
• rounding off to 2 decimal places</td><td>• surds: squaring both sides
validity of the roots (checking and verifying)
• inequalities:
algebraically
graphically
table method
• simultaneous equations:
subject of the formula
substitution</td></tr>
</table>

TOPIC	ACTIVITY
Factorisation	Common factor, solve
	Transpose, factorise, solve
	Remove brackets, transpose, factorise, solve
Quadratic formula	Formula, substitution, answers correct to 2 decimal places/surd form
	Remove brackets, transpose/standard form, correct to 2 decimal places/surd form
Surds	Square both sides, solve, validate
	Transpose, square both sides, solve, validate
Simultaneous equations	Subject of the formula, substitution, standard form, factorise, solve, substitution for the other variable

Exponents	Same base, equate exponents, solve
	Laws of exponents, write as a power, factorise, equate exponents, solve
	Split, factorise, simplify
	Same bases, laws of exponents, equate exponents, solve
Inequalities	Critical values, sketch, answer
	Standard form, factorise, critical values, sketch, solve
	Remove brackets, standard form, factorise, critical values, sketch, solve
Nature of the roots	Use $b^2–4ac \geq 0$, solve
	Formula, substitution, $b^2–4ac \geq 0$, solve
	Rational, irrational, real/non-real

SECTION	CONCEPT	EXAMPLE
Algebra, Equations & Inequalities (± 25)	**Factorisation**	1) $x^2 - 6x = 0$
		2) $(x-4)(x+2) = 0$
		3) $x - 3 = \frac{4}{x}$
		4) $3x^2 - 5x - 2 = 0$ $(where\ \ a\ is\ greater\ than\ 1)$
	Quadratic formula	1) $2x^2 + 3x - 1 = 0$ (ans corr to 2 decimal digits)
		2) $2x^2 + 3x - 1 = 0$ (ans in simplest surd form)
	Inequalities	1) $(x-4)(x+2) > 0$
		2) $(x+1)(2-x) < 0$
		3) $3^x(x-5) < 0$
		4) $x^2(x+5) < 0$
		5) $3x^2 - 5x - 2 \geq 0$ (for both a $>$ 0 and a $<$ 0)
	Exponential Equations	1) $2x^{-\frac{5}{3}} = 64$
		2) $2^{x+2} + 2^x = 20$
		3) $2.3^x = 81 - 3^x$
	Surds	1) $\sqrt{x+1} = x - 1$
		2) $2 + \sqrt{2-x} = x$
	Simultaneous Equations	1) $y = x^2 - x - 6$ and $2x - y = 2$
		2) $2x - y + 1 = 0$ and $x^2 - 3x - 4 = y^2$
		3) $3^{x-10} = 3^{3x}$ and $y^2 + x = 20$

EXAMPLES:

Example 1

$$(x-3)(x+5)=9$$
$$x^2+5x-3x-15-9=0$$
$$x^2+2x-24=0$$
$$(x+6)(x-4)=0$$
$$x=-6 \text{ or } x=4$$

Example 2

Solve for x:

$$\sqrt{x-2}+x=4$$
$$\sqrt{x-2}=4-x$$
$$x-2=(4-x)^2$$
$$x-2=16-8x+x^2$$
$$x^2-8x+16-x+2=0$$
$$x^2-9x+18=0$$
$$(x-3)(x-6)=0$$
$$x=3 \quad \text{or} \quad x=6$$

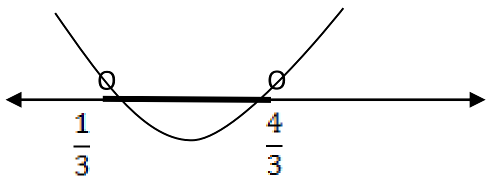
after checking both solutions

$x=3$ is the only solution

Example 3

Solve for x:

$$15x-4>9x^2$$
$$15x-4-9x^2>0$$
$$9x^2-15x+4<0$$
$$(3x-1)(3x-4)<0$$



$$\tfrac{1}{3}<x<\tfrac{4}{3}$$

PRACTICE EXERCISES

<u>QUESTION 1</u>

1.1. Solve for $\boldsymbol{x}$

1.1.1. $\boldsymbol{x(x-4)=5}$ (3)

1.1.2. $4x^2$-20x+1=0 (round off your answer correct to 2 decimal places) (4)

1.1.3. Solve simultaneously for x *and* y in the following system of equations:

$$y - x + 3 = 0$$
$$x^2 - x = 6 + y$$ (6)

QUESTION 2

2.1. Solve for x

2.1.1. $x^2 - 5x = -6$ (3)

2.1.2. $(3x + 1)(x - 4) < 0$ (3)

2.1.3. $2x + \sqrt{x + 1} = 1$ (4)

2.1.4. $12^{5+3x} = 1$ (2)

2.2. Solve for x and y

$$2x - y = 8$$
$$x^2 - xy + y^2 = 19$$ (7)

QUESTION 3

3.1. Solve for x

3.1.1. $(x+2)^2 = 3x(x-2)$ Giving your answer correct to one decimal digit (5)

3.1.2. $x^2 - 9x \geq 36$ (4)

3.1.3. $3^x - 3^{x-2} = 72$ (4)

3.2. Given $(2m - 3)(n + 5) = 0$

Solve for:

3.2.1. n if $m = 1$ (1)

3.2.2. m if $n \neq -5$ (1)

3.2.3. m if $n = -$ (2)

QUESTION 4

4.1. Solve for x

4.1.1. $(x - 3)(x + 1) = 5$ (3)

4.1.2. $9^{2x-1} = \frac{3^x}{3}$ (3)

4.1.3. $2\sqrt{2 - 7x} = \sqrt{-36x}$ (4)

QUESTION 5

5.1. Solve for x:

5.1.1. $10x = 3x^2 - 8$ (3)

5.1.2. $x + \sqrt{x-2} = 4$ (5)

5.1.3. $x(2x - 1) \geq 15$ (5)

5.2. Given: $P = \frac{4^{x+3}+4^x}{8^{x+2}+8^x}$

5.2.1. Simplify P (3)

5.2.2. Hence solve for x: P = 3 (2)

5.3. State whether the following numbers are rational, irrational or non-real.

5.3.1. $\sqrt{3}$ (1)

5.3.2. $\frac{22}{7}$ (1)

5.3.3. The roots of $x^2 + 4 = 0$ (1)

QUESTION 6

6.1 Solve for x:

6.1.1 $2x^2 + 11 = x + 21$ (3)

6.1.2 $3x^3 + x^2 - x = 0$ (5)

6.1.3 $2x + p = p(x + 2)$, stating any restriction (4)

6.1.4 $x^{-1} - x^{-\frac{1}{2}} = 20$

6.2. Solve for x and y simultaneously in the following equations

$2x^2 - 3xy = -4$ and $4^{x+y} = 2^{y+4}$ (6)

QUESTION 7

7.1. Solve for x. Leave the answer in the simplest surd form where necessary

7.1.1. $(2x + 5)(x^2 - 2) = 0$ (3)

7.1.2. $x^2 - 4 \geq 5$ (4)

7.1.3. $12^{2x} = 8.36^x$ (4)

7.2. Solve for x, correct to two decimal places:

$2(x+1)^2 = 9$ (4)

7.3. Solve for x and y simultaneously:

$y = -2x + 7$ and $\frac{y+5}{x-1} = \frac{1}{2}$ (4)

QUESTION 8

8.1. Given $x^2 + 2x = 0$

8.1.1. Solve for x (2)

8.1.2. Hence, determine the positive values of x for which $x^2 \geq -2x$ (3)

8.2. Solve for x:

$2x^2 - 3x - 7 = 0$ (correct to two decimal places) (4)

8.3. Given $k+5 = \frac{14}{k}$

8.3.1. Solve for k (3)

8.3.2. Hence, or otherwise, solve for x if $\sqrt{x+5} + 5 = \frac{14}{\sqrt{x+5}}$ (3)

8.4. Solve for x and y simultaneously if:

$x - 2y - 3 = 0$ *and*

$4x^2 - 5xy + y^2 = 0$ (7)

8.5. The roots of a quadratic equation is given by $x = \frac{-2 \pm \sqrt{4-20k}}{2}$

Determine the values of k for which the equation will have real roots (2)

QUESTION 9

9.1 Solve for x

9.1.1 $2x^2 - 5x - 3 = 0$ (2)

9.1.2 $(x-3)(x-4) \geq 12$ (5)

9.2 Consider: $5x - \frac{3}{x} = 1$

9.2.1 Solve for x correct to two decimal places. (5)

1.2.2 Hence, determine the value of y if $5(2y+1) - \frac{3}{2y+1} = 1$. (3)

9.3 Solve simultaneously for x and y in the following set of equations:

$y = x - 1$ and $y + 7 = x^2 + 2x$ (5)

9.4 Calculate the value(s) of m if the roots of $3mx^2 - 7x + 3 = 0$ are equal. (4)

[24]

QUESTION 10

10.1 Solve for x in each of the following:

10.1.1 $x(2x+5) = 0$ (2)

10.1.2 $2x^2 - 3x = 7$ (Give answer correct to TWO decimal places) (4)

10.1.3 $x - 7 - \sqrt{x-5} = 0$ (4)

10.1.4 $\frac{1}{2}x(3x+1) < 0$ (2)

10.2 Solve for x and y simultaneously:

$$2x + y = 3 \quad and$$
$$x^2 + y + x = y^2$$
(6)

[22]

<u>QUESTION 11</u>

11.1 Solve for x:

11.1.1 $4x^2 = 81$ (2)

11.1.2 (a) $x^2 - 5x = 2$, correct to TWO decimal places. (4)

(b) Hence, or otherwise, solve $(x^2-2)^2 - 5(x^2-2) - 2 = 0$ (3)

11.1.3 $(2-x)(x+4) \geq 0$ (3)

11.1.4 $3^{x+1} - 4 + \frac{1}{3^x} = 0$ (5)

11.2 Solve for x and y simultaneously:

$x + y = 3$

$2x^2 + 2y^2 = 5xy$ (6)

<u>QUESTION 12</u>

12.1 Solve for x:

12.1.1 $3x^2 + 10x + 6 = 0$ (correct to TWO decimal places) (3)

12.1.2 $\sqrt{6x^2 - 15} = x + 1$ (5)

12.1.3 $x^2 + 2x - 24 \geq 0$ (3)

12.2 Solve simultaneously for x and y:

$5x + y = 3$ and $3x^2 - 2xy = y^2 - 105$ (6)

12.3 12.3.1 Solve for p if $p^2 - 48p - 49 = 0$ (3)

12.3.2 Hence, or otherwise, solve for x if $7^{2x} - 48(7^x) - 49 = 0$ (3)

<u>QUESTION 13</u>

13.1 Solve for x:

13.1.1 $x^2 + 9x + 14 = 0$ (3)

13.1.2 $4x^2 + 9x - 3 = 0$ (correct to TWO decimal places) (4)

13.1.3 $\sqrt{x^2 - 5} = 2\sqrt{x}$ (4)

13.2 Solve for x and y if:

$3x - y = 4$ and $x^2 + 2xy - y^2 = -2$ (6)

13.3 Given: $f(x) = x^2 + 8x + 16$

13.3.1 Solve for x if $f(x) > 0$. (3)

PATTERNS, SEQUENCE AND SERIES

There are three types of sequences, namely, Arithmetic sequence (AS), Geometric Sequence (GS) and Quadratics Sequence.

A single number in a pattern or sequence is called a **term**.

Term 1 is written as T_1, term 2 is written as T_2 and so on. The number of the term shows its position in the sequence.

T_{10} is the 10^{th} term in the sequence.

T_n is the n^{th} term in a sequence.

Quadratic Sequences

Examples

1. Consider the sequence: 5 ; 18 ; 37 ; 62 ; 93 ; …
   1.1 If the sequence behaves consistently, determine the next TWO terms of the sequence.
   1.2 Calculate a formula for the nth term of the sequence.
   1.3 Use your formula to calculate n if the n^{th} term in the sequence is 1 278.

Worked Solution

1.1 130 ; 173

1.2

$$\begin{array}{ccccccccccl} 5 & & 18 & & 37 & & 62 & & 93 & & \\ & 13 & & 19 & & 25 & & 31 & & & \text{sequence of first difference} \\ & & 6 & & 6 & & 6 & & & & \text{second difference is constant} \end{array}$$

The second difference is constant $\therefore\ T_n$ is quadratic $\therefore\ an^2 + bn + c = T_n$

$2a = 6$

$a = 3$ $\therefore\ T_n = 3n^2 + bn + c$

$$5 = 3(1)^2 + b(1) + c$$

$$b + c = 2 \qquad \text{... (1)}$$

$$18 = 3(2)^2 + b(2) + c$$

$$2b + c = 6 \qquad \text{... (2)}$$

(2) – (1): $b = 4$

$$c = -2$$

$$\therefore\ T_n = 3n^2 + 4n - 2$$

1.3 $3n^2 + 4n - 2 = 1278$

$3n^2 + 4n - 1280 = 0$

$(3n + 64)(n - 20) = 0$

$n = \frac{-64}{3}$ or $n = 20$

$n = \frac{-64}{3}$ is not valid $\therefore$ $n = 20$

Exercise 4

1. Given the quadratic sequence: -1 ; -7 ; -11 ; p ; …

   1.1 Write down the value of p.

   1.2 Determine the n^{th} term of the sequence.

   1.3 The first difference between two consecutive terms of the sequence is 96.

   Calculate the values of these two terms.

2. Given the following quadratic sequence: -2 ; 0 ; 3 ; 7 ; ...

   2.1 Write down the value of the next term of this sequence.

   2.2 Determine an expression for the n^{th} term of this sequence.

   2.3 Which term of the sequence will be equal to 322?

3. Look at the following sequence and answer the questions that follow:

   10 ; 21 ; 38 ; 61 ; ....

   3.1 Determine the type of sequence.

   3.2 Determine the general term.

   3.3 Which term has a value of 1 245?

ANSWERS TO EXERCISES

Exercise 1

1.1 $-1 + 2 + 5 + ...\ or -1; 2; 5; ...$

1.2 14750

2.1 $n = 42$

2.2 3818

2.3 $\sum_{n=1}^{46} (4n - 11)$

Exercise 2

1.1 $r = 1\frac{1}{4}$

1.2 R22 517,58

2.1 $r = \frac{T_4}{T_3} = \frac{40}{20} = 2$

2.2 $T_n = 5.2^{n-1}$

3.1 $w=7$

3.2 $d=w-1$

$d=7-1=6$

4. 7255

3.1 $t=\frac{1}{3}$

3.2 $1-t=1-\frac{1}{3}=\frac{2}{3}$

3.3 $1\frac{1}{2}$

4.1 $r=1$ (invalid) or $r=2$

4.2 $n=8$

Exercise 3

1.1 $\frac{1}{16}$; 13

1.2

$S_{50}=S_{25}+S_{25}=0{,}9997+1000=1000{,}9997$

2.1 $T_{191}=0$

2.2 $S_{500}=31000$

Exercise 4

1.1 $p=13$

1.2 $T_n=n^2-9n+7$

1.3 $n=52$ $T_{52}=2243$

2.1 The next term of the sequence is 12

2.2 $T_n=\frac{1}{2}n^2+\frac{1}{2}n-3$ $S_{500}=31000$

2.3 The 25th term has a value of 322.

Examples

1.1 Find the 32nd term of sequence 3; 7; 11; ….

1.2 Find the sum of the first 32 terms of the sequence.

Solution

1.1 We may list all 32 terms, but that will take a long time and a lot of space. The simplest or a quickest way is to use the general term, $T_n = a + (n-1)d$

$$T_n = a + (n-1)d$$

$$T_{32} = 3 + (32-1)4$$

$$= 3 + 31(4)$$

$$= 3 + 124 = 127$$

1.2 Similarly, to find the sum we can add all terms together or we can use some formula:

$$S_n = \frac{n}{2}[2a+(n-1)d] \quad or$$
$$S_{32} = \frac{32}{2}[2\times 3+(32-1)4]$$
$$= 16[6+(31)4]$$
$$= 16(130)$$
$$= 2210$$

$$S_n = \frac{n}{2}(a+T_n)$$
$$S_{32} = \frac{32}{2}(3+T_{32})$$
$$= 16(3+127)$$
$$= 16(130)$$
$$= 2210$$

2.1 Consider the sequence: 30 ; 22 ; 14 ; . . .

Find the sum of the first 19 terms of the sequence.

2.2 The 5th term of an arithmetic sequence is 17 and a common difference of 6, determine the first term of the sequence.

Solution

2.1 To find S_{19} without first calculating T_{19} we use the formula:

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{19} = \frac{19}{2}[2\times 30+(19-1)(-18)]$$

$$= \frac{19}{2}[60+(18)(-18)]$$

$$= \frac{19}{2}[60-144]$$

$$= \frac{19}{2}(-188)$$

$$= -798$$

2.2 To find T_1 given T_5, we may subtract d = 6 four times from 17 because $T_5 = a + 4d$, that is,

$$a = 17-6-6-6-6 = -7$$

or $T_5 = a+4d$

$$17 = a+4\times 6$$

$$17-24 = a$$

$$-7 = a$$

Exercises 1

1. Given $\sum_{t=0}^{99}(3t-1)$

   1.1 Write down the first THREE terms of the series

   1.2 Calculate the sum of the series.

2 Given the arithmetic series: $-7-3+1+...+173$

   2.1 How many terms are there in the series?

   2.2 Calculate the sum of the series.

   2.3 Write the series in sigma notation.

3 Given the arithmetic sequence: $w-3; 2w-4; 23-w$

   3.1 Determine the value of w

   3.2 Write down the common difference of this sequence.

4 The arithmetic sequence 4 ; 10 ; 16 ; ... is the sequence of first differences of a quadratic sequence with a first term equal to 3. Determine the 50^{th} term of the quadratic sequence.

2 Geometric Sequences and Series

A geometric progression is one in which there is a ***constant ratio*** between any two consecutive geometric terms. For example:

1; 10; 100; 1000; 10000;….

3; 9; 27; 81; 243;….

The distinct feature of these sequences is that each term, after the first, is obtained by multiplying the previous term by a constant, *r*. In the examples above, '*r*' is 10 and 3 respectively. Again, in the discussion that follows '*a*' is used to represent the first term.

For example, in the second sequence above, the first term, T_1, of the sequence is $a = 3$ and the second term, T_2, of the sequence is $ar = 3 \times 3$. The third term, T_3, is $ar^2 = 3 \times 3^2$.

In the above sequences if we replace the ";" by "+" the sequence becomes a series. For example,

1 + 10 + 100 + 1000 + 10000 + ….

3 + 9 + 27 + 81 + 243 + ….

There are two important formulae that can be used to solve most arithmetic sequence and series problems:

The general or *n*-th term: $T_n = ar^{n-1}$

The sum of the first *n* terms: $S_n = \dfrac{a(r^n - 1)}{r-1}$ or $S_n = \dfrac{a(1-r^n)}{1-r}$

Examples

1. Consider the sequence: 3 ; 12 ; 48 ; 192 ; 768; ...

   1.1 Find the 12th term of the sequence

   1.2 Find the sum of the first eight terms

2. Find the sum of the first eleven terms of the following sequence: 40 ; 8 ; 1,6 ; 0,32 ; 0,064;..

3. A geometric sequence has all its terms positive. The first term is 7 and the third term is 28.

   3.1 Find the common ratio.

   3.2 Find the sum of the first 14 terms.

Solutions

1. The sequence can be written as: 3 ; $3 \times 4 = 12$; $12 \times 4 = 48$; $48 \times 4 = 192$; $192 \times 4 = 768$

1.1 It is a geometric sequence with $a = 3$ and $r = 4$; $\therefore$ $T_n = ar^{n-1}$

$$T_{12} = 3.\ 4^{12-1}$$

$$= 3.\ 4^{11}$$

$$= 3 \times 4\ 194\ 304 = 12\ 582\ 912$$

1.2 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_n = \frac{3(4^8 - 1)}{4 - 1}$$

$$= 4^8 - 1$$

$$= 65\ 535$$

2. $r = \dfrac{8}{40} = \dfrac{1{,}6}{8} = \dfrac{1}{5} = 0{,}2$ and $a = 40$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{11} = \frac{40(1 - 0{,}2^{11})}{1 - 0{,}2}$$

$$= 50(1 - 0{,}2^{11})$$

$$= 50(0{,}9\ 999\ 999\ 795)$$

$$= 50{,}000$$

3. $a = 7$; $T_2 = ar^2 = 28$

3.1 $7r^2 = 28$

$r^2 = 4$

$\therefore \ r = 2$

3.2 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_{14} = \frac{7(2^{14} - 1)}{2 - 1}$$

$= 7(2^{14} - 1)$

$= 7 \times 16\ 383$

$= 114\ 681$

Exercise 2

1. The tuition fees for the first three years of school are R2 000 ; R2 500 ; R3 125. If these tuition fees form a geometric sequence, find:

1.1 Find the common ratio, r, for this sequence

1.2 If fees continue to rise at the same rate, calculate (to the nearest rand) the total cost of tuition fees for the first six years of school.

2. A geometric sequence has $T_3 = 20$ and $T_4 = 40$

Determine:

2.1 The common ratio

2.2 A formula for T_n

3. If $1-5t$, $1-t$ and $t+1$ are the first three terms of a convergent geometric series, calculate:

The value of t.

The common ratio.

The sum to infinity of the series.

4. The first term of a geometric sequence is 3 and the sum of the first 4 terms is 5 times the sum of the first 2 terms. The common ratio is greater than 1.

Calculate:

4.1 The first three terms of the sequence, and

4.2 The value of n for which the sum to n terms will be 765

3 Combined Sequences and Series

Examples

Consider the following sequence of numbers:

2 ; 5 ; 2 ; 9 ; 2 ; 13 ; 2 ; 17 ; …

1. 1.1 Write down the next TWO terms of the sequence, given that the pattern continues.

1.2 Calculate the sum of the first 100 terms of the sequence.

Worked Solution

2.1 2 ; 21

2.2
$$\underbrace{(2+2+...+2)}_{for\ 50\ terms}+\underbrace{(5+9+13+...)}_{for\ 50\ terms}$$
$$=\sum_{i=1}^{50}2+\sum_{i=1}^{50}(4i+1)$$
$$=2(50)+\left[\frac{50}{2}(2(5)+49(4))\right]$$
$$=100+25(10+196)$$
$$=100+5150$$
$$=5250$$

Exercise 3

1. Consider the sequence: $\frac{1}{2};4;\frac{1}{4};7;\frac{1}{8};10;\ ...$

1.1 If the pattern continues in the same way, write down the next TWO terms in the sequence.

1.2 Calculate the sum of the first 50 terms of the sequence.

2. Given: $0;-\frac{1}{2};0;\frac{1}{2};0;\frac{3}{2};0;\frac{5}{2};0;\frac{7}{2};0;....$

Assume that this number pattern continues consistently.

2.1 Write down the value of the 191st term of this sequence.

2.2 Determine the sum of the first 500 terms of this sequence.

EXAMINATION QUESTIONS FROM PAST PAPERS

LIMPOPO SEPT 2013

QUESTION 1

1.1 In the sequence 3; p; q; 24 are the first four terms.

Determine the values of p and q if

1.1.1 The sequence is geometric. (3)L2

1.1.2 The sequence is arithmetic. (3)L2

1.2 Given that $S_n = 33n + 3n^2$

1.2.1 Determine the sum of 10 terms (1)L1

1.2.2 List the first three terms of this series (4)L2

1.3 Determine n which is the number of terms for which the series:

$5 + 4 + \frac{16}{5} + \ldots$ (to n terms) $= \frac{11529}{625}$ (5)L3

1.4 Given $\mathbf{K} = \frac{1}{3}x + (\frac{1}{3}x)^2 + (\frac{1}{3}x)^3 + \ldots\ldots$ which converges.

1.4.1 Determine the value of x (2)L2

1.4.2 Determine K if $x = -2$ (3)L2

QUESTION 2

2.1 Given the sequence 3; 6; 13; 24; ….

2.1.1 Derive the general term of this sequence. (4)L2

2.1.2 Which term of this sequence is the first to be greater than 500. (5)L3

MPUMALANGA SEPT 2013

QUESTION 3

Given: 1; 11; 26; 46; 71; …….

3.1 Determine the formula for the general term of the sequence. (4)L2

3.2 Which term in the sequence has a value of 521? (4)L2

QUESTION 4

4.1 The sum of the third and ninth terms of an arithmetic sequence is 20. The difference between the twelfth and fourth terms of the arithmetic sequence is 32.
Determine the value of the first term, a and the difference, d. (5)L3

4.2 Calculate the value of k if $\sum_{p=1}^{k}(4-\frac{p}{2}) = -4$ (6)L3

4.3 Consider the geometric sequence:

$$2(2t-1)\ ;\ (2t-1)^2\ ;\ \tfrac{1}{2}(2t-1)^3\ ;\ \ldots\quad (t \neq \tfrac{1}{2})$$

Calculate:

4.3.1 the common ratio r. (2)L1

4.3.2 the value(s) of t for which the sequence converges. (3)L3

4.3.3 the sum to infinity of the sequence, if $t = \frac{1}{4}$. (4)L2

NORTH WEST TRIAL 2014

QUESTION 5

5.1 Evaluate:

$$\sum_{n=3}^{20}(15-4n)$$

(4) L2

5.2 A water tank contains 216 litres of water at the end of day 1. Because of a leak, the tank loses one-sixth of the previous day's contents each day.
How many litres of water will be in the tank by the end of:

5.2.1 the 2nd day? (2)L1

5.2.2 the 7th day? (3)L2

5.3 Consider the geometric series: $2(3x-1)+2(3x-1)^2+2(3x-1)^3+\ldots$

5.3.1 For which values of x is the series convergent? (3)L1

5.3.2 Calculate the sum to infinity of the series if $x=\frac{1}{2}$. (4)L2

5.4 2 ; x ; 12 ; y ; . . . are the first four terms of a quadratic sequence.
If the second difference is 6, calculate the values of x and y. (5)L3

WC METRO NORTH DISTRICT TRIAL 2014

QUESTION 6

6.1 The following arithmetic sequence is given: 20 ; 23 ; 26 ; 29; … ; 101

6.1.1 How many terms are there in this sequence? (2)L1

6.1.2 The even numbers are removed from the sequence.

Calculate the sum of the terms of the remaining sequence. (6)L2

6.2 Study the geometric series: $x+\frac{x^2}{3}+\frac{x^3}{9}+\cdots$....

6.2.1 Determine the n-th term in terms of x. (2)L2

6.2.2 Determine the value(s) of x for which the series will converge. (3)L2

6.3 The sum of n terms in a sequence is given by $S_n = -n^2 + 5$.
Determine the 23rd term. (3)L2

QUESTION 7

The sequence 3 ; 9 ; 17 ; 27 ; … is quadratic.

7.1 Determine an expression for the n-th term of the sequence. (4)L2

7.2 What is the value of the first term of the sequence that is greater than 269? (4)L3

KZN TRIAL 2015

QUESTION 8

8.1 Given the combined arithmetic and constant sequences :

3 ; 2 ; 6 ; 2 ; 9 ; 2 ; ...

8.1.1 Write down the next **two** terms in the sequence. (2)L1

8.1.2 Calculate the sum of the first 100 terms of the sequence. (5)L2

8.2 Prove that: $a + ar + ar^2 + ...(to\ n\ terms) = \frac{a(r^n - 1)}{r - 1}; \quad r \neq 1$ (4)L1

QUESTION 9

Given the geometric series: $\frac{24}{x} + 12 + 6x + 3x^2 + ...$

9.1 If $x = 4$, then determine the sum to 15 terms of the sequence. (4)L2

9.2 Determine the values of x for which the original series converges. (3)L2

9.3 Determine the values of x for which the original series will be increasing. (2)L4

QUESTION 10

Given the quadratic sequence: 5 ; 7 ; 13 ; 23 ; . . .

10.1 Calculate the nth term of the quadratic sequence. (4) L2

10.2 Determine between which two consecutive terms of the quadratic sequence the first difference will be equal to 2018. (3)L3

MARITZBURG COLLEGE TRIAL 2015

QUESTION 11

11.1 Given the arithmetic series: $1 + 4 + 7 + ...$

11.1.1 Determine the 65th term of the series. (1)L1

11.1.2 Derive a formula for T_n, the n^{th} term of this series. (2)L1

11.1.3 Calculate k if $1 + 4 + 7 + ...\ (\text{to } k \text{ terms}) = 590$. (4)L3

11. 2 Find the value of $\sum_{p=1}^{\infty} 16\left(\frac{3}{5}\right)^p$. (4)L2

QUESTION 12

12.1 With reference to the sequence, $2;4;8;\alpha$ give the value of α if:

12.1.1 the sequence is geometric. (1)L1

12.1.2 the sequence is quadratic. (2)L2

12.2 Given the quadratic sequence $6\ ;\ 3\ ;-2;-9;...$

12.2.1 Determine the n^{th} term of the sequence. (5)L2

12.2.2 The sum of two consecutive terms of this sequence is -827. Find these terms. (4)L3

12.3 Calculate $\sum_{n=1}^{10}\frac{18}{2^n}$. (3)L2

12.4 For what values of x will the series $2(1-x)+4(1-x)^2+8(1-x)^3+\ ...$ be convergent?(3)L2

FREE STATE TRIAL 2016

QUESTION 13

13.1 Calculate $\sum_{r=4}^{13}3$ (2)L2

13.2 Given the arithmetic sequence: $3;\quad b;\quad 19;\quad 27;\quad ...$

Calculate the value of b. (1)L1

13.2.1 Determine the n^{th} term of the sequence. (2)L1

13.2.2 Calculate the value of the thirtieth term(T_{30}) (2)L2

13.2.3 Calculate the sum of the first 30 terms of the sequence. (2)L2

13.3 The above sequence $3;\quad b;\quad 19;\quad 27;\quad ...$ forms the first differences of a quadratic sequence. The first term of the quadratic sequence is 1.

13.3.1 Determine the fourth term(T_4) of the quadratic sequence. (2)L3

13.3.2 Determine the n^{th} term of the quadratic sequence. (4)L2

13.3.3 Calculate the value of n if $T_n-1=7700$ (3)L2

KZN TRIAL 2016

QUESTION 14

Given the quadratic sequence: 4; 4; 8; 16; . . .

14.1 Calculate the n^{th} term of the quadratic sequence. (4)L1

14.2 Between which two consecutive terms of the quadratic sequence, will the first difference be equal to 28088? (4)L3

QUESTION 15

15.1 Given the combined arithmetic and constant sequences:

$$6\ ;\ 2\ ;\ 10\ ;\ 2\ ;\ 14\ ;\ 2\ ;\ ...$$

15.1.1 Write down the next **TWO** terms in the sequence. (2)L1

15.1.2 Write down the sum of the first 50 terms of the constant sequence. (1)L1

15.1.3 Calculate the sum of the first 100 terms of the sequence. (4)L2

15.2 Prove that: $a + ar + ar^2 + ...(to\ n\ terms) = \dfrac{a(r^n - 1)}{r-1};\quad r \neq 1$ (4)L1

QUESTION 16

16.1 Evaluate $\sum_{p=1}^{\infty}\left(\frac{2}{3}\right)^{p-1}$ (3)L2

16.2 In a series, $S_n = 3n^2 - n$, calculate the value of the fourth term in the series. (4)L2

LIMPOPO TRIAL 2016

QUESTION 17

The 7th term of a geometric series is $\frac{1}{128}$ and the 11th term is $\frac{1}{2048}$. If $r < 0$,

17.1 Determine the first term of the sequence. (4)L3

17.2 Will this sequence converge? Explain. (2)L2

17.3 A new series is formed by taking $T_1 + T_3 + T_5 + \cdots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + ...$

from the above sequence. Calculate the sum to infinity of this new series. (4)L1

QUESTION 18

A quadratic sequence is defined with the general term:

$$T_n = \sum_{k=0}^{n}(6k-2).$$

18.1 Show that $T_3 = 28$ (3)L1

18.2 The sequence defined has the first 4 terms:

2 ; 12 ; 28 ; 50

Show that $T_n = 3n^2 + n - 2$. (4)L1

18.3 Calculate T_{40} of this sequence. (2)L2

18.4 Calculate the first difference between T_n and T_{n-1} where $T_n = 442$. (5)L3

WESTERN CAPE TRIAL 2016

QUESTION 19

19.1 Given:

$$\frac{3x-1}{4};\frac{2x-1}{3};\frac{7x-5}{12}$$

19.1.1 If $x = 5$, determine the values of the first three terms. (1)L1

19.1.2 What type of sequence is this? Give a reason for your answer. (2)L1

19.1.3 Which term will be equal to $-44,5$? (3)L2

19.2 Given the series:

$18 + 6 + 2 + \cdots$

19.2.1 What is the value of the first negative term, if any? Explain your answer. (2)L1

19.2.2 Determine the tenth term, T_{10}. (2)L2

19.2.3 Determine $S_\infty - S_{10}$. (5)L3

QUESTION 20

20.1 Determine the value of:

$$\sum_{k=2}^{33}(1-2k)$$

(3)L2

20.2 $6; 5 + x; -6$ and $6x$ form the first 4 terms of a quadratic sequence.

20.2.1 Show that $x = -3$. (4)L2

20.2.2 Determine an expression for the general term of the sequence. (4)L2

KZN TRIAL 2017

QUESTION 21

21.1 Given below is the combination series of an arithmetic and a constant pattern:

$2+3+5+3+8+3+\ldots$

21.1.1 If the pattern continues, write down the next two terms. (2)L1

21.1.2 Determine the 85th term of the given series. (3)L2

21.1.3 Calculate the sum of the first 85 terms of the series. (3)L3

21.2 Given the series $(x-2)+(x^2-4)+(x^3+2x^2-4x-8)+\ldots$ $(x \neq \pm 2)$.

21.2.1 Determine the values of x for which the series converges. (4)L2

21.2.2 Explain why the series will never converge to zero. (3)L3

QUESTION 22

Given the quadratic sequence: 3; 5; 11; 21; x

22.1 Write down the value of x. (1)L1

22.2 Determine the value of the 48th term. (5)L2

22.3 Prove that the terms of this sequence will never consist of even numbers. (2)L3

22.4 If all the terms of this sequence are increased by 100, write down the general term of the new sequence. (2)L2

WC WINELANDS DISTRICT TRIAL 2017

QUESTION 23

23.1 Which term in this sequence 36; 25; 14; . . . is equal to -52? (3)L2

23.2 Determine:

$$\sum_{k=1}^{8}\frac{1}{4}(2)^{k-1}$$

(3)L2

23.3 In a quadratic pattern, with $T_n = an^2 + bn + c$, the second term is equal to 8 and the first differences of the quadratic sequence are given as: $6; 12; 18; \ldots\ldots\ldots\ldots$

23.3.1 Write down the values of the first four terms of the quadratic sequence. (3)L2

23.3.2 Calculate the value of T_{40} of the quadratic sequence. (5)L3

EDEN & CENTRAL KAROO DISTRICT TRIAL 2018

QUESTION 24

24.1 Prove that in any arithmetic series of which the first term is a and where the constant difference is *d*, the sum of the first *n* terms is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$ (4)L1

24.2 Given the following sequence: $-5\ ;\ -1\ ; 3\ ; 7\ ; \ldots\ldots\ldots\ldots\ldots\ldots ; 35$

24.2.1 Determine the number of terms in the sequence. (3)L2

24.2.2 Calculate the sum of the sequence. (2)L2

24.3 For an arithmetic series consisting of 15 terms, $S_n = 2n - n^2$
Determine:

24.3.1 the first term of the sequence. (2)L2

24.3.2 the sum of the last 3 terms. (3)L2

QUESTION 25

25.1 A quadratic number pattern $T_n = an^2 + bn + c$ has a third term equal to -1, while the first differences of the quadratic sequence are given by: $-12; -8; -4$

25.1.1 Write down the values of the first four terms of the quadratic sequence. (2)L2

25.1.2 Calculate the value of a, b and c. (3)L2

25.2 Consider the geometric series $4 + p + \frac{p^2}{4} + \frac{p^3}{16} + \cdots$

25.2.1 Calculate the value(s) of p for which the series converges. (2)L1

25.2.2 Calculate the value of p if the sum to infinity is 3. (3)L2

KZN TRIAL 2018

QUESTION 26

The first four terms of a quadratic sequence are $9\,;19\;;\;33\;;\;51\;;...$

26.1 Write down the next TWO terms of the quadratic sequence. (2)L1

26.2 Determine the n^{th} term of the sequence. (4)L1

26.3 Prove that all the terms of the quadratic sequence are odd. (3)L3

QUESTION 27

$3-t\;;-t\;;\sqrt{9-2t}$ are the first three terms of an arithmetic sequence.

27.1 Determine the value of *t*. (4)L2

27.2 If $t=-8$, then determine the number of terms in the sequence that will be positive. (3)L2

QUESTION 28

28.1 Given the infinite geometric series $(x-3)+(x-3)^2+(x-3)^3+...$

28.1.1 Write down the value of the common ratio in terms of *x*. (1)L1

28.1.2 For which value(s) of *x* will the series converge? (3)L2

28.2 An arithmetic sequence and a geometric sequence have their first term as 3. The common difference of the arithmetic sequence is *p* and the common ratio of the geometric sequence is *p*. If the tenth term of the arithmetic sequence is equal to the sum to infinity of the geometric sequence, determine the value of *p*. (5)L3

FREE STATE TRIAL 2019

QUESTION 29

29 Given the quadratic sequence 1; 6; 15; 28; …

29.1 Write down the second difference. (1)L1

29.2 Determine the *n*th term. (4)L1

29.3 Calculate which term of the sequence equals 2701. (3)L2

QUESTION 30

30. Given the arithmetic series: 10 + 15 + 20 + 25 + … + 185

30.1 How many terms are there in the series? (3)L2

30.2 Calculate the sum of all the natural numbers from 10 to 185 that are NOT divisible by 5. (6)L3

QUESTION 31

The first four terms of a quadratic sequence are $8 ; 15 ; 24 ; 35 ;...$

31.1 Write down the next TWO terms of the quadratic sequence. (1)L1

31.2 Determine the n^{th} term of the sequence. (4)L2

QUESTION 32

The first three terms of an arithmetic sequence are $2p-3 \; ; \; p+5 \; ; \; 2p+7$.

32.1 Determine the value(s) of p. (3)L2

32.2 Calculate the sum of the first 120 terms. (3)L2

32.3 The following pattern is true for the arithmetic sequence above:

$$T_1 + T_4 = T_2 + T_3$$
$$T_5 + T_8 = T_6 + T_7$$
$$T_9 + T_{12} = T_{10} + T_{11}$$
$$\therefore T_k + T_{k+3} = T_x + T_y$$

32.3.1 Write down the values of x and y in terms of k. (2)L2

32.3.2 Hence, calculate the value of $T_x + T_y$ in terms of k in simplest form. (4)L3

QUESTION 33

33.1 Given: $\sum_{k=1}^{\infty} 5(3^{2-k})$

33.1.1 Write down the value of the first TWO terms of the infinite geometric series. (2)L2

33.1.2 Calculate the sum to infinity of the series. (2)L2

33.2 Consider the following geometric sequence:

$$\sin 30^\circ ; \cos 30^\circ ; \frac{3}{2} ; \ldots ; \frac{81\sqrt{3}}{2}$$

Determine the number of terms in the sequence. (5)L3

FUNCTIONS, INVERSES AND LOGARITHMS

June/Trials: 35 ± 3 marks

LINEAR FUNCTION

$$y = x$$

$$y = ax \qquad\qquad y = ax + p$$

- Intercepts with the axis
- Range and domain

QUADRATIC FUNCTION

$$y = x^2$$

$$y = ax^2 \qquad y = ax^2 + c \qquad y = a(x+p)^2 \qquad y = a(x+p)^2 + q$$

- Intercepts with the axes
- Axis of symmetry
- Turning point
- Range and domain

HYPERBOLIC FUNCTION

$$y = \frac{k}{x}$$

$$y = \frac{k}{x+p} \qquad y = \frac{k}{x} + q \qquad y = \frac{k}{x+p} + q$$

- Two asymptotes
- Axis of symmetry: ***y=x* or *y=-x***
- Intercepts with axes
- Range and domain

EXPONENTIAL FUNCTION

$y = a^x \longrightarrow y = a^x + q \quad where\ a > 0\ and\ a \neq 1 and\ x > 0$

- Intercepts with the axes
- Only One asymptote
- Range and domain

  - UNDERSTAND THE INVERSE OF A FUNCTION
    $y = x + q;\ y = ax^2\ and\ y = a^x\ \ (swap\ x\ and\ y)$
  - THE AVERAGE RATE OF CHANGE.

IMPORTANT DEFINITIONS

A **relation** is just a set of ordered pairs. There is absolutely nothing special at all about the data that are in a relation. In other words, any bunch of data is a relation so long as these numbers comes in pairs.

A set of all the starting point is called "**the domain**" and a set of all ending points is called a range. In other words the domain is what you start with, and the range is what you end with (i.e. x values and y-values respectively).

A **function** is a well behaved relation. For a relation to be a function, there must be one and ONLY one y in the range that corresponds to a given x from the domain.

So we can define a function as a set of ordered pairs in which each element of the domain has one and only one element associated with it in the range.

TYPES OF MAPPINGS

A. Functions

 One to one mapping- x and y values are not repeated.

 Many to one mapping- x values are not repeated but y values are repeated.

B. Non-Functions

 One to many mapping- x values repeat y values do not.

 Many to many mapping- x and y values repeat.

INCREASING AND DECREASING FUNCTIONS

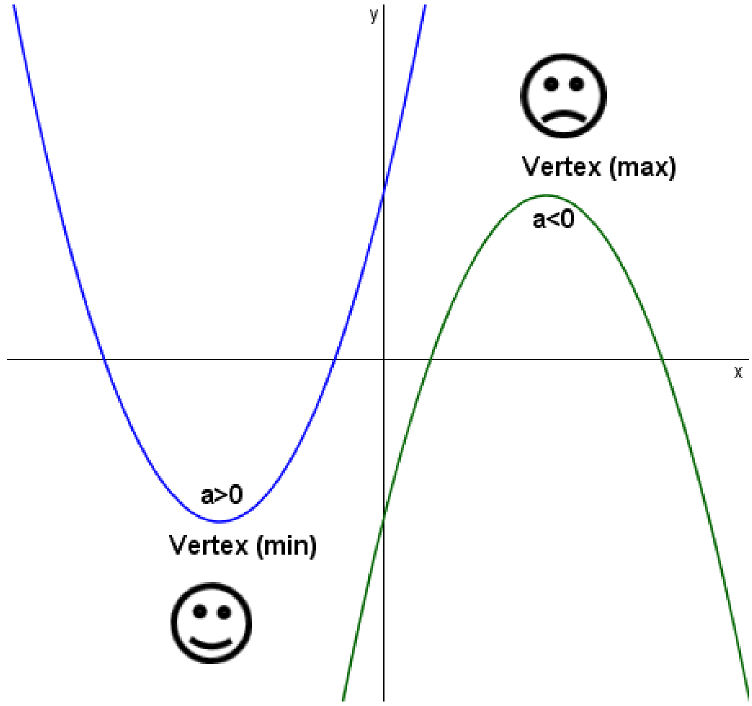
 Increasing- as x increases the y values increase.(positive gradient)

 Decreasing- as x increases the y values decrease. (negative gradient)

TEST FOR A FUNCTION:

Draw a line parallel to y-axis. If the line intersects the graph once only, the graph is a function.

Examples of functions:

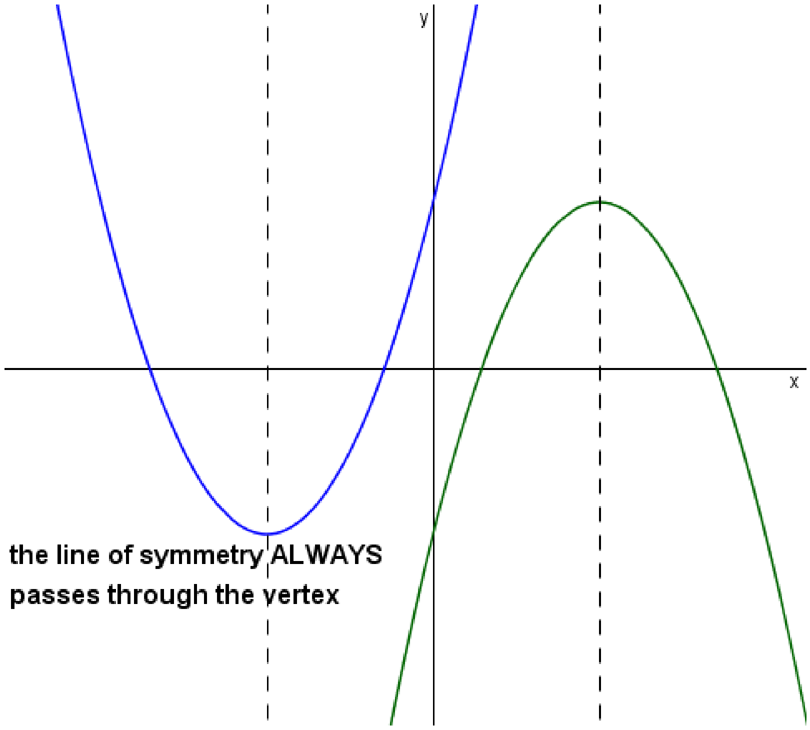


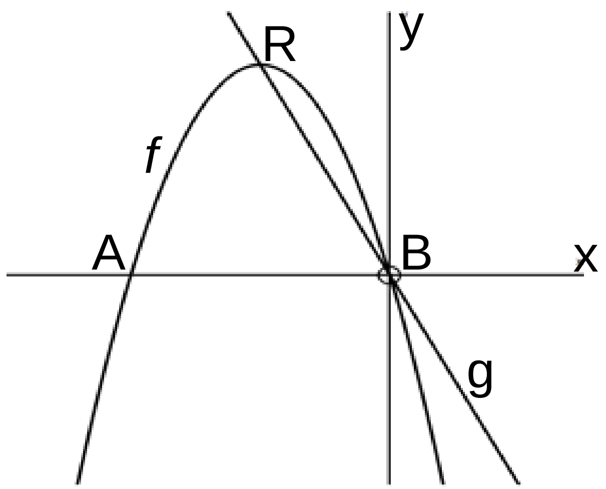
QUADRATIC FUNCTIONS

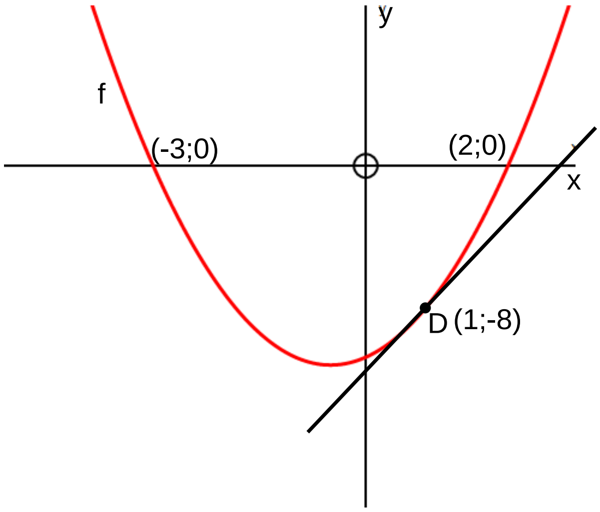
1. The graph of a quadratic function is a **parabola**.
2. A parabola can open up or down.
3. If the parabola opens up, the lowest point is called the vertex.
4. If the parabola opens down, the vertex is the highest point
5. NOTE: if the parabola opened left or right it would not be a function!
6. The standard form of a quadratic function is $\boldsymbol{f(x) = ax2 + bx + c}$
7. The parabola will open up when the value of a is positive.
8. The parabola will open down when the value of a is negative.

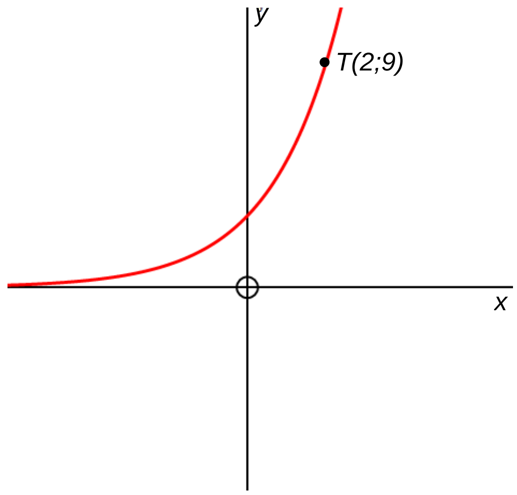
Line of symmetry

1. Parabolas have a symmetric property to them.
2. If we drew a line down the middle of the parabola, we could fold the parabola in half.
3. We call this line the **line of symmetry**.
4. Or, if we graphed one side of the parabola, we could "fold" (or REFLECT) it over, the line of symmetry to graph the other side.



	PRACTICAL EXERCISES **QUESTION 1**	
1.1	Given: $f(x) = x^2 - 2x - 3$	
1.1.1	Calculate the intercepts with axes.	(3)
1.1.2	Calculate the coordinates of the turning point.	(2)
1.1.4	Draw a graph of **f** showing all the intercepts with axes and the turning point.	(3)
1.1.5	Write down the range and domain of **f.**	(2)
1.2	Given: $f(x) = (x-2)^2 - 9$	
1.2.1	Write down the coordinates of the turning point of the graph of **f.**	(1)
1.2.2	Calculate the x and the y intercept of the graph of **f.**	(4)
1.2.3	Draw a neat graph of **f** and show the intercepts of the axes and the turning point.	(3)
1.2.4	Hence write the range and the domain of the function.	(2)
1.2.5	For which values of **x** is f(x) decreasing?	(2)
1.2.6	Use your graph to solve the inequality: $f(x) \leq 0$.	(2)
1.2.7	Write down the equation (in turning point form) of the graph obtained by (a) shifting **f,** 2 units left and 9 units up. (b) reflecting **f** in the y – axis. (c) reflecting **f** in the x – axis.	 (2) (2) (2)
	QUESTION 2. (WESTERN CAPE SEPTEMBER 2005)	
2.1	Sketch below are the graphs of: $f(x) = -(x+2)^2 + 4$ and $g(x) = ax + q$,R is the turning point of **f.** 	
2.2.1	Write down the coordinates of R. (2)	
2.2.2	Calculate the length of AB.	(2)

2.2.3	Determine the equation of **g**.	(2)
2.2.4	For which values of **x** is g(x) > f(x).	(2)
2.2.5	.Write down the equation of the axis of symmetry of **h** if h(x) = f(− x).	(2)
2.2.6	Write down the range of p if p(x) = −f(x)	(2)
	QUESTION 6	
	The graph of $f\left(x\right)=x^2+bx+c\,;a\neq 0$ and $g\left(x\right)=mx+k$ D(1;8) is a common point on **f** and **g. f** intersects the x –axis at (-3;4) and (2;0) **g** is the tangent to f at D.  	
6.1	For which value(s) of **x** is f(x) ≤ 0?	(2)
6.2	Determine the value of a, b and c.	(5)
6.3	Determine the coordinates of the turning points of **f.**	(3)
6.4	Write down the equation of the axis of symmetry of **h** if h(x) = f(x-7) + 2.	(2)
6.5	Calculate the gradient of **g.**	(3)
	QUESTION 5 (DBE NOV. 2018)	
	Given: $f\left(x\right)=\dfrac{-1}{x-1}$	
5.1	Write down the domain of **f.**	(1)
5.2	Write down the asymptotes of **f.**	(2)
5.3	Sketch the graph of f, clearly showing all intercepts with the axes and any asymptotes.	(3)
	QUESTION 4	
	Given: $f\left(x\right)=\dfrac{6}{x+2}-1$	
4.1	Write down the equations of the asymptotes of **g.**	(2)
4.2	Calculate:	

	4.2.1 The y – intercept of **g**.	(1)
	4.2.2 The x – intercept of **g**.	(2)
4.3	Draw the graph of g, show clearly the asymptotes and the intercepts with the axes.	(3)
4.4	Determine the equation of the line of symmetry, that has a negative gradient, in the form y = ……	(3)
4.5	Determine the value(s) of x for which $\frac{6}{x+2}-1\geq -x-3$	(2)
	QUESTION 2	
	Given: $f\left(x\right)=2^{x+1}-8$	
2.1	Write down the equation of the asymptote of **f.**	(1)
2.2	Sketch the graph of **f**, clearly indicate ALL the intercepts with the axes as well as the asymptote.	(3)
2.3	The graph of g is obtained by reflecting the graph of y about the y – axis. Write down the equation of **g.**	(1)
	QUESTION 4	
	Given: $f\left(x\right)=2^{-x}+1$	
4.1	Determine the coordinates of the y – intercepts	(2)
4.2	Sketch the graph of **f**, clearly indicate ALL the intercepts with the axes as well as the asymptote.	(3)
4.3	Calculate the average gradient of **f** between the points on the graph where x = −2 and x = 1.	(3)
4.4	If h(x) = 3f(x), write down the equation of the asymptote of **h**.	(2)
	QUESTION 5	
	The graph of $f\left(x\right)=a^{x};a>1$ is shown below. T(2;9) lies on **f**. 	
5.1	Calculate the value of a.	(2)
5.2	Determine the equation of g if g(x) =f(−x).	(1)
5.3	Determine the value(s) of x for which $f^{-1}\left(x\right)\geq 2$.	(2)

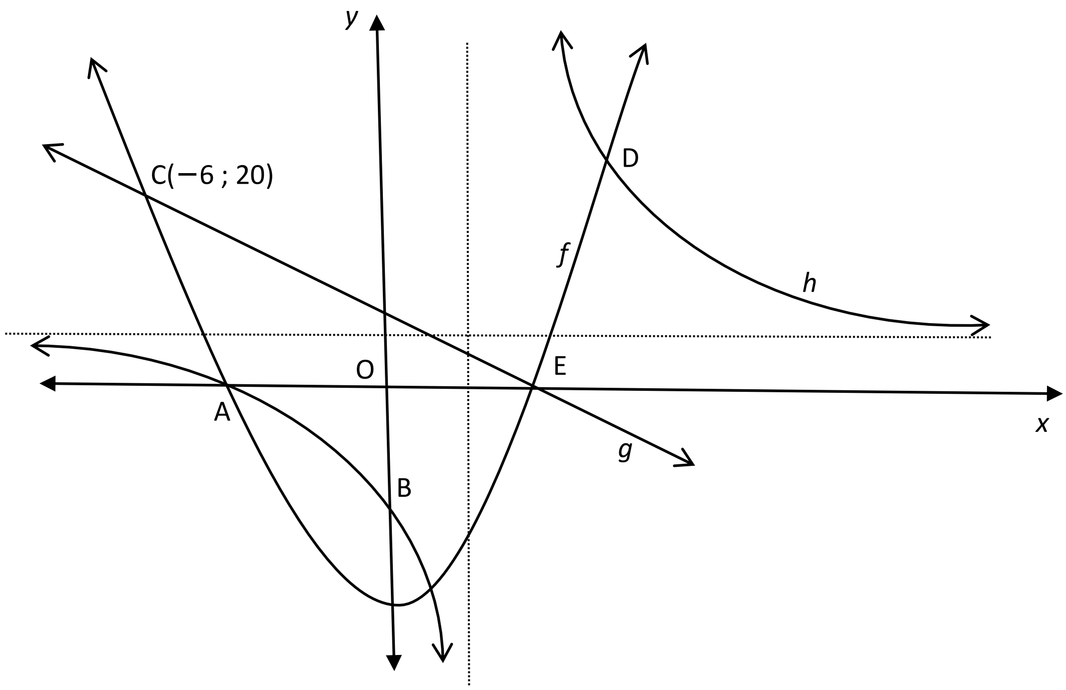
5.4	Is the inverse a function? Explain your answer.	(2)

Example 2:

Sketched below are the graphs of: $g(x) = -2x+8$; $f(x) = x^2 + k$ and

$h(x) = \dfrac{6}{x-2} + 1$

A and B are the *x*- and *y* - intercepts of *h* respectively, C (−6 ; 20) and E are the points of intersection of *f* and *g* .



Calculate the coordinates of A, B and E.

Show that the value of $k = -16$

Determine the domain and the range of *f*

Write down the values of *x* for which $g(x) - f(x) \geq 0$

Determine the equation of the symmetry axis of *h* if the gradient is negative.

Write down the range of *s*, if $s(x) = f(x) + 2$.

Write down the range of *t*, if $t(x) = h(x) + 2$

Solutions:

To answer the above questions, you need to identify all the functions in order to apply the deductions indicated above.

A and B are *x* and *y* intercepts of *g* respectively.

at A, $y=0 \therefore \dfrac{6}{x-2}+1=0$

$$6=-x+2$$
$$4=-x$$
$$\therefore x=-4$$

Thus A$(-4;0)$

at B, $x=0 \therefore y=\dfrac{6}{-2}+1$

$$y=-3+1$$
$$\therefore y=-2$$

Thus B$(0;-2)$

E is the x- intercept of the straight line and the parabola. It is easy and straight forward to use the equation of the straight line to get the coordinates of E.

At E, $y=0; \ \therefore 0=-2x+8$

$$2x=8$$
$$x=4$$

Thus E$(4;0)$

b) C(-6; 20) is on *f* and *g* , substituting the

into

$$y=x^2+k \Rightarrow 20=(-6)^2+k$$
$$\therefore 20-36=k$$
$$k=-16$$

c) Domain is $x \in \mathbb{R}$

Range is $y \geq -16; \ y \in \mathbb{R}$

d) These are values of *x* for which the graph of *g* and *f* intersect or *f* is below *g* .

It is from C(-6 ; 20) and E(4 ; 0)

That is $-6 \leq x \leq 4$

e) For negative gradient, $y=-(x-2)+1$

$$y=-x+2+1$$
$$y=-x+3$$

f) + 2 implies the value of *p* is increased by 2

The range of *s* is $y \geq -16+2$

$$y \geq 14$$

g) + 2 implies the value of *p* is increased by 2

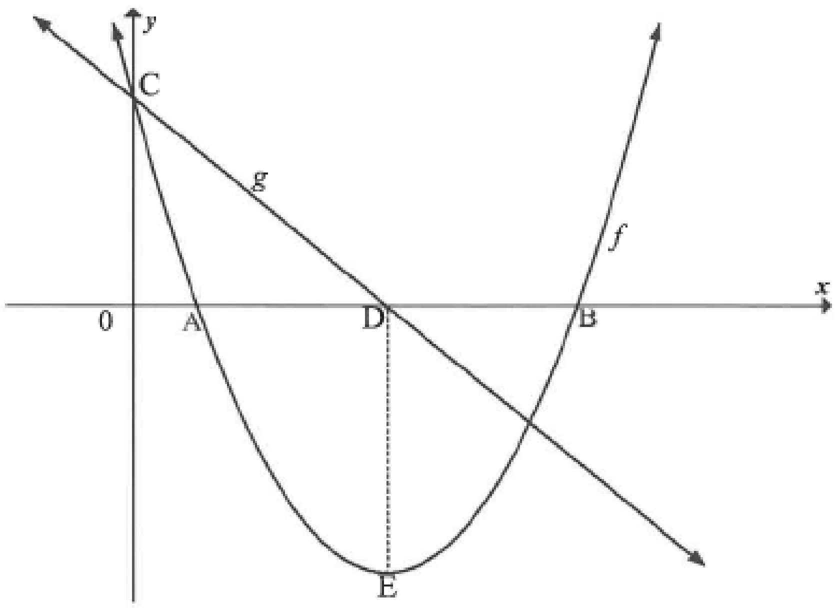
The range of t is $y \neq 1+2; \ y \in \mathbb{R}$

$$y \neq 3; \ y \in \mathbb{R}$$

QUESTION 4

Below are the graphs of $f(x) = (x-4)^2 - 9$ and a straight line g.

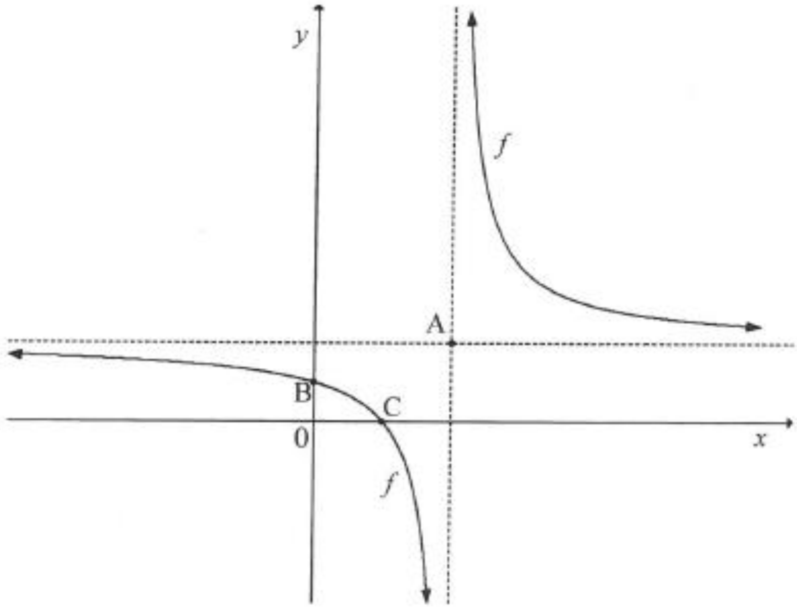
- A and B are the x-intercepts of f and E is the turning point of f.
- C is the y-intercept of both f and g.
- The x-intercept of g is D. DE is parallel to the y-axis.



4.1	Write down the coordinates of E.	(2)
4.2	Calculate the coordinates of A.	(3)
4.3	M is the reflection of C in the axis of symmetry of f. Write down the coordinates of M.	(3)
4.4	Determine the equation of g in the form $y = mx + c$.	(3)
4.5	Write down the equation of g^{-1} in the form $y = \ldots$	(3)

QUESTION 4

The sketch below shows the graph of $f(x)=\dfrac{6}{x-4}+3$. The asymptotes of f intersect at A. The graph f intersects the x-axis and y-axis at C and B respectively.



4.1	Write down the coordinates of A.	(1)
4.2	Calculate the coordinates of B.	(2)
4.3	Calculate the coordinates of C.	(2)
4.4	Calculate the average gradient of f between B and C.	(2)
4.5	Determine the equation of a line of symmetry of f which has a positive y-intercept.	(2)

QUESTION 5

Given: $f(x)=x^2-5x-14$ and $g(x)=2x-14$

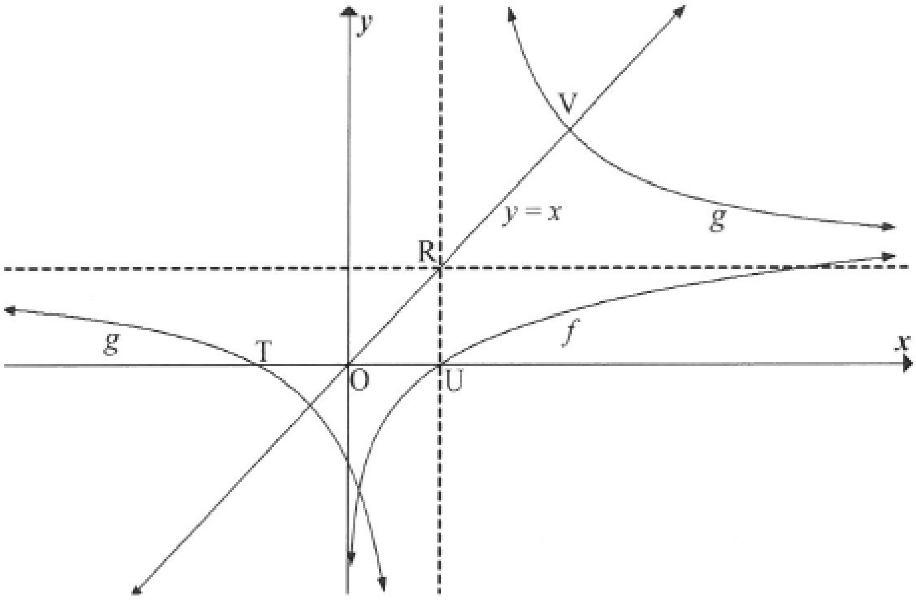
5.1	On the same set of axes, sketch the graphs of f and g. Clearly indicate all intercepts with the axes and turning points.	(6)
5.2	Determine the equation of the tangent to f at $x=2\frac{1}{2}$.	(2)
5.3	Determine the value(s) of k for which $f(x)=k$ will have two unequal positive real roots.	(2)
5.4	A new graph h is obtained by first reflecting g in the x-axis and then translating it 7 units to the left. Write down the equation of h in the form $h(x)=mx+c$.	(2)

[12]

QUESTION 4

The sketch below shows the graphs of $f(x) = \log_5 x$ and $g(x) = \dfrac{2}{x-1} + 1$.

- T and U are the x-intercepts of g and f respectively.
- The line $y = x$ intersects the asymptotes of g at R, and the graph of g at V.



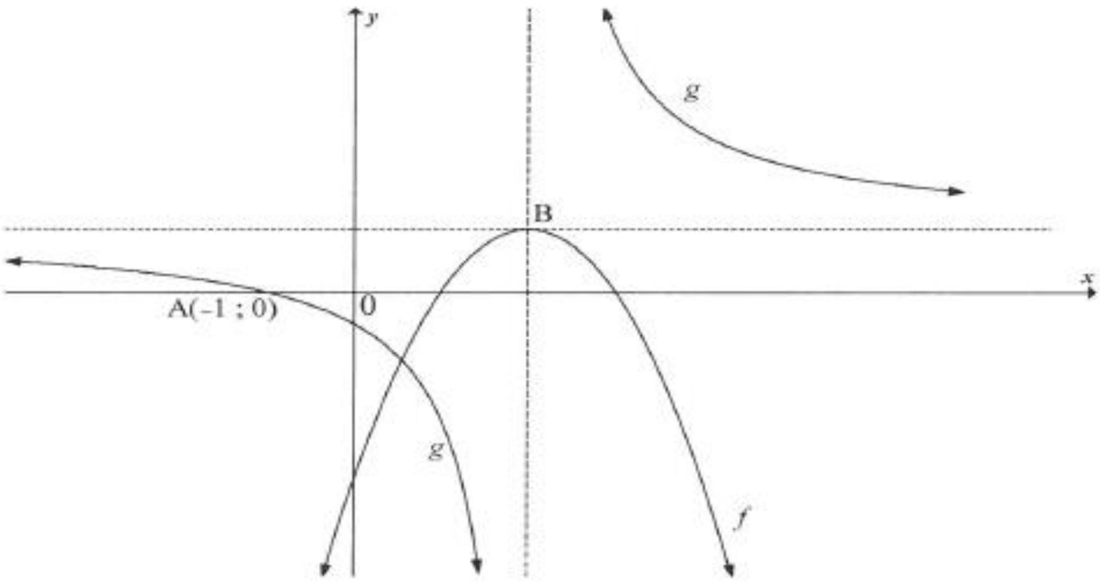
4.1	Write down the coordinates of U.	(1)
4.2	Write down the equations of the asymptotes of g.	(2)
4.3	Determine the coordinates of T.	(2)
4.4	Write down the equation of h, the reflection of f in the line $y = x$, in the form $y = \ldots$	(2)
4.5	Write down the equation of the asymptote of $h(x-3)$.	(1)

QUESTION 5 (DBE November 2016)

QUESTION 5

Sketched below is the parabola f with equation $f(x) = -x^2 + 4x - 3$ and a hyperbola g with equation $(x-p)(y+t) = 3$.

- B, the turning point of f, lies at the point of intersection of the asymptotes of g.
- $A(-1\ ;\ 0)$ is the x-intercept of g.



5.1	Show that the coordinates of B are $(2\ ;\ 1)$	(2)
5.2	Write down the range of f.	(1)
5.3	For which value(s) of x will $g(x) \geq 0$?	(2)
5.4	Determine the equation of the vertical asymptote of the graph of h if $h(x) = g(x+4)$	(1)
5.5	Determine the values of p and t.	(4)

FINANCE, GROWTH AND DECAY: **COMPOUND GROWTH & DECAY**

- Understand compound **growth**: $A = P(1+i)^n$. Used when a **single** investment or value of an item is **growing**. **Inflation** also forms part of this.
- Understand compound **decay**: $A = P(1-i)^n$. Used when the value of an item is **decreasing**. Terms associated with this formula are: **Reducing balance, scrap value, book value, trade-in value**.

Note: **A** is the amount **A**fter growth or decay has occurred. **P** is the amount **B**efore growth or decay happens.

- In each of the two formulae, you must be able to calculate A, P, i and n. See examples below:

NOTE: When dealing with money, interest can be compounded, usually:

✓ Monthly (So, *divide interest rate by 12 and multiply years by 12*)
✓ Quarterly (So, *divide interest rate by 4 and multiply years by 4)*
✓ Half-Yearly or Semi-annually or bi-annually (So, *divide interest rate by 2 and multiply years by 2*)

Example 1: Sandile invests R9000 at 8% p.a **compounded monthly**. How much will he have after 6 years? $A = P(1+i)^n$ $A = 9000\left(1+\frac{0,08}{12}\right)^{6\times 12}$ *note* $\frac{i}{12}$ *and* $n \times 12$, *compounded monthly* $A = R14521,52$	**Example 2**: A photocopying machine has a book value of R22 300 after depreciating at 9,5% p.a. for 5 years. Calculate its original value. $A = P(1-i)^n$ $22300 = P(1-0,095)^5$ $\frac{22300}{(1-0,095)^5} = \frac{P\cancel{(1-0,095)^5}}{\cancel{(1-0,095)^5}}$ $P = R36733,47$
Example 3: John's car worth R150 000 **depreciates** to R95 000 on the **reducing balance** method after 6 years. Calculate the rate of depreciation. $A = P(1-i)^n$ $95000 = 150000(1-i)^6$ $\frac{95000}{150000} = \frac{\cancel{150000}}{\cancel{150000}}(1-i)^6$ $\frac{19}{30} = (1-i)^6$ $\sqrt[6]{\frac{19}{30}} = \sqrt[6]{(1-i)^6}$ $i = 1 - \sqrt[6]{\frac{19}{30}}$ $i = 7,33\%$	**Example 4**: Nzama invested R6000 at 8,5% p.a. **compounded monthly.** How long will it take to grow to R9500? $9500 = 6000\left(1+\frac{0,085}{12}\right)^n$ **note n is not multiplied by 12** $\frac{9500}{6000} = \left(1+\frac{0,085}{12}\right)^n$ $\log\frac{19}{12} = n \times \log(1+\frac{0,085}{12})$ $\frac{\log\frac{19}{12}}{\log\left(1+\frac{0,085}{12}\right)} = \frac{n \times \log\left(1+\frac{19}{12}\right)}{\log\left(1+\frac{19}{12}\right)}$ $n = 65,10464825$ ***Since n was not multiplied by 12, this answer is in months. So we would divide it by 12 if we wanted it in years.***

Note: These examples show you how to calculate each variable (A, P, i & n). When doing a question and are referring, use the formula (Growth: $A = P(1+i)^n$ or **Decay**: $A = P(1-i)^n$**) relevant to the question.**

QUESTION 1: NOV 2014 (REFER TO EXAMPLE 3)

Exactly five years ago, Mpume bought a new car for R145 OOO. The current book value of this car is R72 500. If the car depreciates by a fixed annual rate according to the reducing-balance method, calculate the rate of depreciation. (3)

QUESTION 2: KZN SEPT 2015 (REFER TO EXAMPLE 4)

How long will it take for a motor car to double in value if the annual inflation rate is 8,5%?

(4)

QUESTION 3: FEB 2016 (REFER TO EXAMPLE 1)

Diane invests a lump sum of R5000 in a savings account for exactly 2 years. The investment earns interest at 10% p.a., compounded quarterly.

Calculate the amount in Diane's savings account at the end of the 2 years. (3)

QUESTION 4: KZN JUNE 2016 (REFER TO EXAMPLE 4)

How long would the price of an asset take to reduce by a third of its original value if it depreciates on a reducing balance at a rate of 4,7% p.a.? (4)

QUESTION 5: FEB 2017 (REFER TO EXAMPLE 1 & 4)

On the 2^{nd} day of January 2015 a company bought a new printer for R150 000.

- The value of the printer decreases by 20% annually on the reducing-balance method.
- When the book value of the printer is R49 152, the company will replace the printer.

a) Calculate the book value of the printer on the 2^{nd} day of January 2017. (3)

b) At the beginning of which year will the company have to replace the printer? Show ALL calculations. (4)

QUESTION 6: KZN JUNE 2017 (REFER TO EXAMPLE 3)

A company bought a new machine for R500 000. After 3 years, the machine has a book value of R331 527. Calculate the yearly rate of depreciation if the machine depreciated according to the reducing-balance method. (3)

QUESTION 7: KZN SEPT 2017 (REFER TO EXAMPLE 2 & 4)

a) Samuel invested an amount with TRUST bank. His investment earned interest of 12% p.a. compounded monthly and grew to R8450 at the end of 10 years.
Calculate the amount that Samuel initially invested. (3)

b) If the rate of depreciation remains at constant of 4,7% p.a., calculate the period it will take for an amount to be worth half of what it was originally. The amount depreciates on a reducing-balance basis. (3)

QUESTION 8: NOV 2017 (REFER TO EXAMPLE 3)

Mbali invested R10 000 for 3 years at an interest rate of r% p.a., compounded monthly. At the end of this period, she received R12 146,72. Calculate r, correct to ONE decimal place. (5)

QUESTION 9: KZN SEPT 2018 (REFER TO EXAMPLE 3)

A tractor costing R180 000 depreciates on the reducing balance method to R65 000 at the end of 8 years. Determine the rate at which the tractor is depreciating per annum (3)

QUESTION 10: KZN SEPT 2019 (REFER TO EXAMPLE 2)

A car depreciated at the rate of 13,5% p.a. to R250 000 over 5 years according to the reducing balance method. Determine the original price of the car, to the nearest rand. (3)

Q 11: NOV 2019 (REFER TO EXAMPLE 1) SIMPLE INTEREST: $\boldsymbol{A = P(1 + in)}$

Two friends, Kuda and Thabo, each want to invest R5 000 for four years. Kuda invests his money in an account that pays simple interest at 8,3% per annum. At the end of four years, he will receive a bonus of exactly 4% of the accumulated amount. Thabo invests his money in an account that pays interest at 8,1% p.a., compounded monthly.

Whose investment will yield a better return at the end of four years? Justify your answer with appropriate calculations. (5)

FINANCE, GROWTH AND DECAY: **PRESENT AND FUTURE VALUE**

- Present value: $P_v = \frac{x[1-(1+i)^{-n}]}{i}$. Used in cases of **loans** and purchasing items on **credit** (not hire purchase), then pay off the debt with instalments.
- Future value: $F_v = \frac{x[(1+i)^{n}-1]}{i}$. Used in cases of **regular deposits** into **investments** made to save money for the **future**.

Note: Unlike Present value, in future value we are saving money not paying off debt AND unlike compound growth, deposits are made regularly, not once.

Example 5: Josh wants to buy a house but does not have money, so he decides to take a loan of R250000 at 12,5% p.a. **compounded monthly**. How much must he pay per month if the loan is to be repaid over 11 years? $P_v = \frac{x[1-(1+i)^{-n}]}{i}$ $250000 = \frac{x\left[1-\left(1+\frac{0{,}125}{12}\right)^{-11\times 12}\right]}{\frac{0{,}125}{12}}$	**Example 6:** Andiswa saves R1600 every **half a year**. The bank offers her 8% p.a. **compounded semi-annually**. How much will she have if she does this for a period of 7 years? *NOTE: savings are made on a regular basis, so this is not compound growth. It's future value.* $F_v = \frac{x[(1+i)^{n}-1]}{i}$ $F_v = \frac{1600\left[\left(1+\frac{0{,}08}{2}\right)^{7\times 2}-1\right]}{\frac{0{,}08}{2}}$ $F_v = R29267{,}06$

$\frac{0,125}{12} \times 250000 = x\left[1-\left(1+\frac{0,125}{12}\right)^{-132}\right]$ $\frac{\frac{0,125}{12} \times 250000}{\left[1-\left(1+\frac{0,125}{12}\right)^{-132}\right]} = x$ $x = R3493,86$	She will have $R29267,06$ <u>The numerator can be calculated on its own</u>, The denominator can be calculated on its own Then divide to find the answer.

QUESTION 12: NOV 2014 (REFER TO EXAMPLE 5)

Samuel took out a loan for R500 000 at an interest rate of 12% per annum, compounded monthly. He plans to repay this loan over 20 years and his first payment is made one month after the loan is granted.

Calculate the value of Samuel's monthly instalment. (4)

QUESTION 13: FEB 2015 (REFER TO EXAMPLE 6)

Nomsa started working on 1 January 1970. At the end of January 1970 and at the end of each month thereafter, she deposited R400 into an annuity fund. She continued doing this until she retired on 31December 2013.

a) Determine the total amount of money that she paid into the fund. (2)
b) The interest rate on this fund was 8% p.a., compounded monthly. Calculate the value of the fund at the time that she retired. (5)

QUESTION 14: KZN SEPT 2015 (REFER TO EXAMPLE 1 & 5)

A loan of R350 000, taken on 1 January 2005, is to be repaid in regular fixed instalments at the end of each month. Interest was charged at 13,5% p.a. compounded monthly for 20 years. The client made the first payment on 31 March 2005.

a) Calculate the value of the loan payable on 28 February 2005. (2)
b) Determine the monthly payment that will settle the loan within the 20 years. (4)

QUESTION 15: NOV 2016 (REFER TO EXAMPLE 1 & 5)

On 1 June 2016 a bank granted Thabiso a loan of R250 000 at an interest rate of 15% p.a., compounded monthly, to buy a car. Thabiso agreed to repay the loan in monthly instalments commencing on 1 July 2016 and ending 4 years later on 1 June 2020. However, Thabiso was unable to make the first two instalments and only commenced with the monthly instalments on 1 September 2016.

(a) Calculate the amount Thabiso owed the bank on 1 August 2016, a month before he paid his first monthly instalment. (2)
(b) Having paid the first monthly instalment on 1 September 2016, Thabiso will still pay his last instalment on 1 June 2020. Calculate his monthly instalment. (4)

QUESTION 16: FEB 2017 (REFER TO EXAMPLE 6)

Lerato wishes to apply for a home loan. The bank charges interest at 11% per annum, compounded monthly. She can afford a monthly instalment of R9 000 and wants to repay the loan over a period of 15 years. She will make the first monthly repayment one month after the loan is granted. Calculate, to the nearest thousand rand, the maximum amount that Lerato can borrow from the bank.

(5)

QUESTION 17: NOV 2017(REFER TO EXAMPLE 5)

Nokwe wants to buy a house. She takes out a loan for R950 000. Interest is charged on the loan at 10,5% p.a. compounded monthly.

Calculate the monthly repayments if the loan is repaid over 20 years. (5)

QUESTION 18: NOV 2017(REFER TO EXAMPLE 5)

Piet takes a loan from a bank to buy a car for R235 000. He agrees to repay the loan over a period of 54 months. The first instalment will be paid one month after the loan is granted. The bank charges interest at 11% p.a., compounded monthly.

Calculate Piet's monthly instalment. (4)

QUESTION 19: KZN SEPT 2018 (REFER TO EXAMPLE 5)

Tebogo buys a flat at the beach front for R850 000. She takes out a loan from the bank at an interest rate of 14,25% per annum compounded monthly. Her first instalment will commence in one month after she has taken out the loan.

Calculate the monthly payments over a period of 20 years. (4)

QUESTION 20: NOV 2018 (REFER TO EXAMPLE 5)

Tshepo takes out a home loan over 20 years to buy a house that costs R1 500 000.

Calculate the monthly instalment if interest is charged at 10,5% p.a., compounded monthly.

(4)

QUESTION 21: KZN SEPT 2019 (REFER TO EXAMPLE 5)

Melissa takes a loan of R950 000 to buy a house. The interest is 14,25% p.a., compounded monthly. His first instalment will commence one month after taking out the loan.

Calculate the monthly payments over a period of 20 years. (4)

FINANCE, GROWTH AND DECAY: **SINKING FUND**

When a company or individual purchases a machine, usually, three things happen:

1. The machine just bought **loses value**: $A = P(1 - i)^{n}$
2. As soon as the company or an individual purchases the machine, they will start **saving regularly** to afford a new machine in future: $F_v = \frac{x[(1+i)^{n} - 1]}{i}$
3. The manufacturers of the machine add new and more advanced features, so the new machine in future will be more expensive (**price increases**): $A = P(1 + i)^{n}$

QUESTION 22 (REFER TO EXAMPLE 1 & 6)

Jabulani purchases a car for R250 000. He has planned to replace the car in 6 years' time. The replacement cost of the car is expected to rise at 10% p.a. compounded annually and the rate of depreciation of his current car is 15% p.a. on the reducing-balance method. Jabulani sets up a sinking fund to pay for a new car in 6 years' time. Calculate:

a) The **trade-in** value of Jabulani's car in 6 years' time. (3)
b) The cost of a new car in 6 years' time. (3)
c) How much Jabulani must have invested in 6 years' time (sinking fund) if he will trade-in his old car? (2)
d) The monthly payments into the sinking fund if interest is earned at 7,5% p.a., compounded monthly. (5)

[13]

QUESTION 23 (REFER TO EXAMPLE 1 & 6)

Machinery is purchased at a cost of R550 000 and is expected to rise in cost at 15% per annum compounded annually and depreciate in value at a rate of 8% p.a., compounded annually.

A sinking fund is started to make provision for replacing the old machine. The sinking fund pays 16% p.a., compounded monthly, and monthly payments are made into this fund for 10 years. Determine:

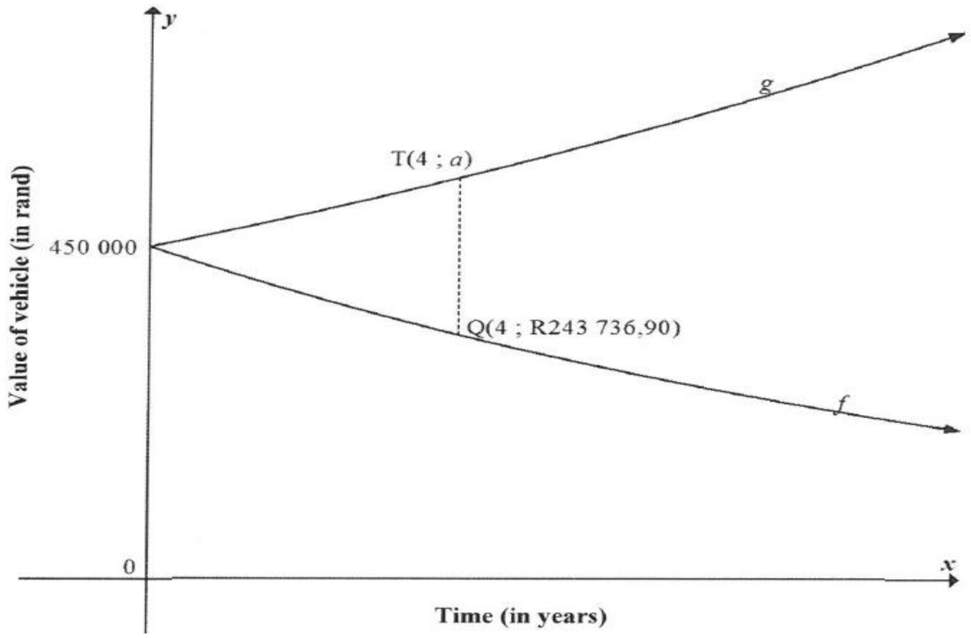
a) The replacement cost in 10 years' time. (3)
b) The **scrap value** of the machine in 10 years' time. (3)
c) The monthly payment into the sinking fund that will make provision for the replacement of the new machine. (5)

[11]

QUESTION 24: NOV 2015 (REFER TO EXAMPLE 1 & 6)

The graph of f shows the book value of a vehicle x years after the time Joe bought it.

The graph of g shows the cost price of a similar new vehicle x years later.



a) How much did Joe pay for the vehicle? (1)

b) Use the reducing-balance method to calculate the percentage annual rate of depreciation of the vehicle that Joe bought. (4)

c) If the average rate of price increase is 8,1% p.a., calculate the value of a. (3)

d) A vehicle that costs R450 000 now, is to be replaced at the end of 4 years. The old vehicle will be used as a trade-in. A sinking fund is created to cover the replacement cost of this vehicle. Payments will be made at the end of each month. The first payment will be made at the end of the 13^{th} month and the last payment will be made at the end of the 48^{th} month. The sinking fund earns interest at a rate of 6,2% p.a., compounded monthly.
Calculate the monthly payment to the fund. (5)

[13]

QUESTION 25: FEB 2017 (REFER TO EXAMPLE 1 & 6)

On the 2^{nd} of January 2015 a company bought a new printer for R150 000.

- The value of the printer decreases by 20% annually on the reducing balance method.
- When the book value of the printer is R49 152, the company will replace the printer.

a) Calculate the book value of the printer on the 2^{nd} of January 2017. (3)

b) At the beginning of which year will the company have to replace the printer? Show ALL calculations. (4)

c) The cost of a similar printer will be R280 000 at the beginning of 2020. The company will use the R49 152 that it will receive from the sale of the old printer to cover some of the costs of replacing the printer. The company sets up a sinking fund to cover the balance. The fund pays interest at 8,5% per annum, compounded quarterly. The first deposit was made on 2 April 2015 and every three months thereafter until 2 January 2020. Calculate the amount that should be deposited every three months to have enough money to replace the printer on 2 January 2020. (4)

DIFFERENTIAL CALCULUS

HINTS

A) **FIRST PRINCIPLES AND RULES**

- Work with and have an intuitive understanding of the concept of limit.
- Differentiate specified functions from first principles.

$f(x) = a$ $\quad f(x) = ax + b$ $\quad f(x) = ax^2 + bx + c$

$f(x) = ax^3$ $\quad f(x) = \frac{a}{x}$ $\quad a, b\ and\ c\ are \in \mathbb{R}$

There are two ways of finding the derivative:

a) First principles
b) Power rule

1. For first principles:
   - Use the formula: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
   - Copy the formula from the formula sheet as it is
   - Do not substitute 0 for h
   - Remove limit when writing the final answer

2. For power rule:
   - If $f(x) = ax^n$ then $f'(x) = n.ax^{n-1}$
   - If the function is *f(x)* then the derivative is $f'(x)$
   - If the function is *y* then the derivative is $\frac{dy}{dx}$
   - If D_x then for the answer remove D_x
   - If $f(x) = c$ where *c* is a constant then the derivative is 0

B) **TANGENTS, CUBIC FUNCTIONS AND OPTIMISATION**

- Determine the equations of tangents to graphs
- Work with an increasing and decreasing function and identify stationary points and point of inflection.
- Factorise cubic polynomials
- Sketch and interpret graphs of cubic function
- Solve practical problems involving optimisation, rates of change and calculus of motion

FIRST PRINCIPLES

Example on determining the derivative from first principles

$$f(x) = 2x^2 + 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{lim}{h \to 0} \frac{2(x+h)^2 + 4 - (2x^2 + 4)}{h}$$

$$= \frac{lim}{h \to 0} \frac{2(x^2 + 2xh + h^2) + 4 - 2x^2 - 4}{h}$$

$$= \frac{lim}{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 4 - 2x^2 - 4}{h}$$

$$= \frac{lim}{h \to 0} \frac{4xh + 2h^2}{h}$$

$$= \frac{lim}{h \to 0} \frac{h(4x + 2h)}{h}$$

$$= 4x$$

WHEN DETERMINING THE DERIVATIVE USING RULES, TAKE NOTE OF:

- **Subject of the formula**

  When determining $\frac{dy}{dx}$*, make y the subject of the formula first and do the derivative with respect to x*

  e.g.

  (a) $y = 3x^5 - 4x^3 + 2x^2 - 5$

  $$\frac{dy}{dx} = 15x^4 - 12x^2 + 4x$$

  (b) $xy = x^2 + y - 1$

  $$xy - y = x^2 - 1$$

  $y(x-1) = (x-1)(x+1)$

  $y = \frac{(x-1)(x+1)}{(x-1)}$

  $y = x + 1$

  $\frac{dy}{dx} = 1$

  (c) $\sqrt{y} = 2x + 1$

  $(\sqrt{y})^2 = (2x+1)^2$

  $y = 4x^2 + 4x + 1$

  $\frac{dy}{dx} = 8x + 4$

- **Multiplication (Products)**
  Determine the product first e.g.

(a) $y = 3x^2.4x^3$

$y = 12x^5$

$\frac{dy}{dx} = 60x^4$

(b) $f(x) = (3x^2 - 2)^2$

$f(x) = 9x^4 - 12x^2 + 4$

$f'(x) = 36x^3 - 24x$

- **Surds**

Change surds into exponential form. $\sqrt[a]{x^n} = x^{\frac{n}{a}}$ *e.g.*

$D_x\left(\sqrt[3]{x^4} + 8\sqrt{x}\right)$

$= D_x\left(x^{\frac{4}{3}} + 8x^{\frac{1}{2}}\right)$

$= \frac{4}{3}x^{\frac{1}{3}} + 4x^{-\frac{1}{2}}$

- **Variable in the denominator (One term)**

Divide each term of the numerator by the denominator e.g.

$g(x) = \frac{6x^5 - 4x^2 + 8x - 7}{2x^2}$

$g(x) = \frac{6x^5}{2x^2} - \frac{4x^2}{2x^2} + \frac{8x}{2x^2} - \frac{7}{2x^2}$

$g(x) = 3x^3 - 2 + 4x^{-1} - \frac{7}{2}x^{-2}$

$g'(x) = 9x^2 - 4x^{-2} + 7x^{-3}$

- **Variable in the denominator (Two or more terms)**
  Factorise e.g.

$y = \frac{3x^2 - 2x - 1}{x - 1}$

$y = \frac{(3x + 1)(x - 1)}{x - 1}$

$y = 3x + 1$

$\frac{dy}{dx} = 3$

PRACTICE EXERCISES

1. Determine the derivative from first principles.

a) $f(x) = 2x + 3$ (4)

b) $f(x) = 5 - 3x$ (5)

c) $f(x) = 3x^2 + 7$ (5)

d) $f(x) = -2x^2 + x$ (5)

e) $f(x) = 2x^3$ (5)

f) $f(x) = \frac{2}{x}$ (5)

g) $f(x) = -\frac{3}{x}$ (5)

2. Use the rules of differentiation to differentiate the following:

(i) Subject of the formula

(a) $xy = x^3 + 2x^2 - 5x$ (3)

(b) $xy + 4y = x^2 - 16$ (3)

(c) $xy = x^2 + 5x - y + 4$ (4)

(d) $y = 8x^3 - 2xy + 1$ (5)

(ii) Multiplication (Products)

(a) $y = 3x^5.6x$ (2)

(b) $D_x[(2x - 5)^2]$ (3)

(c) $y = (2x - 3)\left(4x^{\frac{1}{3}} + 5\right)$ (4)

(iii) Surds

(a) $g(x) = \sqrt[3]{x^2} + 5x^3$ (3)

(b) $h(x) = \sqrt[5]{x} - 4\sqrt{x^4}$ (3)

(c) $y = 5\sqrt{x}.2\sqrt[5]{x^3}$ (4)

(iv) Variable in the denominator (One term)

(a) $y = \frac{2x^4 - 3x^3 + 4x}{x}$ (3)

(b) $y = \frac{2x^2 - 4x + 3\sqrt{x}}{x^2}$ (4)

(c) $y = \frac{3x^3 + 6x^2 - 15x + 2}{3x}$ (4)

(v) Variable in the denominator (Two or more terms)

(a) $y = \frac{3x^2 - 2x - 5}{x + 1}$ (3)

(b) $D_t\left[\frac{t^2 - 1}{2t + 2}\right]$ (3)

(c) $y = \frac{x^3 + 8}{x^2 - 2x + 4}$ (3)

MIXED QUESTIONS FROM PAST PAPERS

Differentiate the following:

(a) $y = \frac{4}{\sqrt{x}} - \frac{x^3}{9}$ (3)

(b) $y = \left(1 + \sqrt{x}\right)^2$ (3)

(c) $y = \frac{8 - 3x^6}{8x^5}$ (4)

(d) $y = x^{-4} + 2x^3 - \frac{x}{5}$ (3)

(e) $y = \frac{2\sqrt{x}+1}{x^2}$ (4)

(f) $f(x) = -3x^2 + 5\sqrt{x}$ (3)

(g) $p(x) = \left(\frac{1}{x^3} + 4x\right)^2$ (4)

(h) $D_x\left(\frac{x^3-1}{x-1}\right)$ (3)

(i) $\sqrt[3]{y} = x + 1$ (5)

(j) $y = \sqrt{x^3} - \frac{5}{x} + \frac{1}{2}\pi$ (4)

(k) $\frac{dy}{da}\ if\ y = ax^2 + a$ (2)

THE EQUATION OF A TANGENT [level 2 and 3]

$$y - y_1 = m(x - x_1)$$
$$f'(x) = m_t \; at \; x = a$$

1.1. Determine the equation of the tangent to $f(x) = x^2 - 6x + 5 \; at \; x = 2$

1.2. The function defined by $g(x) = x^2 - 8x + 20$ is given. Determine:

1.2.1. The point on the curve of g where the gradient of the curve is 4.

1.2.2. The equation of the tangent at this point

1.3. The function g and f defined by $g(x) = -4x + 3 \; and \; f(x) = x^3 - 9x^2 + 20x - 8$ are given. Determine:

1.3.1. The gradient of the tangent to the curve of *f* at the point (1;4)

1.3.2. One point on the curve of *f* where the tangent to the curve of *f* will be parallel to line g.

2. SKETCHING THE GRAPH OF A CUBIC FUNCTION

CURVE SKETCHING

This brings together skills learned in **polynomials** and in **stationary points**.

Steps: □ axis intercepts $(x : y = 0; \; y : x = 0)$

□ stationary points and their nature (type)

** **e.g.18** Sketch the following curves:

a) $f(x) = x^2 - 6x - 7$

b) $y = x^3 - x^2 - x + 1$

c) $y = -x^3 + 3x^2$

» **Solution** a) $y - \text{int}: \; f(0) = -7$

$x - \text{ints}: \; x^2 - 6x - 7 = 0 \Rightarrow (x+1)(x-7) = 0$

$\Rightarrow \quad x = -1; 7$

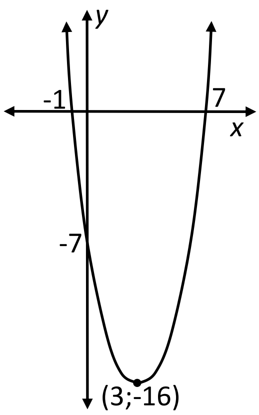
S.P.'s: $f'(x) = 0 \Rightarrow 2x - 6 = 0$

$\Rightarrow \quad x = 3$

$f(3) = 3^2 - 6.3 - 7 = -16 \quad \therefore \text{SP}(3; -16)$

$f''(x) = 2 ... > 0 \;\; \therefore$ concave up

$\therefore$ minimum $(3; -16)$



b) $y-\text{int}: 1$

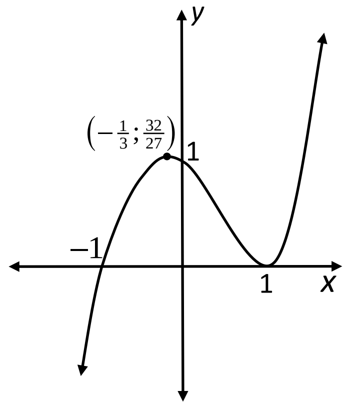
$x-\text{ints}:$ $\boxed{\textit{Use factor theorem.}}$

$f(1)=1^3-1^2-1+1=0 \quad \therefore (x-1)$ is a factor

$x^3-x^2-x+1=(x-1)(x^2+mx-1)$ $\boxed{\textit{or long division}}$

$mx^2-x^2=-x^2 \Rightarrow m=0$

$$(x-1)(x^2-1)=0$$
$$(x-1)(x-1)(x+1)=0$$
$$\Rightarrow \qquad x=1;-1$$



S.P.'s: $\dfrac{dy}{dx}=0 \Rightarrow \quad 3x^2-2x-1=0$

$$\Rightarrow (3x+1)(x-1)=0$$
$$\Rightarrow \qquad x=-\tfrac{1}{3};1$$

$y=(-\tfrac{1}{3})^3-(-\tfrac{1}{3})^2-(-\tfrac{1}{3})+1$ or $y=1^3-1^2-1+1$

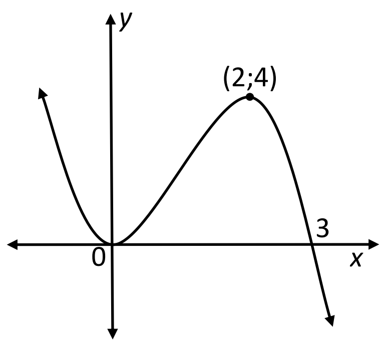
$=\tfrac{32}{27}$ $\qquad\qquad =0$

$$\frac{d^2y}{dx^2}=6x-2\begin{cases}<0 \text{ at } x=-\tfrac{1}{3} & \text{concave down} \quad \therefore \text{local max at } (-\tfrac{1}{3};\tfrac{32}{27})\\ >0 \text{ at } x=1 & \text{concave up} \quad \therefore \text{local min at } (1;0)\end{cases}$$

c) $y-\text{int}: 0$

$x-\text{ints}:$ $-x^3+3x^2=0 \Rightarrow -x^2(x-3)=0$

$$\Rightarrow \qquad x=0;3$$



S.P.'s: $\dfrac{dy}{dx}=0 \Rightarrow -3x^2+6x=0$

$$\Rightarrow -3x(x-2)=0$$
$$\Rightarrow \qquad x=0;2$$

$y=-0^3+3.0^2$ or $y=-2^3+3.2^2$

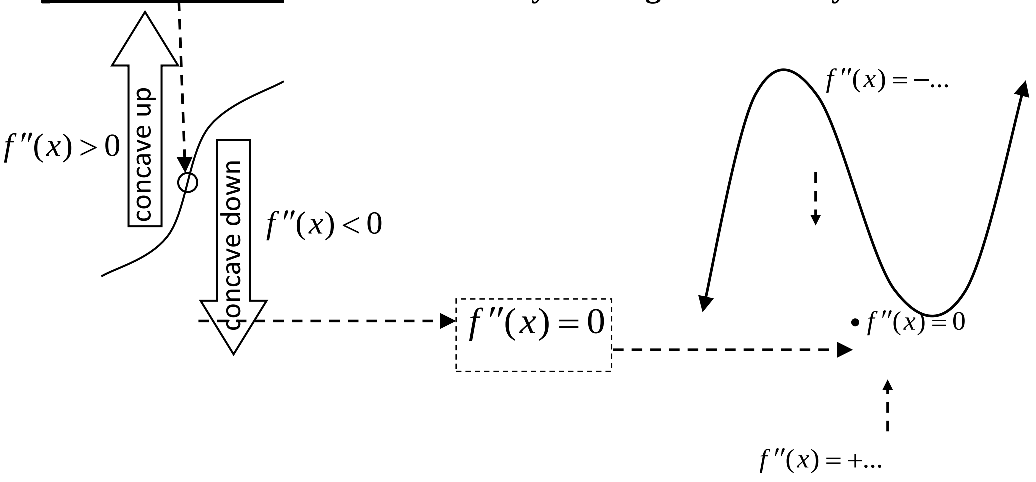
$=0$ $\qquad\qquad =4$

$$\frac{d^2y}{dx^2}=-6x+6\begin{cases}>0 \text{ at } x=0 & \text{concave up} \quad \therefore \text{local max at } (0;0)\\ <0 \text{ at } x=2 & \text{concave down} \quad \therefore \text{local min at } (2;4)\end{cases}$$

More about points of inflection

A point of inflection is not necessarily a stationary point … rather …

A <u>point of inflection</u> is characterised by a change in concavity.



**** e.g.19** Find the coordinates of the point(s) of inflection of:

a) $f(x) = x^3 - x^2 - x + 1$

b) $y = -x^3 + 3x^2$ »

Solution

a) $f'(x) = 3x^2 - 2x - 1$ and $f''(x) = 6x - 2$

At pt of inflection, $f''(x) = 0 \Rightarrow 6x - 2 = 0$

$\Rightarrow \quad x = \frac{1}{3}$

Sub. in $f(x): f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 1$

$= \frac{16}{27}$

$\therefore$ Point of inflection $\left(\frac{1}{3}; \frac{16}{27}\right)$

b) $\frac{dy}{dx} = -3x^2 + 6x$ and $\frac{d^2y}{dx^2} = -6x + 6$

At pt of inflection, $\frac{d^2y}{dx^2} = 0 \Rightarrow -6x + 6 = 0$

$\Rightarrow \quad x = 1$

Sub. $: y = -(1)^3 + 3(1)^2$

$= 2$

$\therefore$ Point of inflection $(1; 2)$

2.1. Solve for x: **[level 2]**

2.1.1. $x^3 - 12x^2 + 36x = 0$

2.1.2. $x^3 - 2x^2 - 4x + 8 = 0$

2.1.3. $x^3 - x^2 - 10x - 8 = 0$

2.2. Determine the coordinates of the turning point of the following functions**: [level 2**]

2.2.1. $h(x) = x^3 - 12x^2 + 36x$

2.2.2. $f(x) = x^3 - 2x^2 - 4x + 8$

2.2.3. $g(x) = x^3 - x^2 - 10x - 8$

2.3. Sketch the following functions: **[level 3]**

2.3.1. $h(x) = x^3 - 12x^2 + 36x$

2.3.2. $f(x) = x^3 - 2x^2 - 4x + 8$

2.3.3. $g(x) = x^3 - x^2 - 10x - 8$

2.3.4. $p(x) = -x^3 - 4x^2 + 3x + 18$

EXAM TYPE QUESTIONS

QUESTION 1

1.1. Given $f(x) = 1 - 4x^2$

1.1.1. Determine $f'(x)$ from first principle (5)

1.1.2. Hence, calculate the gradient of a tangent of $f\ at\ x = 2$ (2)

1.2. Determine:

1.2.1. $\frac{dy}{dx}$ if $y = (2 - x)^2$ (3)

1.2.2. $f'(x)$ if $f(x) = \sqrt[5]{x^2} + \frac{1}{4x^4}$ (4)

QUESTION 2

2.1. Determine $f'(x)$ from first principle if $f(x) = 3x^2 - x$ (5)

2.2. Determine $\frac{dy}{dx}$ if:

2.2.1. $y = (\ x + x^{-2})^2$ (4)

2.2.2. $y = \sqrt[5]{x^4} - \frac{1}{10}x^5$ (3)

2.3. Given: $f(x) = x^2 - \frac{4}{x^2}$.

2.3.1. Determine the gradient of the tangent to *f* at the point where $x = 2$ (3)

2.3.2. Determine the equation of the tangent to f at $x = 2$. (3)

QUESTION 3

3.1. Given: $f(x) = 5 - 2x^2$

3.1.1. Determine $f'(x)$ from first principles. (5)

3.1.2. The line $g(x) = -\frac{1}{8}x + p$ is a tangent to the graph of *f* at the point A. Determine the coordinate of A (4)

3.2. Determine:

3.2.1. $D_x[\ 3x - 3x^2 - \frac{3}{x}]$ (4)

3.2.2. $\frac{dy}{dx}$, if $x = \frac{5}{4} \cdot \sqrt[5]{y^2}$ (2)

3.3. Given: $f(x) = x^3 - 2x^2$

3.4. Determine the equation of the tangent to f at the point where $x = 2$. (6)

QUESTION 4

4.1. If $f(x) = -\frac{3}{x}$

Determine the derivative of f(x) from first principles (4)

4.2. If $f(x) = (x + 3)^2$, determine:

4.2.1. $f'(x)$ (3)

4.2.2. the gradient of the tangent to *f(x)* at $x = 3$ (2)

4.3. Determine:

$$D_x\left[6\sqrt[5]{x^2}+\frac{4}{x}-\pi x^3\right]$$ (4)

4.4. It is given that $f(x)=ax^3-24x+b$ has a local minimum at (-2;17).

Calculate the values of a and b. (4)

QUESTION 5

Given: $f(x)=(x-1)^2(x+3)$

5.1. Determine the turning points of *f*. (5)

5.2. Draw a neat sketch of f showing all intercepts with the axes as well as the turning points (4)

5.3. Determine the coordinates of the point where the concavity of f changes (3)

5.4. Determine the value(s) of k, for which f(x)=k has three distinct roots. (2)

5.5. Determine the equation of the tangent to f that is parallel to the line

$y=-5x$ *if* $x<0$. (6)

Question 6

Given: $f(x)=-x^3+x^2+8x-12$

6.1 Calculate the *x*-intercepts of the graph of *f*. (5)

6.2 Calculate the coordinates of the turning points of the graph of *f*. (5)

6.3 Sketch the graph of *f*, showing clearly all the intercepts with the axes and turning points. (3)

6.4 Write down the *x*-coordinate of the point of inflection of *f*. (2)

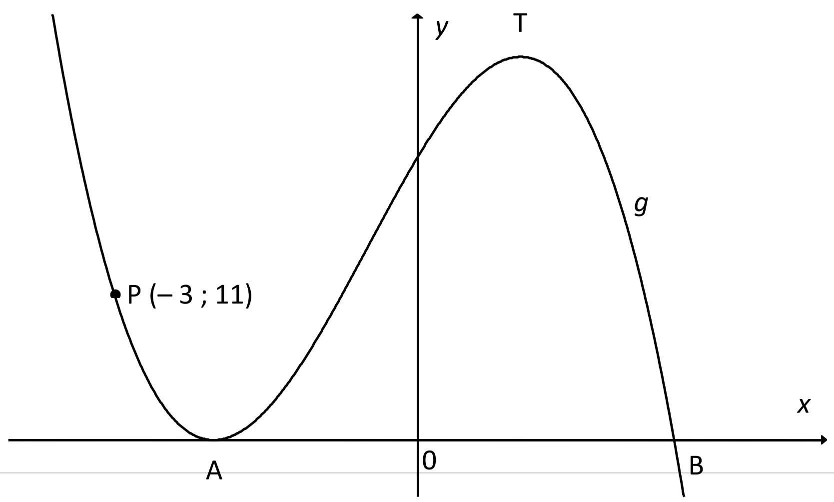
6.5 Write down the coordinates of the turning points of $h(x)=f(x)-3$. (2)

QUESTION 7

Sketched below is the graph of $g(x)=-2x^3-3x^2+12x+20=-(2x-5)(x+2)^2$

A and T are turning points of g. A and B are the x-intercepts of g.

P(– 3 ; 11) is a point on the graph.



7.1 Determine the length of AB. (2)

7.2 Determine the x-coordinate of T. (4)

7.3 Determine the equation of the tangent to g at P(– 3 ; 11), in the form $y = \ldots$ (5)

7.4 Determine the value(s) of k for which $-2x^3 - 3x^2 + 12x + 20 = k$ has three distinct roots. (3)

7.5 Determine the x-coordinate of the point of inflection. (4)

QUESTION 8

A particle moves along a straight line. The distance, s, (in metres) of the particle from a fixed point on the line at time t seconds ($t \geq 0$) is given by $s(t) = 2t^2 - 18t + 45$.

8.1 Calculate the particle's initial velocity. (Velocity is the rate of change of distance.) (3)

8.2 Determine the rate at which the velocity of the particle is changing at t seconds. (1)

8.3 After how many seconds will the particle be closest to the fixed point? (2)

[6]

QUESTION 9

The number of molecules of a certain drug in the bloodstream t hours after it has been taken is represented by the equation $\mathrm{M}(t) = -t^3 + 3t^2 + 72t$, $0 < t < 10$.

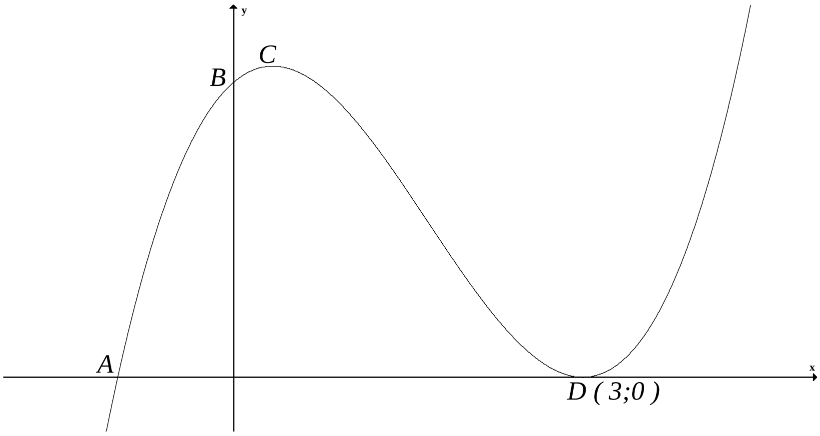
9.1. Determine the number of molecules of the drug in the bloodstream 3 hours after the drug was taken. (2)

9.2. Determine the rate at which the number of molecules of the drug in the bloodstream is changing at exactly 2 hours after the drug was taken. (3)

9.3. How many hours after taking the drug will the rate at which the number of molecules of the drug in the bloodstream is changing, be a maximum? (3)

QUESTION 10

In the diagram, the graph of f (x) = $x^3 - 5x^2 + 3x + 9$ is drawn. A and D (3 ; 0) are the x –intercepts of f and B the y – intercept with the axis respectively. C and D are the turning points of f.



10.1 Determine :

10.1.1 The coordinates of A. (3)

10.1.2 The coordinates of C. (5)

10.1.3 The x – coordinate of the point of inflection of f. (2)

10.2 Use the graph to determine the values of x for which :

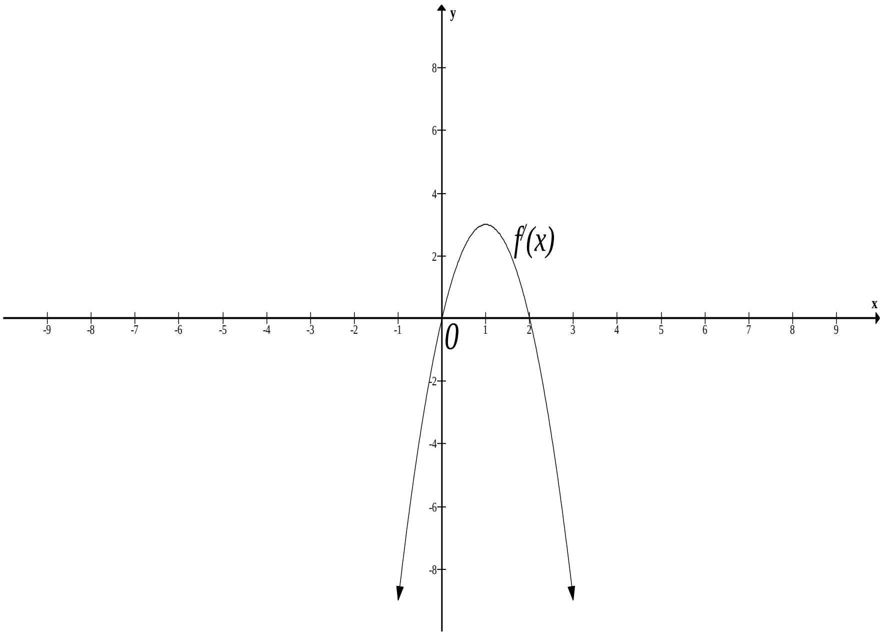
10.2.1 $f'(x) < 0$ (3)

10.2.2 The graph is concave up (2)

10.3 For which values of k will $f(x) = k$ only have one root? (2)

QUESTION 11

11.1 For a certain function $f(x)$, the first derivative is given as $-3x^2 + 6x$



11.1.1 Calculate the coordinates of the stationery points of $f(x)$. (3)

11.1.2 Determine the value of x at the point where the concavity of the curve changes. (1)

11.2 If it is further given that $f(x) = ax^3 + bx^2 + cx - 4$ and $f(2) = 0$. Draw a rough sketch of $f(x)$, clearly showing the coordinates of the turning points. (3)

Probability and the Counting Principle

To the Learner

You must have learnt the following in earlier Grades

Basic Probability

dependent and independent events;

the product rule for independent events: $P(A \text{ and } B) = P(A) \times P(B)$**;**

the sum rule for mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$**;**

the identity: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$**;**

the complementary rule: $P(\text{not } A) = 1 - P(A)$**.**

solving of probability problems (where events are not necessarily independent) by using Venn-diagrams, tree diagrams, two-way contingency tables and other techniques.

In Grade 12 you learnt the following

The fundamental counting principle

application of the Basic Counting Principle to solve probability problems

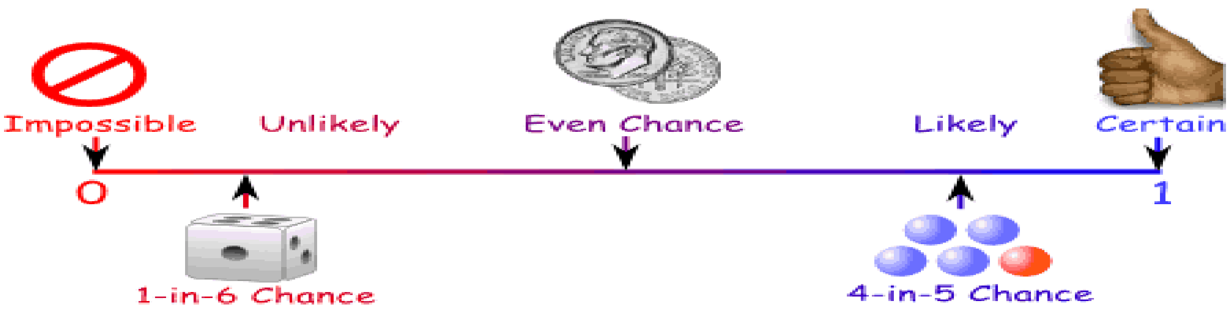
Factorial: n! = n x (n-1) (n-2) x…x3 x 2 x 1 e.g. 4! = 4 x 3 x 2 x1

Basic Probability

Probability = $\frac{\text{Number of ways an event can happen}}{\text{Total number of outcomes}}$

Probability ranges between 0 (impossible event) and 1(certainty) and can be expressed in fractional form $\left(\frac{a}{b}\right)$, decimal or as a percentage e.g. $\frac{1}{4} = 0.25 = 50\%$

The probability line



Example 1: There are 4 King cards in a deck of 52 cards. What is the probability of choosing a King?

Solution: Probability of choosing a King (K) = P(K) = $\frac{4}{52} = \frac{1}{13}$

Example 2

I toss a coin. What is the probability of getting a head?

Solution: P(H) = $\frac{1}{2}$

Example 3

What is the probability of obtaining:

1. exactly two heads in three spins of a coin?
2. a 4 when throwing a dice?
3. a sum of 5 when throwing two dice?

Solution:

1. $E_2 = \{HHT;HTH;THH\}$ P(2 heads) $= \frac{n(E_2)}{n(S)} = \frac{3}{8}$

2. $E_3 = \{4\}$ P(4) $= \frac{n(E_3)}{n(S)} = \frac{1}{6}$

3. $E_4 = \{(1;4);(2;3);(3;2);(4;1)\}$ P(sum of 5) $= \frac{4}{36} = \frac{1}{9}$

INDEPENDENT EVENTS:

If the occurrence of event A cannot affect the outcome of event B then A and B are independent events. e.g. spinning a coin twice … the outcome of the second spin has nothing to do with the outcome of the first spin.

For independent events A and B: **P(A and B) = P(A)$\times$P(B)**

Example: A coin is spun and a dice is thrown. What is the probability that we obtain a head and a 4?

Solution: P(head and 4) = P(head) $\times$ P(4) $= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

MUTUALY EXCLUSIVE EVENTS:

Mutually exclusive events are events which exclude one another. i.e. if A occurs then B cannot occur and vice versa.

Example: If tossing a dice once, and Event A is obtaining a 6 and Event B is obtaining a 2.

Therefore it is impossible to obtain Event A and Event B together.

For Mutually Exclusive Events: **P(A and B) = 0**.

If A and B are mutually exclusive then n(A or B) = n(A) + n(B) hence **P(A or B) = P(A) + P(B)**

Otherwise the general rule applies ie . **P(A or B) = P(A) + P(B)**

COMPLEMENTARY EVENTS:

If a set U= $\{1; 2; 3; 4\ ; 5; 6\}$ and A{1; 2; 3} then the complement of A= A' ={2; 4 ; 6}

Obviously **P(A') = 1 - P(A)** = $\frac{3}{6}$ = also P(A) = 1 – P(A')

THE COUNTING PRINCIPLE AND PROBABILITY:

FUNDAMENTAL COUNTING PRINCIPLE:

It states that if there are **m** ways an event can occur and **n** ways another event can occur then both events can occur in

m x n ways in which the events can occur.

Example 1:

If ice cream comes in 5 different flavours and there are 4 toppings. How many different recipes /combinations can you make?

Solution: 5 x 4 = 20

Example 2:

You want to buy a Toyota car and the advertisement says there are 5 different colours(white , blue, silver, black and red), 3 models(1.6 Prestige, 1.4 Esteem and 1.6 Sprinter) and 2 body types (hatchback & sedan). How many different combinations/options can you choose from?

Solution: 5 x 3 x 2 = 30

Example 3:

A menu for a social event is given below. A person attending a function and he must choose only ONE item from each category, that is starters, main course and dessert.

MENU		
STARTERS	MAIN COURSE	DESSERT
Soup	Fried Chicken	Ice-cream
Salad	Fillet steak	Fruit salad
	Chicken Curry	
	Vegetable Curry	

3.1 How many different meal combinations can be chosen?

3.2 Sipho wishes to have soup as his starter and chicken as his main course. How many different meal combinatios does he have?

Solution:

Starters can happen in two ways, Main Course can happen in 4 ways and Dessert in 2 ways.

3.1 $2 \times 4 \times 2 = 16$ Meal Combiations

3.2 $1 \times 2 \times 2 = 4$ Meal Combinations

Example 4: WHEN REPETATION I`S ALLOWED

How many 4 – Digit Codes (Using digits 0 – 9) are possible if each digit may be used any number of times?

Solution:

In this example, the choice of each digit is regarded as an event. Since there are 4 digits, we have 4 events.

For each event, we have 10 options, (0, 1, 2, 3, 4, 5, 6, 7, 8 or 9):

$\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$

$10 \times 10 \times 10 \times 10 = 10\ 000$ Possibilities

$= 10^4$ Possibilities

Number of Possibilities = $\boxed{n^r}$

n = the number of options to choose from

r = the number of times we choose from these options

Example 5: WHEN REPETION IS NOT ALLOWED

When choosing froma group of different objects multiple times, and repetition is not allowed, the number of objects we choose from is reduced by 1 each time a selection is made.

How many 4 digit codes (using digits 0 – 9) are possible if no digit may be used more than once?

Solution:

We will start with 10 options for the first digit and then reduce it by 1 for each digit after that:

$\underline{10}$ $\underline{9}$ $\underline{8}$ $\underline{7}$

$10 \times 9 \times 8 \times 7 = 5040$ Possibilities.

Example 6:

How many ways can the letters of the word ACTION be arranged,

6.1 letters may not repeat.

6.2 letters may repeat.

Solution:

6.1 There are 6 letters in the word ACTION. We will start wit 6 options and reduce it by 1 for each digit after that.

$\underline{6}$ $\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

= 720 Possibilities.

6.2 There are 6 letters in the word ACTION. There are 6 possible letters that can be used for the first letter. For the second letter there are still 6 letters that can be used since repeating letters is allowed. Therefore,

$\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$ $\underline{6}$

$6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^6 = 46\,656$

Example 7:

Consider the letters of the word NEEDED. How many word arrangements can be made with this word?

Solution:

The two letters E (3 times) and D (2 times) are repeated.

The total possible arrangements is:

$\frac{6!}{3! \times 2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60$

Example 8:

In a school there are 8 prefects, 5 Girls and 3 Boys. They are seated on the stage for an assembly.

1. How many different ways can they be seated on the stage?
2. How many ways can they be seated if all boys and all girls are to sit next to each other?
3. If the prefects are randomly seated what is the probability of all the boys and girls being seated next to each other?
   (In other words all the boys together and all the girls together.)

Solution:

1.

SEAT	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
CHOICES	8	7	6	5	4	3	2	1

Number of different seating arrangements = $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 8\,! = 40\ 320$

2. In Order to solve tis problem we will treat boys as one seat and the girls as another seat.

SEAT	**BOYS**			**GIRLS**				
CHOICES	3	2	1	5	4	3	2	1

3 Boys can be arranged in 3! and Girls can be arranged in 5! ways.

Boys and Girls can swop places, Girls could be seated on the left and Boys on the right, therefore there

are 2! Ways groups of boys and girls can be arranged.

$\therefore\ Number\ of\ ways\ Boys\ and\ Girls\ can\ be\ seated\ together = 3!\ \times 5!\ \times 2! = 1440$

3. It is important to remember that the probability of an event E occurring is:

$$\boldsymbol{P(E) = \frac{n(E)}{n(S)}}$$

$\therefore\ P(all\ the\ boys\ and\ girls\ are\ seated\ together) = \frac{1440}{40320} = \frac{1}{28} = 3{,}57\%$

MIXED PRACTICE EXERCISES WITH SOLUTIONS:

QUESTION 1

1.1 A three-digit code is made up of the numbers 4,7 or 9. How many different codes are possible if

1.1.1 The codes are repeated (1)

1.1.1 Each digit is used only once. (1)

1.2 Six different Mathematics books and five different Physical Sciences books have to be arranged on a shelf. How many arrangements can be made if all the Mathematics books are put together and all the Physical Sciences are put together? (3)

1.3 Survey shows that most people are carriers of a deadly disease. 100 people are tested and the findings are that 3 out of 45 females and 7 males tested positive.

Determine:

1.3.1 The probability that a male from those tested is positive (1)

1.3.2 The number of males in SA, with a population estimated at 50 million who are likely to be positive (2)

1.3.3 The probability that the person randomly tested is positive, given that the person is a female. (2)

1.3.4 The probability that the person randomly tested is a male given that he is positive. (3)

[13]

QUESTION 2

The events A and B are independent. P(A) = 0, 4 and P (B) = 0, 5

Determine:

2.1 P (A and B) (2)

2.2 P (A or B) (2)

2.3 P (not A and not B) (3)

[7]

QUESTION 3

Three boys and four girls go to the cinema. They are seated together in one row as a group.

3.1 Calculate the total number of possible ways of arranging the whole group. (2)

3.2 Determine the total number of ways of arranging the group if the girls want to be seated together. (3)

3.3 Calculate the probability that the girls are seated together. (1)

3.4 Determine the probability that they will all be seated girl, boy, girl, boy etc. until all seven are seated. (3)

[9]

QUESTION 4

The table below represents the favourite sport for 120 learners.

	Boys	**Girls**	**Total**
Golf	32	12	44
Tennis	18	23	41
Squash	20	15	35
Total	**70**	**50**	**120**

4.1 Determine the probability that (if a learner is selected at random):

4.1.1 the learner who is selected, is a boy who plays tennis or squash. (2)

4.1.2 a learner who plays tennis is selected. (1)

4.1.3 a learner who is a girl is selected. (1)

4.2 Are the events of choosing golf as a favourite sport and being a boy, independent? Show ALL calculations to support your answer. (4)

[8]

QUESTION 5

5.1 Five South Africans, three British citizens, four Germans and two Americans arrive for a congress.

5.1.1 In how many ways can they be arranged if they sit next to each other in a row? (1)

5.1.2 In how many ways can they be arranged so that those of the same nationality sit together in a row? (2)

5.2 Consider the letters of the word: PROBABILITY

5.2.1 How many unique word arrangements can be made from this word if the repeated letters are treated as identical? (3)

5.2.2 How many unique word arrangements can be made if the word ends with the letter R? (2)

[8]

QUESTION 6

6.1 If $P(A) = \frac{3}{8}$ and $P(B) = \frac{1}{4}$, find:

6.1.1 $P(A \text{ or } B)$ if A and B are mutually exclusive events. (1)

6.1.2 $P(A \text{ or } B)$ if A and B are independent events. (3)

6.2 A car park has 14 VOLKSWAGEN cars and 18 BMW's. There are no other cars. During the afternoon two cars are stolen – one early afternoon, the other later. Determine the probability that:

6.2.1 Both cars were BMW's. (4)

6.2.2 The first one stolen was a BMW and the second one a Volkswagen. (2)

6.3 Eight boys and seven girls are to be seated randomly in a row. What is the probability that:

6.3.1 The row has a girl at each end? (3)

6.3.2 The row has girls and boys sitting in alternate positions? (3)

[16]

QUESTION 7

7.1 Consider the letters of the word **MERAFONG**

7.1.1 How many unique word arrangements can be made using all the letters? The letters may repeat. (2)

7.1.2 Calculate the probability that the word arrangement will start with a letter R and end with a letter N. (2)

7.1.3 Calculate the probability that the vowels (a, e, o) will be placed next to each other in any of the word arrangements. (3)

7.2 The probability that during a given weekend a PSL team player will play in an international game (S) is 0,7. The probability that the player will play in a charity golf tournament (G) during that weekend is 0,4 and the probability that the player will participate in both activities is 0,3.

7.2.1 Draw a Venn diagram to represent the given information. (4)

7.2.2 Calculate:

a) $P(S')$ (1)

b) $P(S \text{ and } G)'$ (1)

c) $P(S \text{ or } G)$ (1)

[14]

QUESTION 8

8.1 A study on eating chocolate and gender yielded the following results.

	Eating Chocolate	**Not Eating Chocolate**	**TOTAL**
Male	45	25	70
Female	35	45	80
TOTAL	80	70	150

8.1.1 How many people participated in this study? (1)

8.1.2 Calculate the following probabilities:

(a) P (male) (1)
(b) P (Eating Chocolate) (1)

8.1.3 Are the events being a male and eating chocolate independent? Justify your answer with relevant calculations. (3)

8.2 Four - digit codes (not beginning with 0), are to be constructed from the set of digits $\{1 ; 3 ; 4 ; 6 ; 7 ; 8 ; 0\}$.

8.2.1 How many four - digit codes can be constructed, if repetition of digits is allowed? (2)

8.2.2 How many four - digit codes can be constructed, if repetition of digits is not allowed? (2)

8.2.3 Calculate the probability of randomly constructing a four - digit code which is divisible by 5 if repetition of digits is allowed. (3)

[13]

QUESTION 9

9.1 M and K are two events. If P(K) = 0, 5; P (K and M) = 0, 2 and P (not K and not M) = 0, 3

9.1.1 Draw a Venn diagram to represent the above events. (4)

9.1.2 Are events M and K independent or not. Explain. (3)

9.2 Koketso and Marvin are to sit with five other friends in a church in one row. There are seven empty chairs in that row.

9.2.1 In how many different ways can they all sit together? (2)

9.2.2 In how many different ways can they sit if Koketso and Marvin should not sit next to each other? (4)

9.2.3 Koketso first sat on a chair at one end of the row. He now decides to change his initial position. What is the probability that he will sit on a chair at the other end of the row. (2)

[15]

QUESTION 10

At a certain technical school all Grade 9 learners have to choose subjects by the end of the year. All Grade 9 learners have to choose between Mathematics and Technical Mathematics. Thereafter they have to choose between Electrical Technology, Mechanical Technology and Civil Technology. The probability that a learner from this school will choose Mathematics is $\frac{4}{7}$. If a learner chooses Mathematics as a subject, the probability that he will choose Electrical Technology as a subject is $\frac{5}{10}$ and the probability that he will choose Mechanical Technology is $\frac{3}{10}$. If a learner chooses Technical Mathematics, the probability that he will choose Electrical Technology as a subject is $\frac{4}{10}$ and the probability that he will choose Mechanical Technology is $\frac{5}{10}$.

10.1 Draw a tree diagram to represent the above information. Indicate on your diagram the probabilities associated with each branch as well as all the outcomes. (4)

10.2 If a learner is selected randomly, what is the probability that the learner:

10.2.1 will choose Technical Mathematics and Mechanical Technology. (2)

10.2.2 will choose Electrical Technology as a subject. (3)

[9]

QUESTION 11

11.1 The digits 0 to 9 are used to form codes.

11.1.1 Determine the number of different 6-digit codes that can be formed if repetition of digits is allowed. (1)

11.1.2 Determine the number of 6-digit codes that can be formed that starts with a 9 and ends with a 2 if repetition of digits is not allowed. (2)

11.2 The digits 0 to 9 are used to form 10-digit codes. Determine the number of 10-digit codes that can be formed if the 2 and the 3 may not appear next to each other and if repetition of digits is not allowed. (3)

[6]

QUESTION 12

12.1 A group of 540 people with green or blue eyes were randomly selected in order to determine whether or not green or blue eyes are dependent on gender. The results are tabulated below:

	Male	**Female**	**Total**
Green eyes	183	147	**330**
Blue eyes	117	93	**210**
Total	**300**	**240**	**540**

12.1.1 If a person is selected at random, determine the probability that it will be a female with green eyes. (2)

12.1.2 After analysing the results, a grade 11 learner concludes that the probability of having green eyes is independent of gender. Is he correct? Substantiate your answer with relevant calculations. Give all answers correct to 2 decimal places. (5)

12.2 The letters in the word CURRICULUM are rearranged to form other words. Assume that all words have meaning.

12.2.1 How many different letter arrangements (words) are possible? (3)

12.2.2 What is the probability that a word will start and end with the letter U? (4)

EXAM QUESTIONS FROM PREVIOUS PAPERS

DBE/ Nov 2010

QUESTION 1

A school organised a camp for their 103 Grade 12 learners. The learners were asked to indicate their food preferences for the camp. They had to choose from chicken, vegetables and fish.

The following information was collected:

- 2 learners do not eat chicken, fish or vegetables
- 5 learners eat only vegetables
- 2 learners only eat chicken
- 21 learners do not eat fish
- 3 learners eat only fish
- 66 learners eat chicken and fish
- 75 learners eat vegetables and fish

Let the number of learners who eat chicken, vegetables and fish be x.

1.1	Draw an appropriate Venn diagram to represent the information.		(7)
1.2	Calculate x.		(2)
1.3	Calculate the probability that a learner, chosen at random:		
	1.3.1	Eats only chicken and fish, and no vegetables.	(2)
	1.3.2	Eats any TWO of the given food choices: chicken, vegetables and fish.	(2) **[13]**

DBE /Nov 2011

QUESTION 5

The digits 0, 1, 2, 3, 4, 5 and 6 are used to make 3 digit codes.

5.1	How many unique codes are possible if digits can be repeated?	(2)
5.2	How many unique codes are possible if the digits cannot be repeated?	(2)
5.3	In the case where digits may be repeated, how many codes are numbers that are greater than 300 and exactly divisible by 5?	(3) **[7]**

QUESTION 6

Complaints about a restaurant fell into three main categories: the menu (M), the food (F) and the service (S). In total 173 complaints were received in a certain month. The complaints were as follows:

- 110 complained about the menu.
- 55 complained about the food.
- 67 complained about the service.
- 20 complained about the menu and the food, but not the service.
- 11 complained about the menu and the service, but not the food.
- 16 complained about the food and the service, but not the menu.
- The number who complained about all three is unknown.

6.1 Draw a Venn diagram to illustrate the above information. (6)

6.2 Determine the number of people who complained about ALL THREE categories. (3)

6.3 Determine the probability that a complaint selected at random from those received, complained about AT LEAST TWO of the categories (that is. menu, food and service). (3)

[12]

DBE/Nov 2016

QUESTION 11

A survey was conducted among 100 boys and 60 girls to determine how many of them watched TV in the period during which examinations were written. Their responses are shown in the partially completed table below.

	WATCHED TV DURING EXAMINATIONS	DID NOT WATCH TV DURING EXAMINATIONS	TOTALS
Male	80	a	
Female	48	12	
Totals	b	32	160

11.1 Calculate the values of a and b. (2)

11.2 Are the events 'being a male' and 'did not watch TV during examinations' mutually exclusive? Give a reason for your answer. (2)

11.3 If a learner who participated in this survey is chosen at random, what is the probability that the learner:

11.3.1 Watched TV in the period during which the examinations were written? (2)

11.3.2 Is not a male and did not watch TV in the period during which examinations were written? (2)

[8]

DBE / NOV 2016/ QUESTION 12:

The digits 1 to 7 are used to create a four-digit code to enter a locked room. How many different codes are possible if the digits may not be repeated and the code must be an even number bigger than 5 000? **[5]**

QUESTION 4

4.1 A survey of 80 students at a local library indicated the reading preferences below:

44 read the *National Geographic* magazine

33 read the *Getaway* magazine

39 read the *Leadership* magazine

23 read both *National Geographic* and *Leadership* magazines

19 read both *Getaway* and *Leadership* magazines

9 read all three magazines

69 read at least one magazine

4.1.1 How many students did not read any magazine? (1)

4.1.2 Let the number of students who read *National Geographic* and *Getaway*, but not *Leadership*, be represented by x. Draw a Venn diagram to represent reading preferences. (5)

4.1.3 Hence show that $x = 5$. (3)

4.1.4 What is the probability, correct to THREE decimal places, that a student selected at random will read at least two of the three magazines? (3)

4.2 A smoke detector system in a large warehouse uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0,95. The probability that it will be detected by device B is 0,98 and the probability that it will be detected by both devices simultaneously is 0,94.

4.2.1 If smoke is present, what is the probability that it will be detected by device A or device B or both devices? (3)

4.2.2 What is the probability that the smoke will not be detected? (1)

[16]

QUESTION 6

The data below was obtained from the financial aid office at a certain university.

	RECEIVING FINANCIAL AID	NOT RECEIVING FINANCIAL AID	TOTAL
Undergraduates	4 222	3 898	8 120
Postgraduates	1 879	731	2 610
TOTAL	6 101	4 629	10 730

6.1 Determine the probability that a student selected at random is ...

6.1.1 receiving financial aid. (2)

6.1.2 a postgraduate student and not receiving financial aid. (2)

6.1.3 an undergraduate student and receiving financial aid. (2)

6.2 Are the events of being an undergraduate and receiving financial aid independent? Show ALL relevant workings to support your answer. (4)

[10]

QUESTION 7

Consider the digits 1, 2, 3, 4, 5, 6, 7 and 8 and answer the following questions:

7.1 How many 2-digit numbers can be formed if repetition is allowed? (2)

7.2 How many 4-digit numbers can be formed if repetition is NOT allowed? (3)

7.3 How many numbers between 4 000 and 5 000 can be formed? (3)

[8]

QUESTION 12

12.1 $P(A) = \frac{1}{4}$ and $P(A \text{ or } B) = \frac{1}{3}$. Determine P(B), as a simplified fraction, if:

12.1.1 A and B are mutually exclusive events. (3)

12.1.2 A and B are independent events. (5)

12.2 Consider the word: EINSTEIN.

12.2.1 How many different letter arrangements are possible if all the letters are used? (2)

12.2.2 What is the probability that identical letters will be grouped together? (4)

[14]

Data Handling

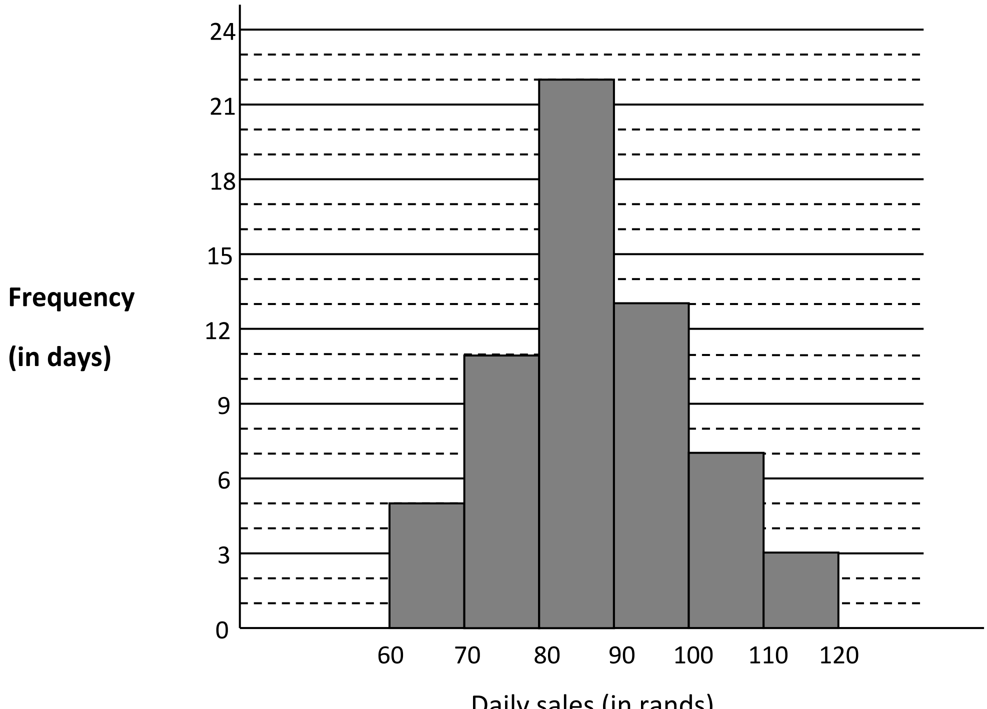
HINTS TO LEARNERS:

1. Clarity with respect to
   1.1 the concepts *MEAN* and *MEDIAN*
   1.2 an understanding of the concept *STANDARD DEVIATION*
   1.3 the meaning of the standard deviation away from the *MEAN*
   1.4 calculation (use of correct calculator keys)
2. Familiarise yourself on how to use the calculator for statistical functions.
3. Understand clearly the number of data within one, two or three standard deviation of the mean.
4. Learners must attempt every question in this section, and it is possible to easily score marks.

Example 1

A street vendor has kept a record of sales for November and December 2007.

The daily sales in rands is shown in the histogram below.



(a) Complete the cumulative frequency table for the sales over November and December.

(b) Draw an ogive for the sales over November and December.

(c) Use your ogive to determine the median value for the daily sales. Explain how you obtain your answer.

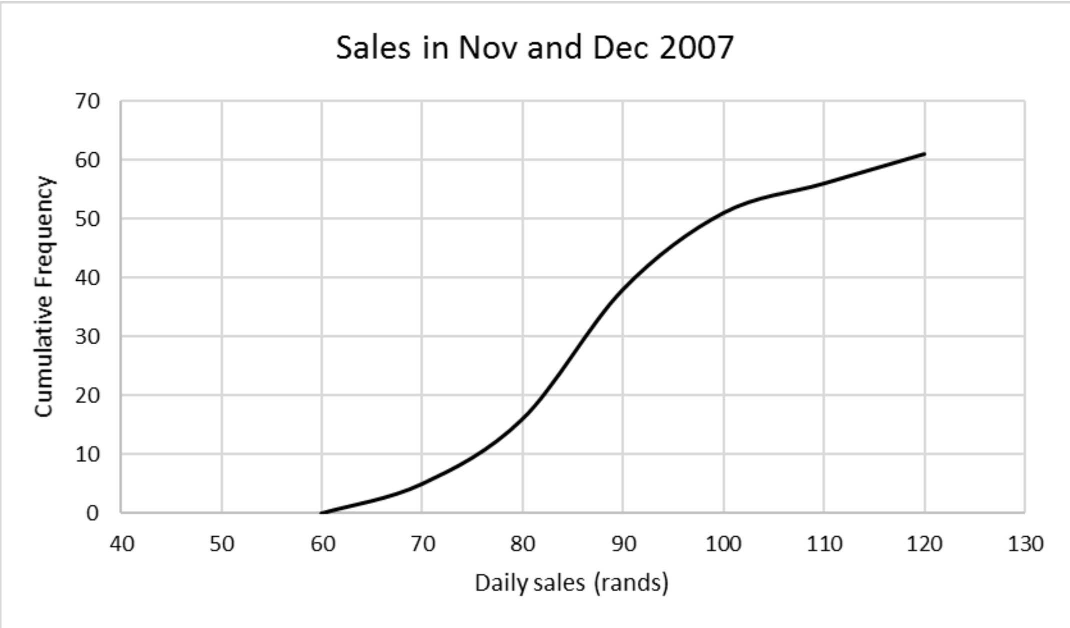
(d) Estimate the interval of the upper 25% of the daily sales.

Solutions:

(a)

Daily sales (in Rand)	Frequency	Cumulative Frequency
60 £ *rand* < 70	5	5
70 £ *rand* < 80	11	16
80 £ *rand* < 90	22	38
90 £ *rand* < 100	13	51
100 £ *rand* < 110	7	58
110 £ *rand* < 120	3	61

(b



Hints:

- x-coordinate - use upper limit of each interval
- y-coordinate – cumulative frequency
- if the frequency of the first interval is not 0, then include an interval before the given one and use 0 as its frequency

(c) *Median = R87. There are 61 data points, so the median in on the 31st position. On the y-axis put a ruler at 31; move horizontally until you touch the graph, then move vertically down to read the x-coordinate.*

(d) *The upper 25% lies above 75%. 75% of 61 = 45,75. Read 45,75 from the y-axis across to the graph and down to the x-axis. Therefore the upper 75% of sales lies in the :*

96 £ *rand* < 120

Example 2:

The data below shows the energy levels, in kilocalories per 100 g, of 10 different snack foods.

440 520 480 560 615 550 620 680 545 490

(a) Calculate the mean energy level of these snack foods.
(b) Calculate the standard deviation.
(c) The energy levels, in kilocalories per 100 g, of 10 different breakfast cereals had a mean of 545,7 kilocalories and a standard deviation of 28 kilocalories. Which of the two types of food show greater variation in energy levels? What do you conclude?

Solution

(a) Mean $= \dfrac{5500}{10} = 550$

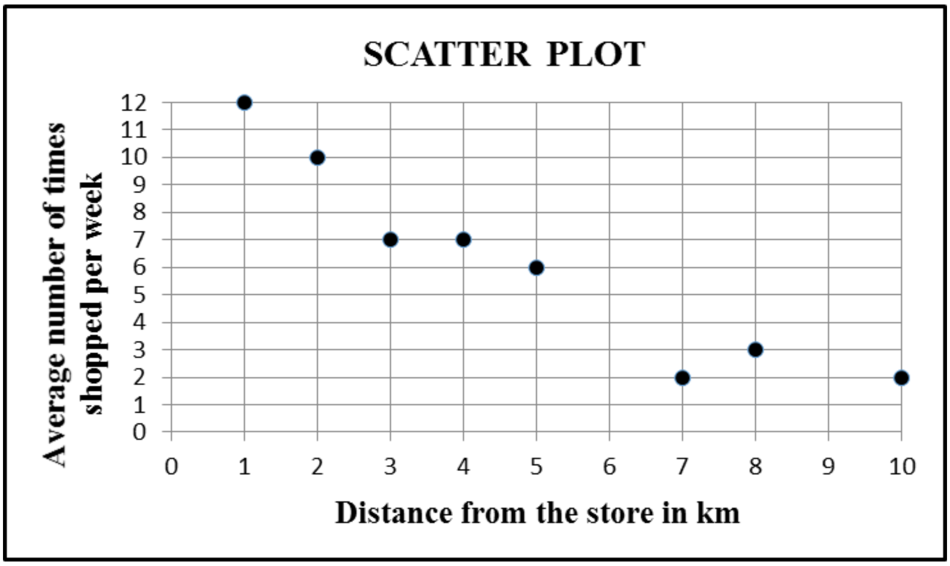
(b) $\sigma = 69{,}03$ kilocalories

(c) Snack foods have a greater variation. The standard deviation for snack foods is 69,03 kilocalories whilst the standard deviation for breakfast cereals is 28 kilocalories. i.e. energy levels of breakfast cereals is spread closer to the mean than in those of the snack food.

Example 3: Nov 2016

A survey was conducted at a local supermarket relating the distance that shoppers lived from the store to the average number of times they shopped at the store in a week. The results are shown in the table below.

Distance from the store in km	1	2	3	4	5	7	8	10
Average number of times shopped per week	12	10	7	7	6	2	3	2



(a) Use the scatter plot to comment on the strength of the relationship between the distance a shopper lived from the store and the average number of times she/he shopped at the store in a week.

(b) Calculate the correlation coefficient of the data.

(c) Calculate the equation of the least squares regression line of the data.

(d) Use your answer at QUESTION (c) to estimate the average number of times that a shopper living 6 km from the supermarket will visit the store in a week.

(e) Sketch the least squares regression line on the scatter plot.

(a) *Strong*

(b) $r = -0{,}95\ (-0{,}9462..)\ldots$

Using CASIO fx 82 ZA plus calculator

- Mode
- 2: STAT
- 2: A + B x
- Enter the x values first:
  1 =; 2 =; 3 =; 4 =; 5 =; 7=; 8=; 10=
- Use arrows to move right to y column and up to start next to 1.
- Enter y values:
  12 =; 10 =; 7 =; 7=; 6=; 2=; 3=; 2=
- Press (orange) AC button
- Press SHIFT STAT (at 1)
- Press 5: Reg

Press 3: r = and get $r = -0{,}95\ (-0{,}9462..)\ldots$

To get equation of regression line:

- Press (orange) AC button
- Press SHIFT STAT (at 1)
- Press 5: Reg
- Press 1: A = and get 11,7132…

This is the y-intercept of the regression line

- Press orange AC button
- Press SHIFT STAT
- Press 5: Reg
- Now press 2: B = and get -1,1176…

This is the gradient of the regression line

Answer:

The least squares regression line:

$\hat{y} = -1{,}12x + 11{,}71$ (correct to 2 decimal places)

Using SHARP EL-W53HT

- Mode
- 1: STAT
- 1: LINE
- Enter the values in coordinate form:
- 1 (x,y) 12change; 2 (x,y) 10 change;
- 3 (x,y) 7 change; 4 (x,y) 7 change;
- 5 (x,y) 6 change; 7 (x,y) 2 change;
- 8 (x,y) 3 change; 10 (x,y) 2 change

Press On: It goes back to Stat 1 (LINE)
Press ALPHA (÷): r _ appears on the screen
Press =: the value of r appears on the screen.

To get equation of regression line:
Press ALPHA(a): a appears on the screen
Perss =: the value of a appears 11,7132…
This is the y-intercept of the regression line

To get equation of regression line:
Press ALPHA(b): b appears on the screen
Perss =: the value of b appears -1,1176…
This is the gradient of the regression line

Answer:

The least squares regression line:

$\hat{y} = -1{,}12x + 11{,}71$ (correct to 2 decimal places)

(c) $a = 11{,}71\ \ (11{,}7132\ldots)$

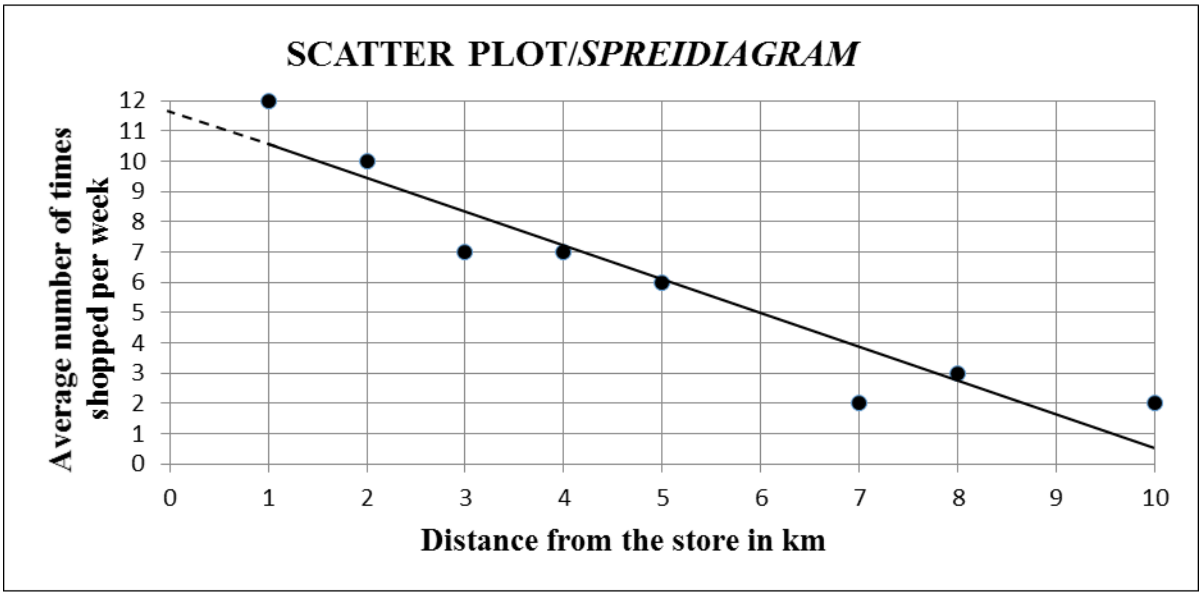
$b = -1{,}12\ \ (-1{,}1176\ldots)$

$\hat{y} = -1{,}12x + 11{,}71$

(d) $\hat{y} = -1{,}12(6) + 11{,}71$

$= 5$ times

(e)



<u>EXERCISES</u>

QUESTION 1-MAY/JUNE 2016

On a certain day a tour operator sent 11 tour buses to 11 different destinations. The table below shows the number of passengers on each bus.

8	8	10	12	16	19	20	21	24	25	26

1.1 Calculate the mean number of passengers travelling in a tour bus. (2)

1.2 Write down the five number summary of the data. (3)

1.3 Draw a box and whisker diagram for the data. (3)

1.4 Refer to the box and whisker diagram and comment on the skewness of the data set. (1)

1.5 Calculate the standard deviation of this data set. (2)

1.6 A tour is regarded as popular if the number of passengers on a tour bus is one standard deviation above the mean. How many destinations were popular on this particular day? (2)

QUESTION 2- JUNE/JULY 2015

The data below shows the ages in years of people who visited the library between 08:00 and 09:00 on a certain morning.

3	4	4	5	23	29	32	36	40	47	56	66	68	76	82

2.1 Determine:

2.1.2 The mean of the visitors. (2)

2.1.2 The median of the data. (1)

2.1.3 The interquartile range of the data (3)

2.2 Draw a box and whisker diagram for the data. (3)

2.3 By making reference to the box and whisker diagram, comment on the skewness of the data set. (1)

3.1 Complete the cumulative frequency table for the sales over November and December. (2)

3.2 Draw an ogive for the sales over November and December. (3)

3.3 Use your ogive to determine the median value for the daily sales. Explain how you obtain your answer. (2)

3.4 Estimate the interval of the upper 25% of the daily sales. (2)

The data below shows the energy levels, in kilocalories per 100 g, of 10 different snack foods.

440 520 480 560 615 550 620 680 545 490

5.1 Calculate the mean energy level of these snack foods. (2)

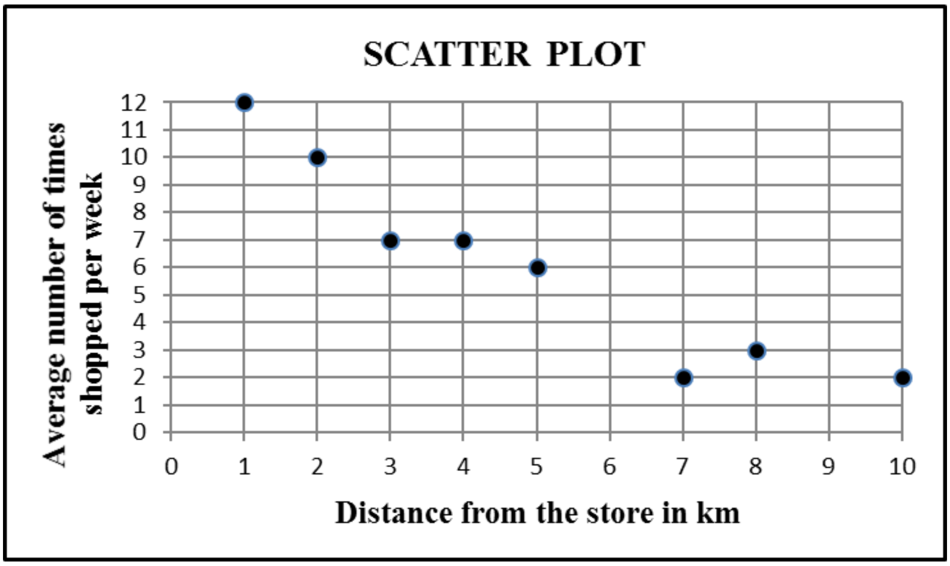
5.2 Calculate the standard deviation. (2)

5.3 The energy levels, in kilocalories per 100 g, of 10 different breakfast cereals had a mean of 545,7 kilocalories and a standard deviation of 28 kilocalories. Which of the two types of food show greater variation in energy levels? What do you conclude? (2)

QUESTION 6 –NOV 2016

A survey was conducted at a local supermarket relating the distance that shoppers lived from the store to the average number of times they shopped at the store in a week. The results are shown in the table below.

Distance from the store in km	1	2	3	4	5	7	8	10
Average number of times shopped per week	12	10	7	7	6	2	3	2



6.1 Use the scatter plot to comment on the strength of the relationship between the distance a shopper lived from the store and the average number of times she/he shopped at the store in a week. (1)

6.2 Calculate the correlation coefficient of the data. (1)

6.3 Calculate the equation of the least squares regression line of the data. (3)

6.4 Use your answer at QUESTION 6.3 to estimate the average number of times that a shopper living 6 km from the supermarket will visit the store in a week. (2)

6.5 Sketch the least squares regression line on the scatter plot. (2)

QUESTION 7-Feb/ March 2016

A company recorded the number of messages sent by e-mail over a period of 60 working days. The data is shown in the table below.

NUMBER OF	NUMBER OF DAYS	
$10 < x \leq 20$	2	
$20 < x \leq 30$	8	
$30 < x \leq 40$	5	
$40 < x \leq 50$	10	
$50 < x \leq 60$	12	
$60 < x \leq 70$	18	
$70 < x \leq 80$	3	
$80 < x \leq 90$	2	

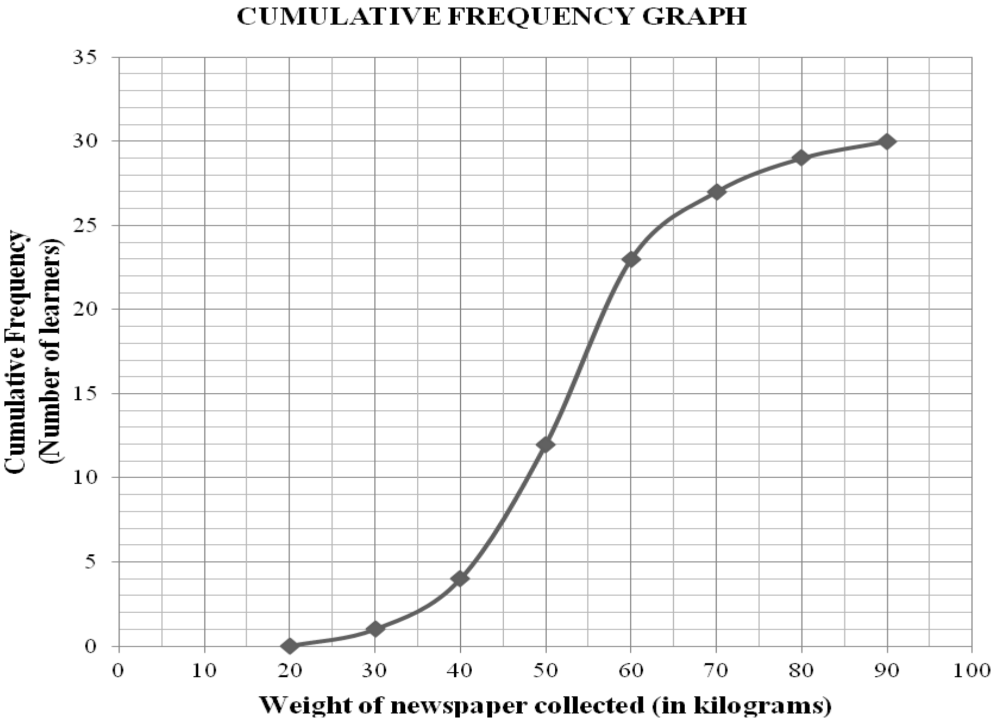
7.1 Estimate the mean number of messages sent per day, rounded off to TWO decimal places. (3)

7.2 Draw a cumulative frequency graph (ogive) of the data. (3)

7.3 Hence, estimate the number of days on which 65 or more messages were sent. (2)

QUESTION 8

As part of an environmental awareness initiative, learners of Greenside High School were requested to collect newspapers for recycling. The cumulative frequency graph (ogive) below shows the total weight of the newspapers (in kilograms) collected over a period of 6 months by 30 learners.



8.1 Determine the modal class of the weight of the newspapers collected. (1)

8.2 Determine the median weight of the newspapers collected by this group of learners. (2)

8.3 How many learners collected more than 60 kilograms of newspaper? (2)

QUESTION 9- NOV 2015

A group of 30 learners each randomly rolled two dice once and the sum of the values on the uppermost faces of the dice was recorded. The data is shown in the frequency table below.

Sum of the values	Frequency
2	0
3	3
4	2
5	4
6	4
7	8
8	3
9	2
10	2
11	1
12	1

9.1 Calculate the mean of the data. (2)

9.2 Determine the median of the data. (2)

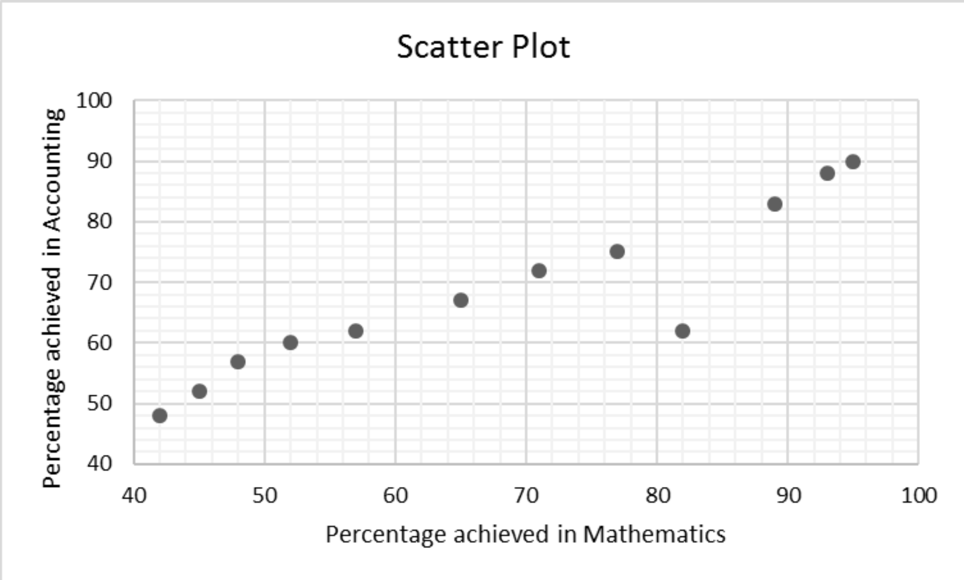
9.3 Determine the standard deviation of the data. (2)

9.4 Determine the number of times that the sum of the recorded values of the dice is within ONE standard deviation from the mean. Show your calculations. (3)

QUESTION 10

At a certain school only 12 candidates take Mathematics and Accounting. The marks, as a percentage, scored by these candidates in the preparatory examinations for Mathematics and Accounting are shown in the table and scatter plot below.

Mathematics	52	82	93	95	71	65	77	42	89	48	45	57
Accounting	60	62	88	90	72	67	75	48	83	57	52	62



10.1 Calculate the mean percentage of the Mathematics data. (2)

10.2 Calculate the standard deviation of the Mathematics data. (2)

10.3 Determine the number of candidates whose percentages in Mathematics lie within ONE standard deviation of the mean. (2)

10.4 Calculate the equation of the least squares regression line for the data. (3)

10.5 If a candidate from this group scored 60% in the Mathematics examination but was absent for the Accounting examination, predict the percentage that this candidate would have scored in the Accounting examination, using your equation in QUESTION 10.4. (Round off your answer to the NEAREST INTEGER. (2)

10.6 Use the scatter plot and identify any outlier(s) in the data. (1)

QUESTION 11

A restaurant wants to know the relationship between the number of customers and the number of chicken pies that are ordered.

number of customers (x)	5	10	15	20	25	30	35	40
number of chicken pies (y)	3	5	10	10	15	20	20	24

11.1 Determine the equation of the regression line correct to two decimal places. (3)

11.2 Determine the value of r, the correlation coefficient. Describe the type and strength of the correlation between the number of people and the number of chicken pies ordered. (2)

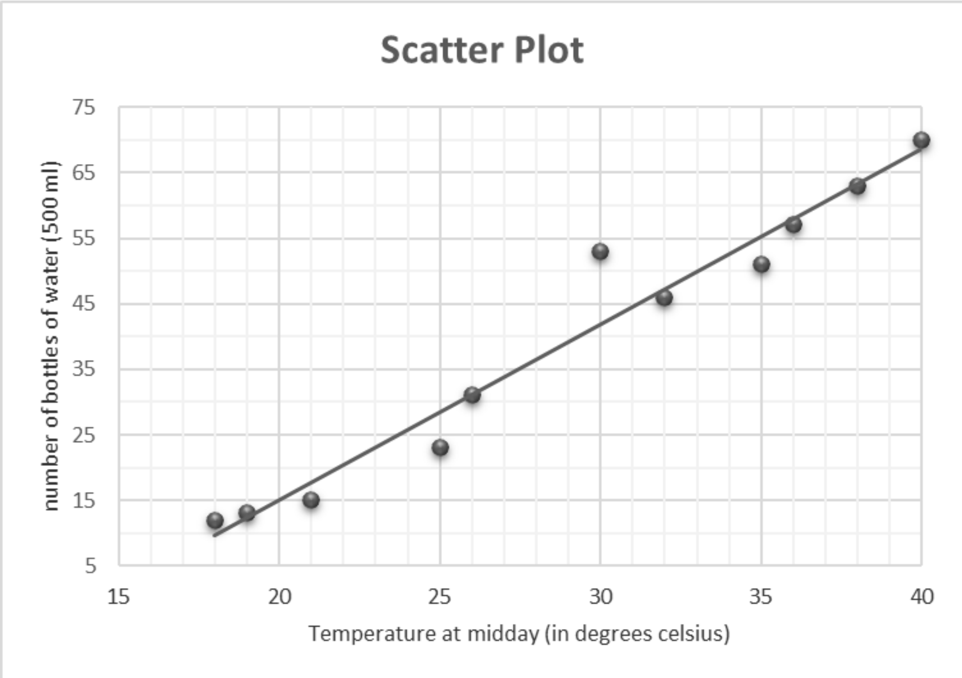
11.3 Determine how many chicken pies 100 people would order. (2)

11.4 If they only have 12 pies left, how many people can they serve? (2)

QUESTION 12

On the first school day of each month information is recorded about the temperature at midday (in ^{0}C) and the number of 500 *ml* bottles of water that were sold at the tuck shop of a certain school during the lunch break. The data is shown in the table below and represented on the scatter plot. The least squares regression line for this data is drawn on the scatter plot.

Temperature at midday (in ^{0}C)	18	21	19	26	32	35	36	40	38	30	25
Number of bottles of water (500 *ml*)	12	15	13	31	46	51	57	70	63	53	23



12.1 Identify an outlier in the data. (1)

12.2 Determine the equation of the least squares regression line. (3)

12.3 Estimate the number of 500 *ml* bottles of water that will be sold if the temperature is 28^{0}C at midday. (2)

12.4 Refer to the scatter plot. Would you say that the relation between the temperature at midday and the number of 500 *ml* bottles of water sold is weak or strong? Motivate your answer. (2)

12.5 Give a reason why the observed trend for this data cannot continue indefinitely. (1)

QUESTION 13

The number of learners absent from 11 weekend classes in a year were recorded as follows.

10 13 15 17 18 23 24 26 28 28 29

13.1 Determine the range of the above data. (1)

13.2 Calculate the average number of the learners absent from a weekend class. (2)

13.3 Calculate the standard deviation of the above data (3)

13.4 Determine the number of weeks where the attendance of the learners lies outside one standard deviation from the mean. (2)

[8]

QUESTION 14- FS 2018

The tuck shop sells cans of soft drinks. The Environmental Club decided to have a can-collection project for three weeks to make learners aware of the effects of litter on the environment. The data below shows the number of cans collected on each school day of the three week project.

58	83	85	89	94
97	98	100	105	109
112	113	114	120	145

14.1 Determine the lower and upper quartiles of the data. (2)

14.2 Use the scaled line in the ANSWER BOOK to draw a box and whisker diagram for this set of data. (3)

14.3 Comment on the skewness in the distribution of the data. (1)

14.4 Calculate the mean number of cans collected over the three week period. (2)

14.5 Calculate the standard deviation of the number of cans collected. (1)

14.6 On how many days did the number of cans collected lie outside one standard deviation of the mean? Show all calculations. (3)

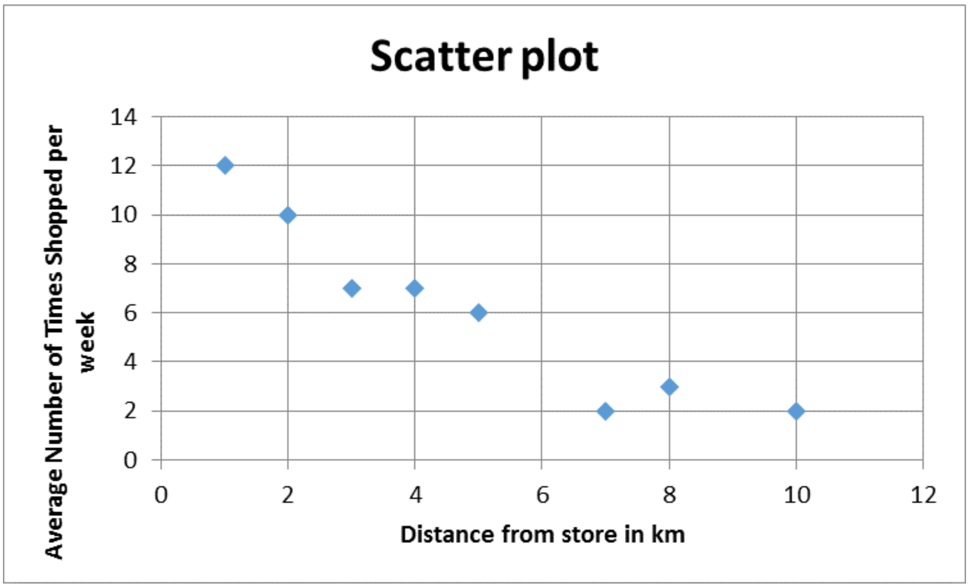
[12]

QUESTION 15

A survey was conducted at a local supermarket to establish the relationship between the distance (in kilometres) that shoppers stay from the store and the average number of times that they shop at the store in a week. The results are shown in a table below.

Distance from store *in km*	1	2	3	4	5	7	8	10
Average number of times shopped per week	12	10	7	7	6	2	3	2

The above data is represented in a scatter plot below.



15.1	Calculate the correlation coefficient of the data.	(2)
15.2	Comment on the strength of the relationship between the distance a shopper lived from the store and the average number of times she/he shopped at the store in a week.	(1)
15.3	Calculate the equation of the least squares regression line of the data.	(3)
15.4	Use your answer in QUESTION 16.3 to predict the average number of times that a shopper living 6 *km* from the supermarket will visit the store in a week.	(2)

ANALYTICAL (COORDINATE) GEOMETRY

LEARNING HINTS

Mathematical language and terminology must be learnt in more detail

1. *Learners should then follow the method laid out below:*

   - Select the correct formula from the data sheet
   - Label the ordered pairs using the correct two points, $e.g.\quad A \text{ and } B.$
   - Substitute correctly and accurately into your chosen formula and **use brackets** where necessary to avoid operations that require isolated expressions and negative signs.
   - Emphasis on the application of distributive law of multiplication.
     $e.g -(x+1) = -x-1.$

2. Often Analytical Geometry questions *follow on*, (scaffolding). Look out for that, as you might have already calculated or proven an aspect before, that you will require for the next sub-question :
   - Even if you failed to show or prove in the previous question, accept that as true in the follow up questions.
3. *Use the diagram more effectively.*
   $e.g.$ Highlight the sides you are going to use for proving perpendicular lines, so you can see clearly which points you are going to use for the substitution.

   You must answer the question, and remember to conclude, exactly what you were asked to show / prove / conclude.

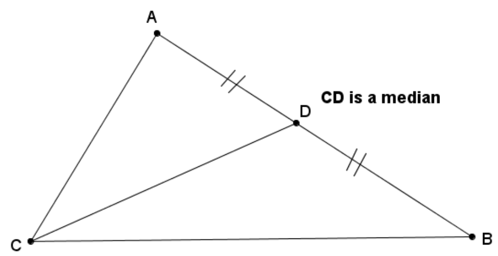
4. *Learners need to know the properties of all geometric figures e.g. triangles and quadrilaterals*
5. Learners need to be able to determine whether a particular point is inside, outside or on the circle by comparing that distance and the radius.
6. Practice exercises are often required to teach the above points.
7. Grade 11 work must ***NOT*** be ignored, *e.g.*

NOTE:

Always **refer to a diagram** when doing problems involving Analytical Geometry. A diagram helps you to visualise the problem accurately.

Median: The median of a triangle bisects the opposite side of a triangle.

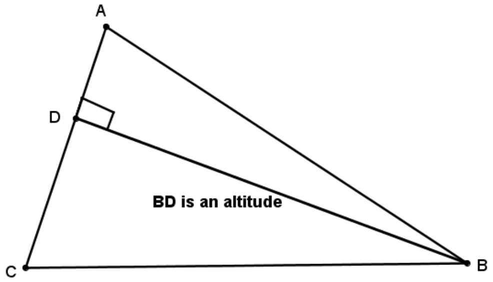
The median of a triangle bisects the area of a triangle.



To find the equation of the median:

- Determine the coordinates of D using the formula for the midpoint
- Use the coordinates of C and D to find the equation.

Altitude: The altitude of a triangle is perpendicular to the opposite side of a triangle.



To find the equation of an altitude:

- Determine the gradient AC= m_{AC}
- Determine the m_{BD} using the fact that $m_{BD} \times m_{AC} = -1$
- Use the coordinates of B and m_{BD} to find the equation.

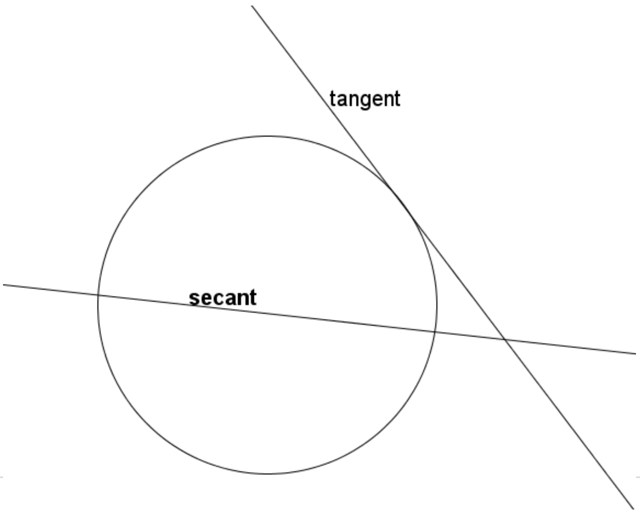
Perpendicular bisector: The perpendicular bisector of a line segment is also found in a triangle.

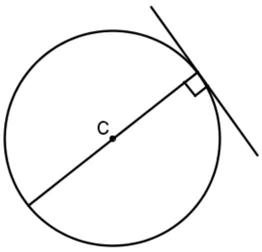
To find the equation of the perpendicular bisector:

- Determine the coordinates of M using the midpoint formula.
- Find the gradient AB i.e. m_{AB}
- Determine m_{DP} using the fact that DP⊥AB $\therefore m_{DP} \times m_{AB} = -1$
- Use the coordinates of M and m_{DP} to find the equation.

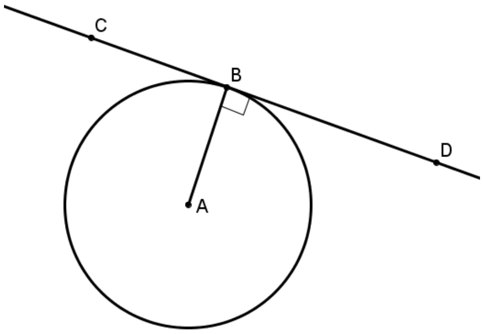
CRITICAL NOTES OF CONCEPTS INTERGRATION:

From Euclidean Geometry: the angle between the tangent and the diameter at the point of contact is 90° (A radius is perpendicular to a diameter at the point of contact)

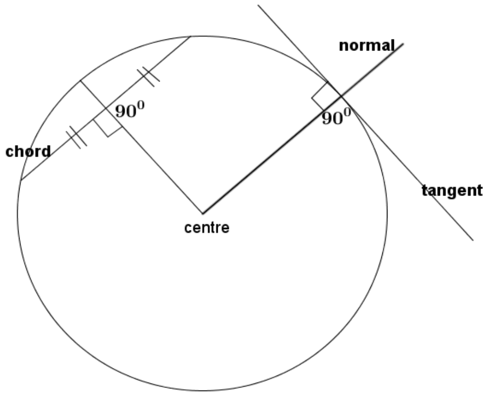




The equation of the tangent to a circle

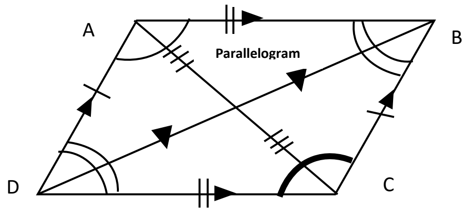
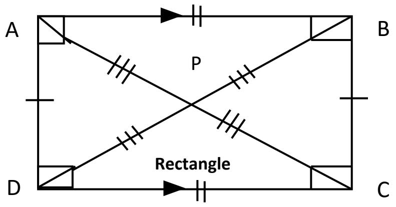
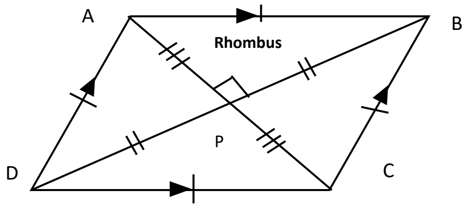
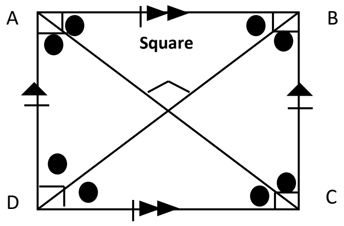
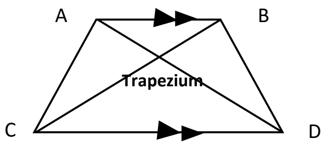
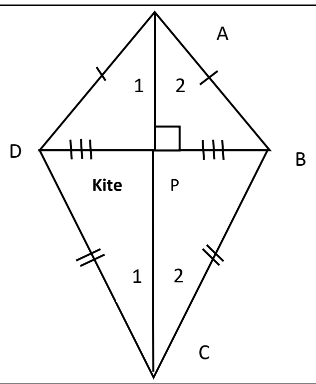


- Determine the gradient AB, m_{AB}
- Determine the gradient CD, using AB⊥CD
- Use the gradient CD and the point B to find the equation CD
  The following need to be revised and be remembered for the purposes of understanding and answering questions in Analytical Geometry.



- The line from the centre of the circle perpendicular to a chord bisects the chord.
- The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- A tangent to a circle is perpendicular to the radius at the point of contact.

PROPERTIES OF QUADRILATERALS

	1. Opposite sides parallel. 2. Opposite sides equal. 3. Opposite angles are equal. 4. Co-interior angles on same side of a transversal are supplementary. 5. Diagonals bisect each other.
	1. All properties of parallelogram 2. Has 4 right angles. 3. Diagonals are equal.
	1. All properties of parallelograms. 2. Has 4 equal **sides,** 3. Diagonals bisect opposite angles. 4. Diagonals each other at right angle.
	1. All properties of parallelogram, rectangle, and rhombus 2. 4 equal sides and 4 equal (right) angles.
	1. One pair of parallel sides.
	1. 2 pairs of adjacent sides equal. 2. 1 pair of opposite angles equal 3. The long diagonal bisects the short one at right angle. 4. Diagonals bisect opposite angles. $\hat{A}_1 = \hat{A}_2$ $\hat{C}_1 = \hat{C}_2$

FUNDAMENTAL COMPETENCE

Represent geometric figures on a Cartesian co-ordinate system by understanding that a point is determined by coordinates in the form of $(x_1;y_1),(x_2;y_2),(x_3;y_3)$, etc.

Use a Cartesian co-ordinate system to apply the following formulae:

(a) the **distance** between the two points:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

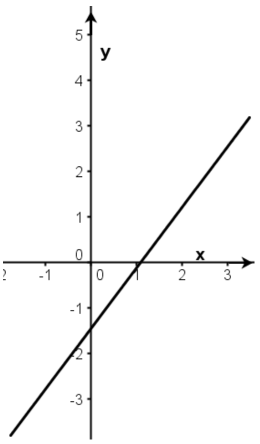
(b) the **gradient of the line segment** joining the points (including collinear points) or the **inclination** of a line.

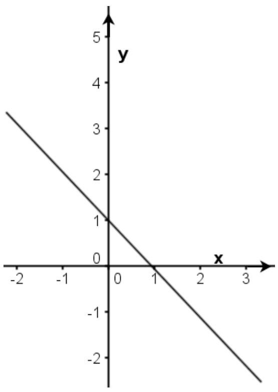
$m=\dfrac{y_2-y_1}{x_2-x_1}$ or $\boldsymbol{m=\tan\theta}$ where $\boldsymbol{\theta}$ **is the anlge that a line makes with a positive direction** of the $x\ axis$.

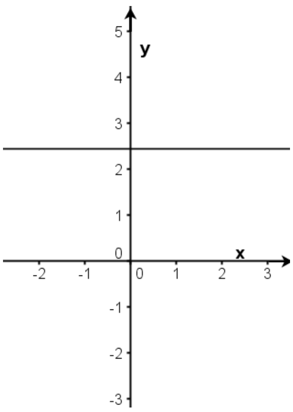
- *The lines joining **collinear points** have the **same gradient**.*

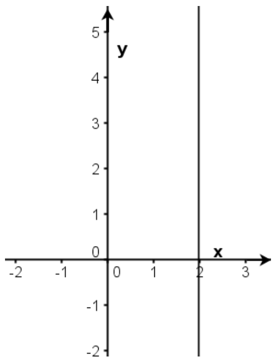
- The gradient or slope of a straight line through the points A$(x_1;y_1)$ and B$(x_2;y_2)$ in which $x_1\neq x_2$ is given by: Gradient of AB$=m_{AB}=\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_1}{x_2-x_1}=\frac{increase\ in\ y}{increase\ in\ x}$

$\boldsymbol{m>0}$	$\boldsymbol{m<0}$	$\boldsymbol{m=0}$	$\boldsymbol{m}$**- undefined**
$\boldsymbol{\tan\theta>0}$	$\boldsymbol{\tan\theta<0}$	$\boldsymbol{\tan\theta=0}$	$\boldsymbol{\tan\theta}$ **is undefined**
		Horizontal line	**Vertical line**









Perpendicular and Parallel Lines

- The product of the gradients of perpendicular lines is **-1,** i.e. $m_1\times m_2=-1$
- $m_1\times m_2=-1$ cannot be applied when one of the lines is parallel to the y −axis
- When two lines are parallel, then $m_1=m_2$

If AB∥PQ, then $m_{AB}=m_{PQ}$ (AB and PQ not parallel to the $y-axis$)

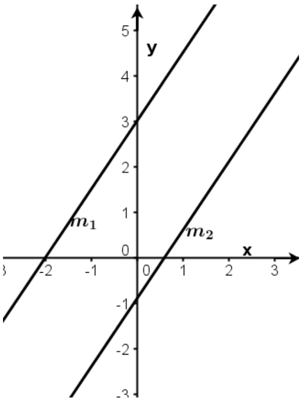
If $m_{AB}=m_{PQ}$,then AB∥PQ

If AB⊥AQ, then $m_{AB} \times m_{AQ} = -1$ (AB and PQ not parallel to the $y - axis$)

If $m_{AB} \times m_{AQ} = -1$,then AB⊥AQ

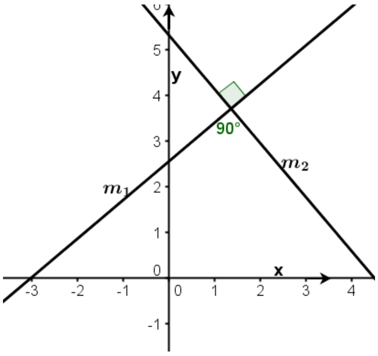
Parallel Lines

$\boldsymbol{m_1 = m_2}$



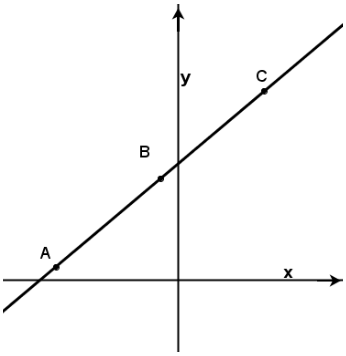
Perpendicular Lines

$\boldsymbol{m_1 \times m_2 = -1}$



Collinear Points

$\boldsymbol{m_{AB} = m_{BC}}$



(c) the co-ordinates of the **mid-point** of the line segment joining the points:

$$\left(\frac{x_1 + x_2}{2};\frac{y_1 + y_2}{2}\right)$$

(d) the equation of a line **through two given points** and the equation of a line through **one point and parallel or perpendicular** to a given line:

$y - y_1 = m(x - x_1)$ or $y = mx + c$

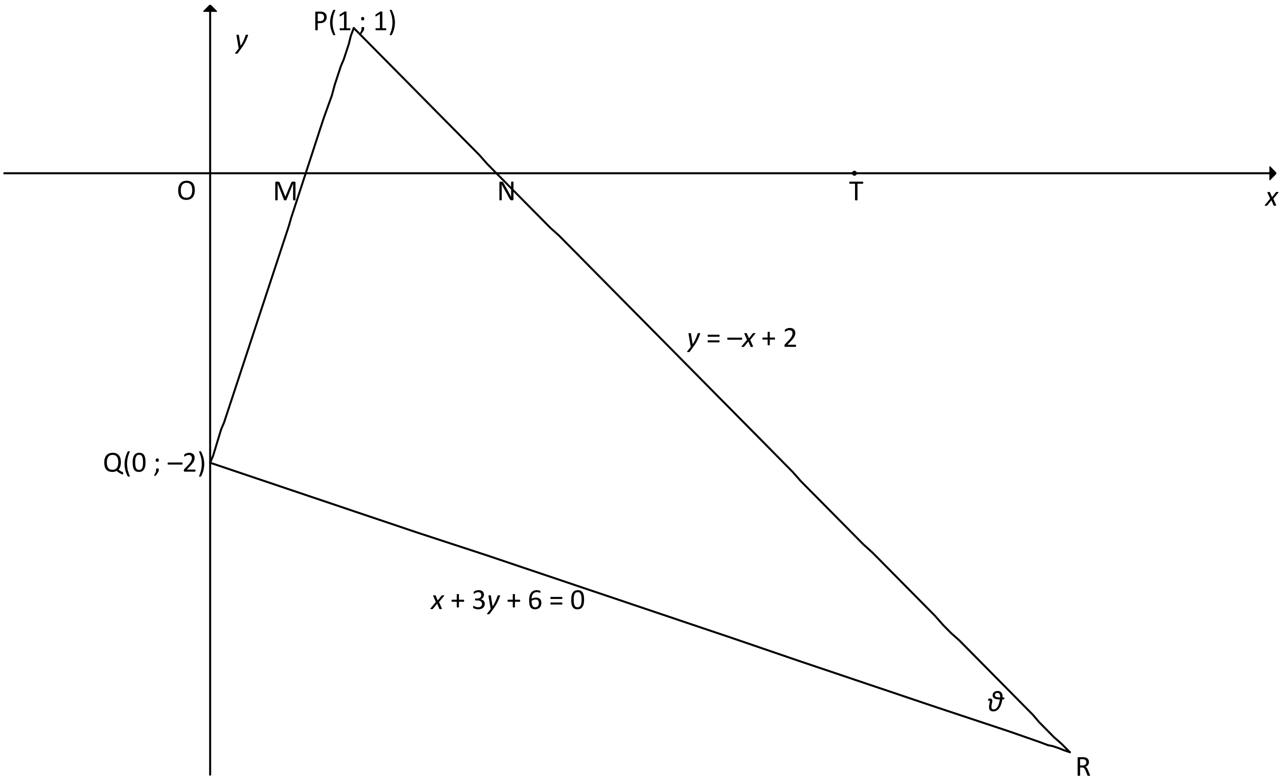
viz. : If $AB \parallel CD$, then $m_{AB} = m_{CD}$

If $AB \perp CD$, then $m_{AB} \times m_{CD} = -1$

WORKED EXAMPLES (METHODOLOGY)

ALL THE ACTIVITIES IN BRACKETS 1 AND 2 ARE INFORMED BY THE FUNDAMENTALS OF THE FOLLOWING WORKED EXAMPLES:

In the diagram below, P(1 ; 1), Q(0 ; –2) and R are the vertices of a triangle and $P\hat{R}Q = \theta$. The x-intercepts of PQ and PR are M and N respectively. The equations of the sides PR and QR are $y = -x + 2$ and $x + 3y + 6 = 0$ respectively. T is a point on the x-axis, as shown.



1.1 Determine the gradient of QP.

Solution:

$$m_{PQ} = \frac{1-(-2)}{1-0}$$

$$= 3$$

1.2 Prove that $P\hat{Q}R = 90°$.

Solution:

QR: $y = -\frac{1}{3}x - 2$

$$\therefore m_{QR} = -\frac{1}{3}$$

$m_{\mathrm{PQ}} \times m_{\mathrm{QR}} = 3 \times -\frac{1}{3}$

$= -1$

$\therefore \mathrm{PQ} \perp \mathrm{QR} \qquad \therefore \mathrm{P\hat{Q}R} = 90°$

1.3 Determine the coordinates of R.

Solution:

$-\frac{1}{3}x - 2 = -x + 2$

$\frac{2}{3}x = 4$

$x = 6$

$y = -4$

$\therefore \mathrm{R}(6 ; -4)$

1.4 Calculate the length of PR. Leave your answer in surd form.

Solution

$\mathrm{PR} = \sqrt{(1-6)^2 + (1-(-4))^2}$

$= \sqrt{50} = 5\sqrt{2}$

1.5 Determine the equation of a circle passing through P, Q and R in the form $(x-a)^2 + (y-b)^2 = r^2$.

Solution

PR is a diameter [chord subtends 90°]

Centre of circle$l$: $\left(\frac{1+6}{2} ; \frac{1-4}{2}\right)$

$= \left(3\frac{1}{2} ; -1\frac{1}{2}\right)$

$r = \frac{\sqrt{50}}{2}$ OR $\frac{5\sqrt{2}}{2}$ OR 3,54

$\therefore \left(x - \frac{7}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{50}{4}$ OR $\frac{25}{2}$ OR 12,5

1.6 Determine the equation of a tangent to the circle passing through P, Q and R at point P in the form $y = mx + c$.

Solution

m of radius = –1

$\therefore m$ of tangent = 1

Equation of tangent:

$$y - y_1 = (x - x_1)$$
$$y - 1 = x - 1$$

$\therefore y = x$

1.7 Calculate the size of θ.

Solution:

$\tan \mathrm{P\hat{N}T} = m_{\mathrm{PR}} = -1$

$\therefore \mathrm{P\hat{N}T} = 135°$

$\tan \mathrm{P\hat{M}T} = m_{\mathrm{PQ}} = 3$

$\therefore \mathrm{P\hat{M}T} = 71{,}57°$

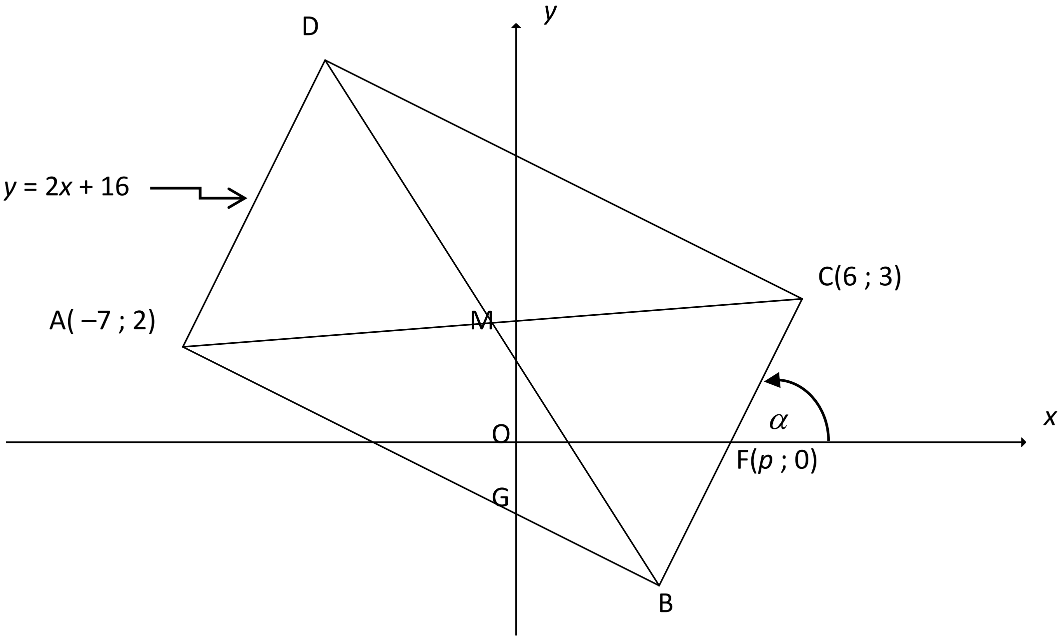
$\hat{\mathrm{P}} = 63{,}43°$ [ext ∠ of Δ]

$\therefore \theta = 26{,}57°$ [sum of ∠s in Δ]

QUESTION 2

In the diagram, A (–7 ; 2), B, C(6 ; 3) and D are the vertices of rectangle ABCD.

The equation of AD is y = 2x + 16. Line AB cuts the y-axis at G. The x-intercept of line BC is F(p ; 0) and the angle of inclination of BC with the positive x-axis is α . The diagonals of the rectangle intersect at M.



2.1 Calculate the coordinates of M.

2.2 Write down the gradient of BC in terms of p.

2.3 Hence, calculate the value of p.

2.4 Calculate the length of DB.

2.5 Calculate the size of α.

2.6 Calculate the size of $O\hat{G}B$

2.7 Determine the equation of the circle passing through points D, B and C in the form

$(x-a)^2+(y-b)^2=r^2$

2.1 $\left(-\frac{1}{2};\frac{5}{2}\right)$

2.2 $-\frac{3}{p-6}$

2.3 $p=4\frac{1}{2}$

2.4 $\sqrt{170}$

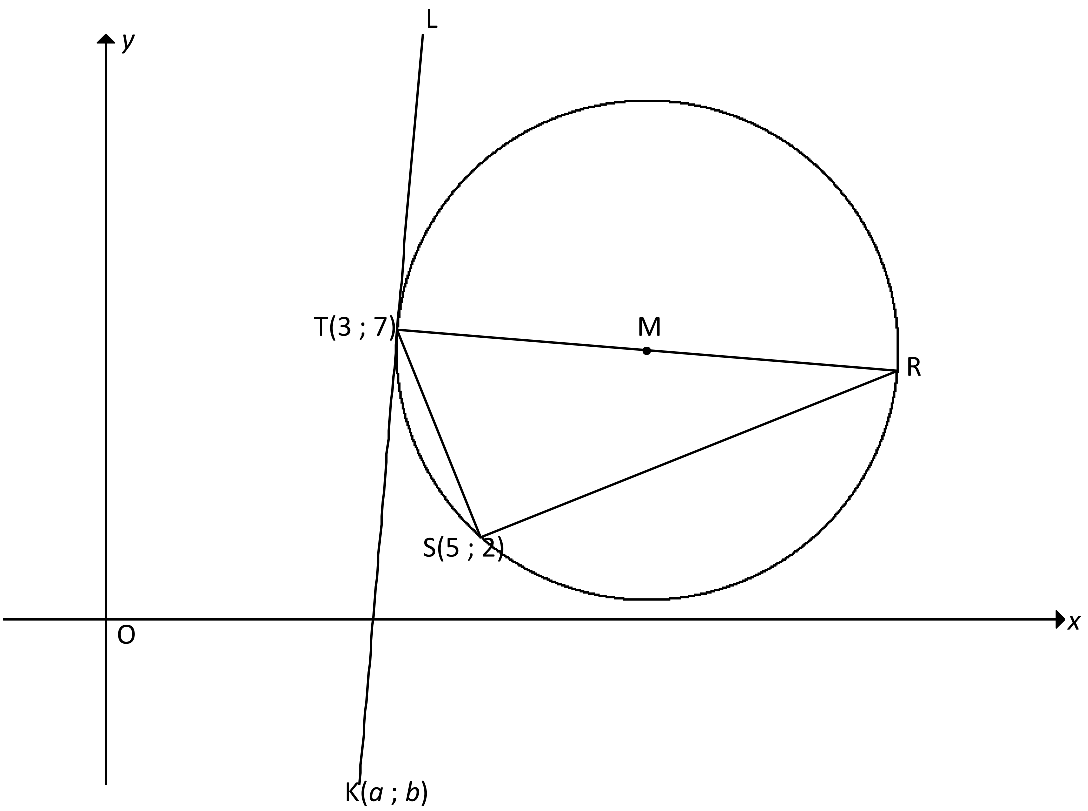
2.5 $\alpha = 63{,}43^0$

2.6 $116{,}57^0$

2.7 42.5

QUESTION 4

In the diagram, M is the centre of the circle passing through T(3 ; 7), R and S(5 ; 2). RT is a diameter of the circle. K(a ; b) is a point in the 4th quadrant such that KTL is a tangent to the circle at T.



4.1 Give a reason why $T\hat{S}R = 90°$.

4.2 Calculate the gradient of TS.

4.3 Determine the equation of the line SR in the form y = mx + c.

4.4 The equation of the circle above is $(x-9)^2 + \left(y - 6\frac{1}{2}\right)^2 = 36\frac{1}{4}$.

4.4.1 Calculate the length of TR in surd form.

4.4.2 Calculate the coordinates of R.

4.4.3 Calculate $\sin R$.

4.4.4 Show that b = 12a – 29.

4.4.5 If TK = TR, calculate the coordinates of K.

ANSWERS

4.1 $\angle$ in semi-circle/ $\angle$ at centre = 2$\angle$ on circle

4.2 $-\frac{5}{2}$

4.3 $y=\frac{2}{5}x$

4.4.1 $\sqrt{145}$

4.4.2 $R(15;6)$

4.4.3 $\frac{\sqrt{5}}{5}$

4.4.4 $b=12a-29$

4.4.5 $k(2;-5)$

1^ST^ BRACKET OF ACTIVITIES

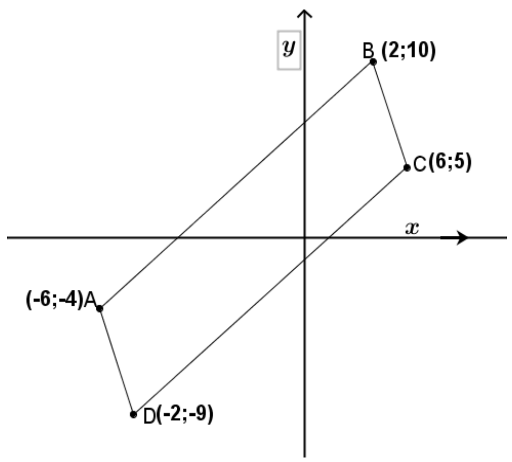
1. Determine the distance between the points given below. Leave answers in surd form.

(a) $(3;0)$ and $(0;3)$

(b) $(4;0)$ and $(2;3)$

2. The distance between A (x; 15) and B (-7; 3) is **13 units**. Calculate the **possible values** of x.

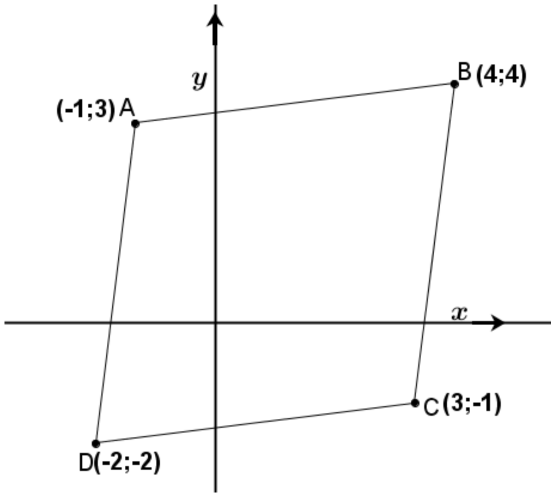
3. Given the following diagram:



(a) Prove that **ABCD** is a parallelogram using the lengths of the sides.

(b) Prove that **ABCD** is a parallelogram using the diagonals.

4. In the diagram below, ABCD is rhombus:

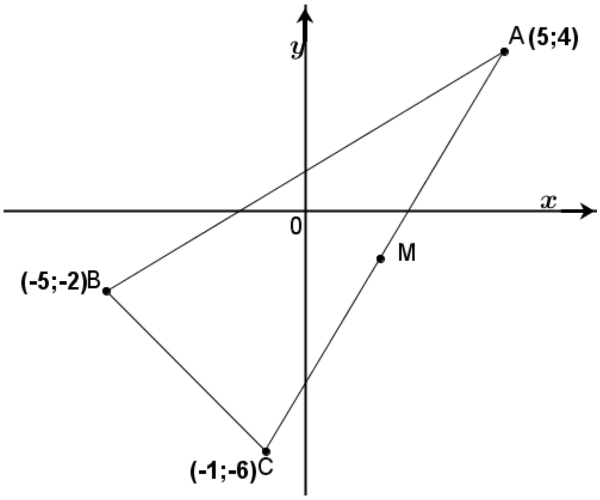


4.1 Determine the gradients of AC and BD.

4.2 Show that AB = DC and AD = BC

5. Find the 4th vertex of a parallelogram PQRT, if the three given vertices are P $(6; -3)$, Q$(3; 3)$ and C $(-2; 1)$

6. Use the diagram below to answer the questions that follow:



(a) Calculate the coordinates of M the midpoint of AC.

(b) Determine the gradient BC.

(c) Determine the equation of the line parallel to BC that passes through M.

(d) Give the coordinates of P, the midpoint of AB.

(e) Calculate the length of BC.

(f) Prove that: $BC = 2PM$.

7. The points A $(-5; 9)$, B $(-3; y)$ and C $(2; -5)$ are given.

(a). Determine the value of y if A; B and C are collinear.

(b). Determine the value of y if BC⊥ AC

8. Calculate the value of t, if AB is parallel to the line which passes through the points

C$(2; 3)$ and

D$(-2; -5)$ if A$(-3; t)$ and B$(0; -2)$

9. Prove that the following points are on the same line.

N$(-7; -8)$, A$(-1; 2)$ and G$(2; 7)$

10. Find the missing coordinates if these points are collinear.

H $(-3; 3)$, O$(-1; y)$, B$(2; -7)$

11. Determine the equation of a straight line passing through the points:

(a) $(3; 7)$ and $(-6; 1)$

(b) $(8; t)$ and $(t; 8)$

12. Determine the equation of the straight line:

(a) passing through the point $(-1; \frac{10}{3})$ and with gradient $m = \frac{2}{3}$

(b) parallel to the x −axis and passing through the point $(0; 11)$

(c) perpendicular to the x −axis and passing through the point $(-\frac{3}{2}; 0)$

(d) with undefined gradient and passing through the point $(4; 0)$

(e) with $m = 3a$ and passing through the point $(-2; -6a + b)$

13. Determine whether or not the following two lines are parallel:

(a) $y + 2x = 1$ and $-2x + 3 = y$

(b) $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$

(c) $y = 2x - 7$ and the line passing through $(1; -2)$ and $(\frac{1}{2}; -1)$

(d) $y + 1 = x$ and $x + y = 3$

14. Determine the equation of the straight line that passes through the point $(1; -5)$ and is parallel to the line $y + 2x - 1 = 0$

15. Determine the equation of the straight line that passes through the point $(-2; \frac{3}{5})$ and is parallel to the line with the angle of inclination $\theta = 145^\circ$

16. Determine whether or not the following two lines are perpendicular.

(a) $y - 1 = 4x$ and $4y + x + 2 = 0$

(b) $10x = 5y - 1$ and $5y - x - 10 = 0$

(c) $x = y - 5$ and the line passing through $(-1; \frac{5}{4})$ and $(3; -\frac{11}{4})$

(d) $y = 2$ and $x = 1$

(e) $\frac{y}{3} = x$ and $3y + x = 9$

17. Determine the equation of a straight line that passes through the point $(-2; -4)$ and is perpendicular to the line $y + 2x = 1$

18. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with an angle of inclination $\theta = 135^\circ$.

For tan$\theta = m$ where $m \geq 0$,$\theta = tan^{-1}m$

19. Determine the gradient correct to one decimal place of each of the following straight lines, given that the angle of inclination is equal to:

(a) 60° (b) 135° (c) 0° (d) 54° (e) 90°

20. Calculate the angle of inclination correct to one decimal place for each of the following:

(a) a line with $m = \frac{3}{4}$

(b) $2y - x = 6$

(c) the line passes through the points A $(-4; -1)$ and B $(2; 5)$

(d) $y = 4$

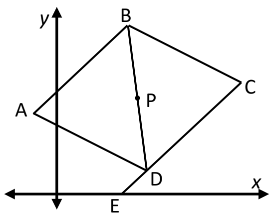
21. Determine the equation of a straight line passing through the point $(3; 1)$ and with an angle of inclination equal to $135°$

22. Determine the acute angle between the line passing through the points A $(-2; \frac{1}{5})$ and B $(0; 1)$ and the line passing through the points C $(1; 0)$ and D $(-2; 6)$

23. Determine the angle between the line $y + x = 3$ and the line $x = y + \frac{1}{2}$

24. Find the angle between the line $y = 2x$ and the line passing through the points P $(-1; \frac{7}{5})$ and Q $(0; 2)$

25. ABCD is a parallelogram with A$(-1;4)$, B$(3;6)$, C$(x;y)$ and D$(4;1)$.



Determine:

(a) the gradient of AB.

(b) the midpoint P of BD.

(c) the coordinates of C.

(d) the equation of CD.

(e) the coordinates of E if E is the x-intercept of the line CD.

(f) the inclination of line AE

(g) the size of $A\hat{E}D$.

(h) the length of BC

CIRCLES:

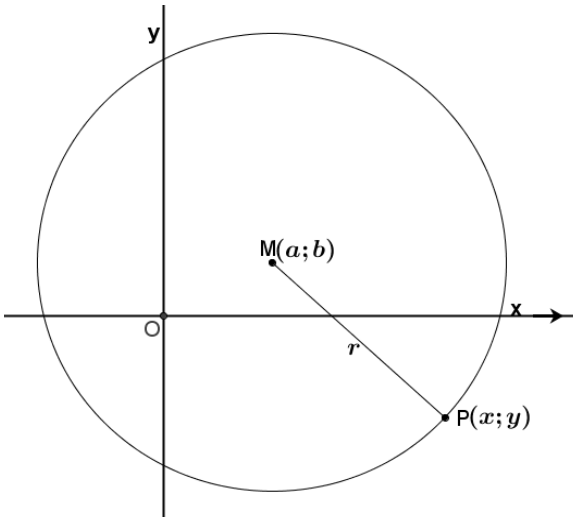
Definition of a Circle: The set of all points equidistant from a fixed point is called a centre.

The equation of a circle with a centre $(a; b)$ and radius r

The equation of a circle with centre $(a; b)$ and radius **r** can be found as follows:

Let P $(x; y)$ be any point on the circle with centre M $(a; b)$ and radius **r**

- the equation of a Circle (any centre); $(x-a)^2+(y-b)^2=r^2$
- the equation of a Circle whose centre is at the origin : $x^2+y^2=r^2$



$MP=r$

$MP^2=r^2$

$(x-a)^2+(y-b)^2=r^2$ Distance formula

NOTE: Learners are expected to know and use as an axiom: ***"the tangent to a circle is perpendicular to the radius drawn to the point of contact."***

(All straight line formulae apply to tangent because of it being a straight line)

2ND BRACKET OF ACTIVITIES

1. Determine the coordinates of the centre of the circle and the radius for each of the following circles:

(a) $x^2+y^2=36$

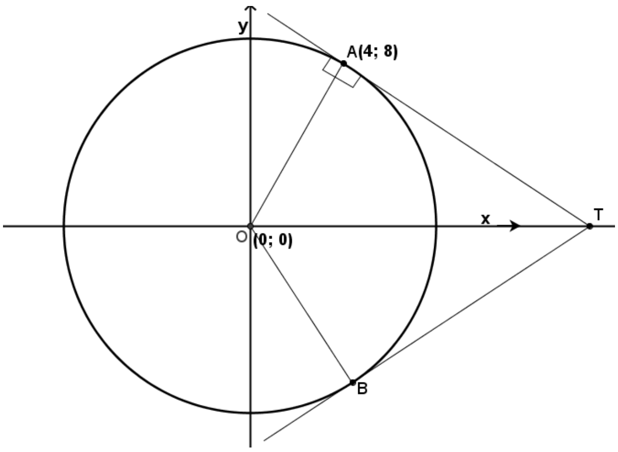
(b) $(x-2)^2+y^2=12$

(c) $x^2+(y-5)^2=18$

(d) $x^2+y^2-6x+10y=18$

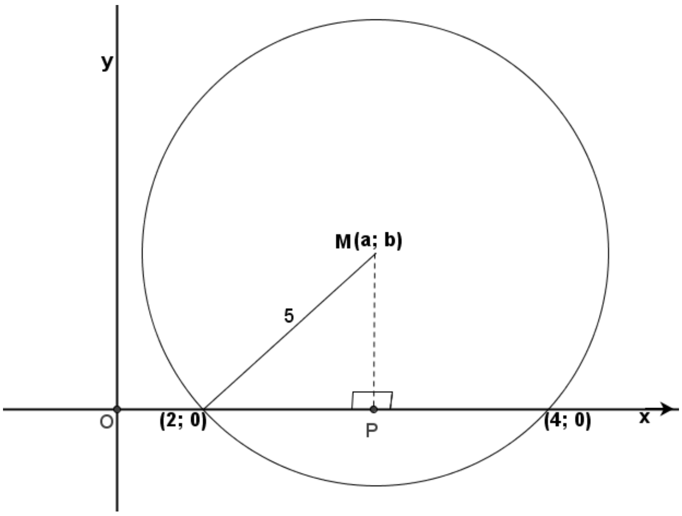
(e) $x^2+y^2-x-2y-5=0$

2. In the diagram below TA and TB are tangents to the circle centre O.

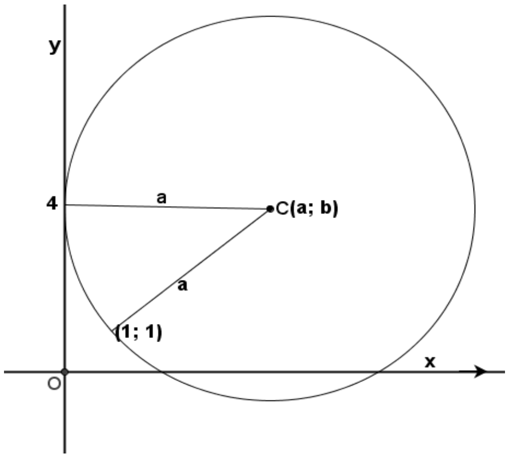


(a) Determine the equation of the circle.
(b) Determine the equation of TA.
(c) Determine the equation of TB.

3. From the diagram below, determine the equation of the circle if the radius is 5 units and the centre is at M $(a; b)$.



4. Using the diagram below, find the equation of the circle if the centre is at C$(a; b)$ and the circle touches the $y-$axis at $(0; 4)$



5. The circle with the equation $x^2 + y^2 = 169$ cuts the y −axis at M and N, and the x − axis at P and Q. A line with the equation $y = x - 7$ cuts the circle at A and B.

(a) Write down the coordinates of M; N; P and Q.
(b) What are the equations of the tangents at M and N.?
(c) Find the coordinates of A and B.
(d) Find the length of AB.

6. A circle with centre $(0; 0)$ passes through A $(3; -4)$ and B $(-5; 0)$
(a) Find the equation of the circle.
(b) Find the equation of the tangent at A.
(c) Find the equation of the radius that passes through the mid-point of AB.

7. Calculate the equation of the tangent to the circle $x^2 - 2x + y^2 + 4y = 5$ at the point $(-2; -1)$

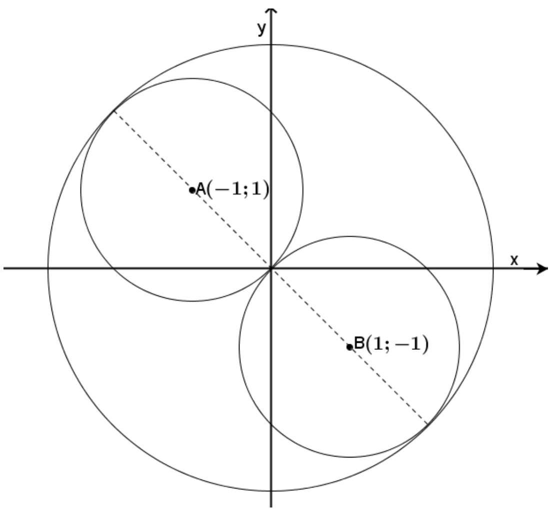
8. Determine the length of the tangent from the point H $(6; -2)$ to the point of intersection with the circle $x^2 + y^2 - 6x + 2y + 8 = 0$

9. Show that the circles with equations $x^2 + y^2 - 16x - 8y + 35 = 0$ and $x^2 + y^2 = 5$ touch each other.

10. Given the circles: $x^2 + y^2 + 2x - 6y + 9 = 0$ and $x^2 + y^2 - 4x - 6y + 9 = 0$

Show that the two circles touch externally.

11. The equation of a bigger circle is $x^2 + y^2 = 8$ and the smaller circles are centred at A and B respectively.



(a) Write down the equations of the circles centred at A and B respectively.

(b) Prove that the circle centred at A and the bigger circle touch internally.

(c) Give the equation of the bigger circle if it is translated 2 units up and 3 units left.

12. The line $x = 8$ cuts the circle $x^2 + y^2 = 100$ at two points.

(a) Calculate the y-coordinates of the two points.

(b) Find the co-ordinates of the points that lie on the line $x = -8$ using symmetry

13. The circle with equation $x^2 + y^2 = 169$ cuts the y-axis at M and N and the x-axis at P and Q.

A line with equation $y = x - 7$ cuts the circle at A and B.

(a) Write down the co-ordinates of M, N, P and Q.

(b) Calculate the co-ordinates of A and B.

(c) Calculate the length of the chord AB.

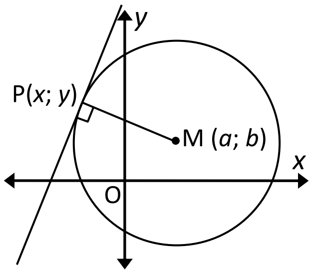
14. The equation of a circle with radius $3\sqrt{2}$ is $x^2 - 6x + y^2 + 2y - m = 0$.

(a) Determine the coordinates of the centre of the circle.

(b) Find the value of m.

TYPICAL EXAMINATIONS QUESTIONS

1. The drawing on the right shows a tangent drawn to a circle at the point P (x; y). Write down the gradient of:



(a) MP in terms of a, b, x and y.

(b) the tangent in terms of a, b, x and y.

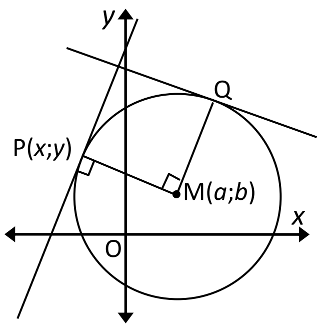
2. Let the centre of the circle in question 18 be M(3; 2) and P the point (-3; 4).

(a) Calculate the length of the radius MP.

(b) Write down the equation of the circle.

(c) Calculate the gradient of the radius MP.

(d) Find the equation of the tangent through the point P.



3. The radius MP and the tangent as described in question 19 through an angle of 90° so that the new point of contact is Q.

(a) Write down the gradient and the equation of the new radius MQ.

(b) Calculate the co-ordinates of the new point(s) of contact.

(c) Write down the gradient of the new tangent.

(d) Find the equation of the new tangent through Q.

4. O and A are the centres of two circles with equal radii.

The two circles touch each other at point K and OKA is a straight line.

(a) Calculate the co-ordinates of K.

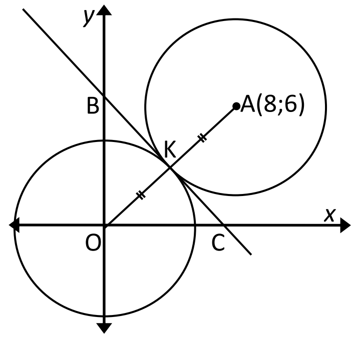
(b) Write down the equation of circle O.

(c) Write down the equation of circle A.

(d) Find the equation of BKC, a common tangent to both circles.

(e) Does OC = OB? Explain.

(f) What kind of shape will OCAB be? Explain.



5. A tangent and the circle $x^2 + y^2 = 25$ touch at (3;–4).

(a) Find the equation of the tangent at (3;–4).

(b) Find the point on the circle where another tangent will be parallel to the tangent in question 5(a).

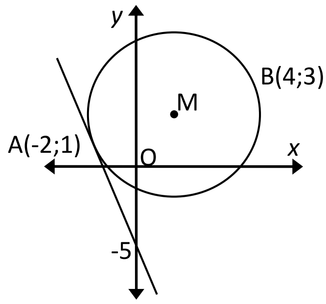
6. The equation of the tangent to the circle at A is $3x + y + 5 = 0$.

Find the:

(a) equation of the perpendicular bisector of AB, and deduce that AB is a diameter of the circle

(b) equation of MA

(c) co-ordinates of M equation of the circle.



7. (a) Find the equation of the circle, with centre the origin, through $A(-2;4)$.

(b) Find the equation of the tangent to the circle at A.

(c) If the tangent cuts the $x-$axis at B, find the length of AB.

(d) Find the equation of the other tangent to the circle from B, if C is the point of contact of the tangent to the circle.

(e) Show that $AB = BC$.

8. Find the equation of a circle with centre $C(2;3)$ and which touches the $x-$axis.

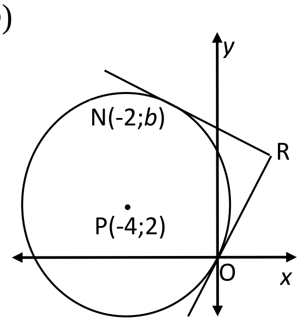
Find the equation of the circle touching the $x-$axis at $(3;0)$, passing through $(1;2)$.

9. The straight line $y = -2x + c$ cuts the circle $(x+1)^2 + y^2 = 20$ at $A(x;y)$.

(a) Determine the equation of the radius through A.

(b) Determine the coordinates of A and hence the value of C.

10. A circle with centre $P(-4;2)$ has the points $O(0;0)$ and $N(-2;b)$ on the circumference. The tangents at O and N meet at R.



Determine:

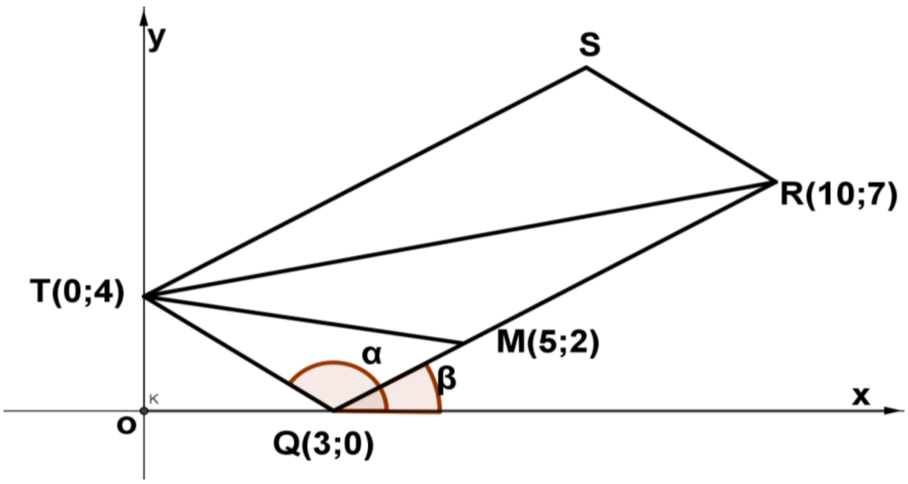
(a) the equation of the circle

(b) the value of *b*.

(c) the equation of OR.

(d) the coordinates of R.

11. In the diagram, Q (3;0), R (10;7), S and T(0;4) are vertices of a parallelogram QRST. From T a straight line is drawn to meet QR at M (5;2). The angles of inclination of TQ and RQ are α and β respectively.



(a) Calculate the gradient of TQ

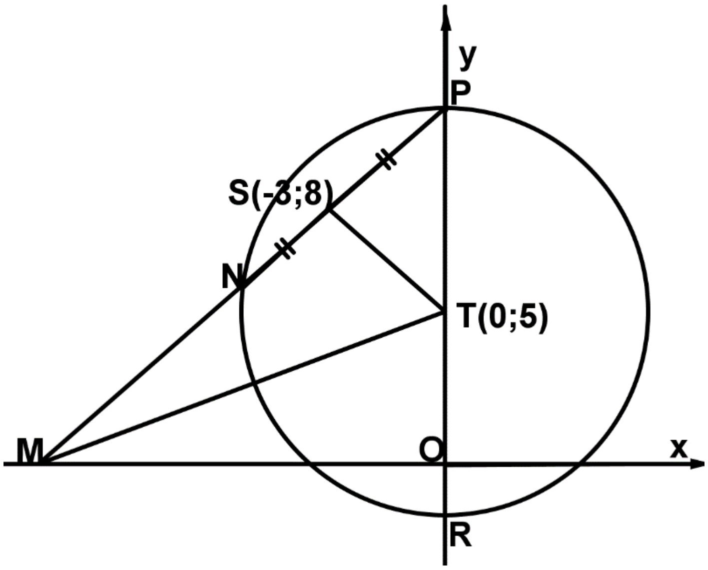
(b) Calculate the length of RQ. Leave your answer in a surd form.

(c) F (k; -8) is a point in a Cartesian plane such that T, Q and F are collinear. Calculate the value of k.

(d) Calculate the coordinates of S.

(e) Calculate the size of $T\hat{S}R$

12. In the diagram, the circle, having the centre T (0;5), cuts the y-axis at P and R. the line through P and



S (-3;8) intersects the circle at N and the x-axis at M . NS = PS. MT is drawn.

(a) Give a reason why TS $\perp$ NP.

(b) Determine the equation of a line passing through N and P in the form $y = mx + c$.

(c) Determine the equations of the tangents to the circle that are parallel to the x-axis.

(d) Determine the length of MT

(e) Another circle is drawn through the points S, T and M. Determine, with reasons, the equation of this circle in the form $(x-a)^2 + (y-b)^2 = r^2$.

TRIGONOMETRY

Trigonometry is the study of the relationship between the sides and angles of triangles.
The word trigonometry means ‘measurement of triangles’

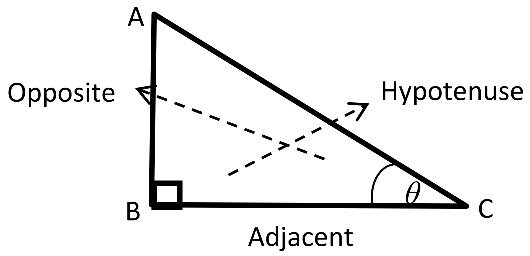
- **Please make sure that you know the names of the sides of a right-angled triangle.**

The trigonometric ratios

Using θ as the **reference angle** in ΔABC

- The side opposite the 90° is the hypotenuse side, therefore side AC is the hypotenuse side.
- The side opposite θ is the opposite side, therefore AB is the opposite side.
- The side adjacent to θ is called the adjacent side, therefore BC is the adjacent side.

Trigonometry involves the *ratios* of the *sides* of *right triangles*.



$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

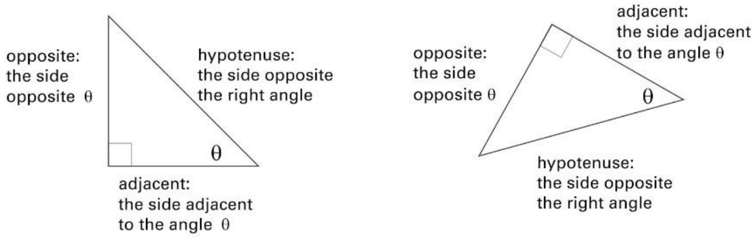
$$\tan\theta = \frac{opposite}{adjacent}$$

PRE – KNOWLEDGE NEEDED:

- Right-angled triangle
- Adjacent; Opposite and Hypotenuse sides in a right-angled triangle
- Pythagoras theorem

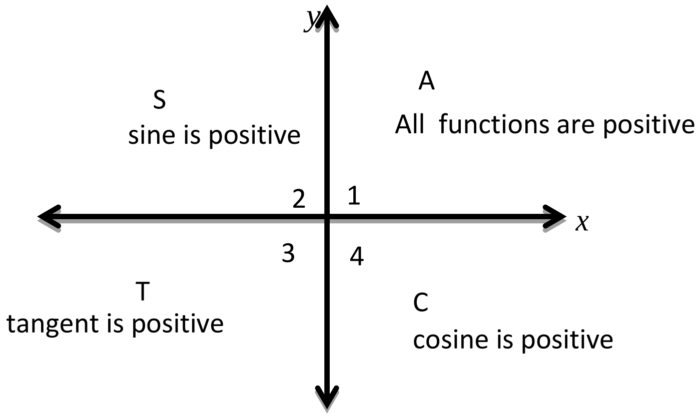
We work with the ratios of the sides of the triangle:

- The ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is called **sine θ** (abbreviated to **sin θ**)
- The ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called **cosine θ** (abbreviated to **cos θ**)
- The ratio $\frac{\text{opposite}}{\text{adjacent}}$ is called **tangent θ** (abbreviated to **tan θ**)



1. THE CARTESIAN PLANE AND IDENTITIES

The Cartesian plane is divided into 4 quadrants by two coordinate axes. These 4 quadrants are labelled 1, 2, 3 and 4 respectively.

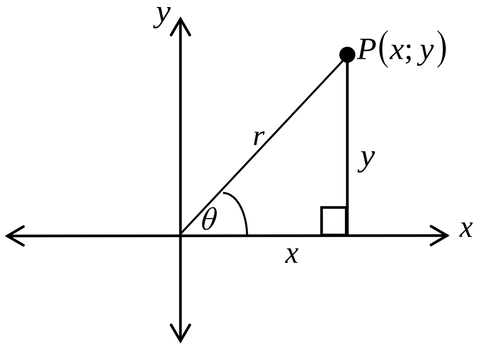


When working with ratios on the Cartesian plane we will make use of the symbols x, y and r (radius).

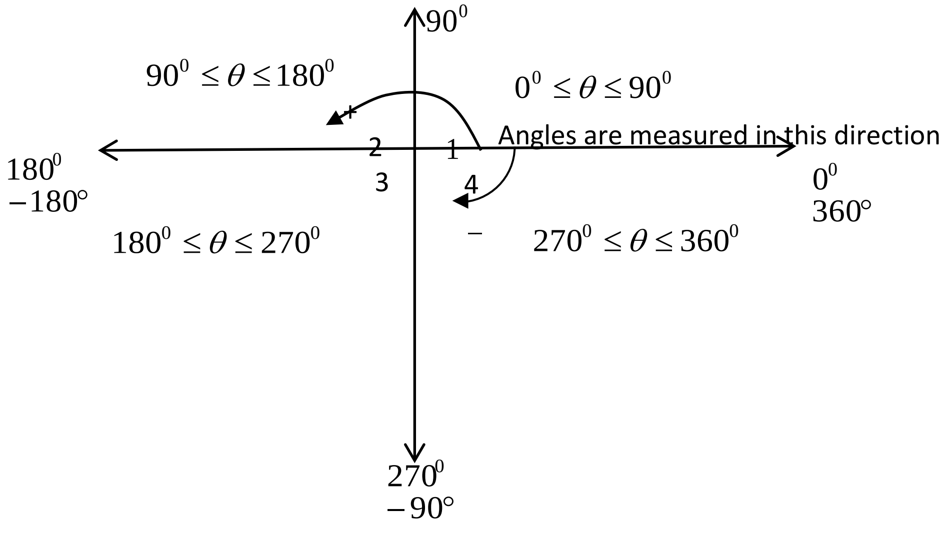
$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



We can find any trigonometric functions if the co-ordinates of the terminal side is given.



2. REDUCTION FORMULAE

- $0^0 \leq \theta \leq 360^0$ use:
- $\sin(180^0 + \theta) = -\sin\theta$ $\quad\quad$ $\sin(180^0 - \theta) = \sin\theta$
- $\cos(180^0 \pm \theta) = -\cos\theta$
- $\sin(360^0 - \theta) = -\sin\theta$ $\quad\quad$ $\cos(360^0 - \theta) = \cos\theta$

- $\tan(180^0-\theta)=-\tan\theta$ $\tan(180^0+\theta)=\tan\theta$
- $\tan(360^0-\theta)=-\tan\theta$

➢ Co-functions:

$\sin(90^0\pm x)=\cos x$ $\cos(90^0+x)=-\sin x$ $\cos(90°-x)=\sin x$

➢ $\theta>360^0$: reduce angles to less than 360^0 by subtracting multiples of 360^0.

➢ $\theta<0^0$: $\sin(-x)=-\sin x$; $\cos(-x)=\cos x$ $\tan(-x)=-\tan x$

➢ The special angle $30^0; 45^0; 60^0$

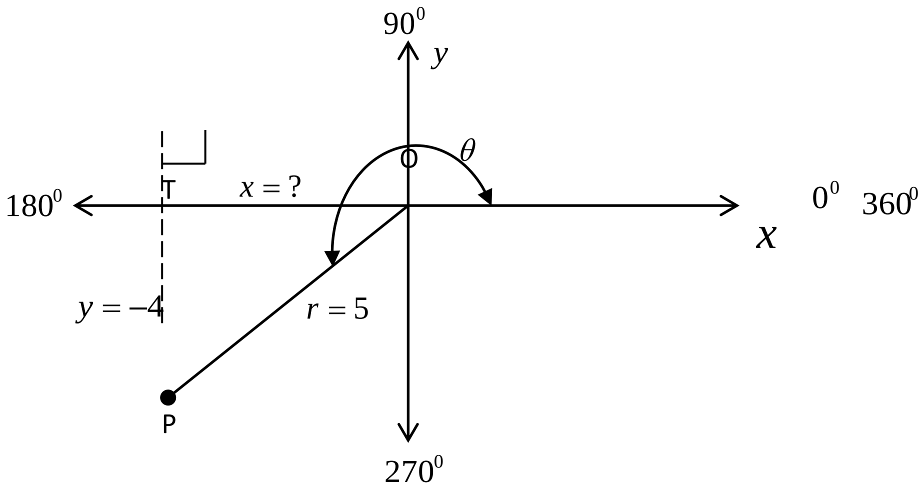
Example 1.

If $\sin\theta=\frac{-4}{5}$ and $90^0\le\theta\le270^0$. Determine

1.1 $\cos^2\theta$

1.2 $5\sin\theta-3\cos\theta$

Solution



You must first calculate the length of the other side,

$$x^2+y^2=r^2$$

$$x^2+(-4)^2=(5)^2$$

Using Pythagoras: $x^2=25-16$

$$x^2=9$$

$$x=\pm3$$

$\therefore x=-3$ (x is negative in the 3rd quadrant)

1.1 $\cos^2\theta=\left(\frac{-3}{5}\right)^2=\frac{9}{25}$

1.2

$$5\sin\theta-3\cos\theta=5\left(\frac{-4}{5}\right)-3\left(\frac{-3}{5}\right)$$

$$=-4+\frac{9}{5}$$

$$=-\frac{11}{5}$$

$$=-2\frac{1}{5}$$

Example 2

If $\tan 127^0 = -P$, express each of the following in terms of P.

2.1 $\sin 127^0$

2.2 $\cos 127^0$

2.3 $\dfrac{\sin^2 127^0}{\cos^2 127^0}$

Solutions

$\tan 127^0 = \dfrac{P}{-1} = \dfrac{y}{x}$; Since $x^2 + y^2 = r^2$; $\quad$ *i.e* $(-1)^2 + P^2 = r^2$

$$\sqrt{1+P^2} = \sqrt{r^2}$$

$$\sqrt{1+P^2} = r$$

2.1 $\sin 127^0 = \dfrac{P}{\sqrt{1+P^2}}$

2.2 $\cos 127^0 = -\dfrac{1}{\sqrt{1+P^2}}$

2.3 $\dfrac{\sin^2 127^0}{\cos^2 127^0} = \tan^2 127^0 = P^2$

Example 3.

Simplify without using a calculator: $\dfrac{\sin 315^0 . \tan 210^0 . \sin 190^0}{\cos 100^0 . \sin 120^0}$

Solution. $\dfrac{\text{Sin}(360^0 - 45^0).\tan(180^0 + 30^0).\sin(180^0 + 10^0)}{\cos(90^0 + 10^0).\sin(180^0 - 60^0)}$

$$\frac{(-\sin(45^0))(\tan(30^0))(-\sin(10^0))}{(-\sin(10^0)).\sin(60^0)}$$

$$\frac{(-\frac{\sqrt{2}}{2})(\frac{1}{\sqrt{3}})}{\frac{\sqrt{3}}{2}}$$

$$\frac{-\sqrt{2}}{3}$$

Example 4.

Simplify the following to a single trigonometric ratio: $\dfrac{\text{Sin}(180^0 - x) - 2\cos(90^0 - x).\cos x}{2\cos^2(360^0 + x) - \cos(-x)}$

Solution : $\dfrac{\text{Sin}(180^0 - x) - 2\cos(90^0 - x).\cos x}{2\cos^2(360^0 + x) - \cos(-x)}$

$$\frac{\sin x - 2\sin x.\cos x}{2\cos^2 x - \cos x}$$

$$\frac{\sin x - 2\sin x.\cos x}{2\cos^2 x - \cos x}$$

$$\frac{\sin x\,(1 - 2\cos x)}{-\cos x(1 - 2\cos x)}$$

$$\frac{\sin x}{-\cos x}$$

$$-\tan x$$

ACTIVITIES

N.B 2.1 to 2.5 are based on example 1 and 2

N.B 2.6 to 2.7 are based on example 3 and 4

2.1. Given that $\sin 23° = \sqrt{k}$, determine, in its simplest form, the value of each of the following in terms of k, WITHOUT using a calculator:

2.1.1 $\sin 203°$ (3)

2.1.2 $\cos 23°$ (2)

2.1.3 $\tan(-23°)$ (2)

2.2 Given: $\sin 16° = p$

Determine the following in terms of p, **without using a calculator**.

2.2.1 $\sin 196°$ (3)

2.2.2 $\cos 16°$ (2)

2.3. If $cos25^0 = m$, then, without the use of a calculator, determine the value(s) of the following in terms of m.

2.3.1 $\sin 25^0$ (2)

2.3.2 $cos\ 50^0$ (3)

2.3.3 $cos55^0$ (4)

2.4. Given: $\cos 2B = \frac{3}{5}$ and $0° \le 2B \le 90°$,Determine, **without using a calculator**, the value of EACH of the following in its simplest form:

2.4.1 $\sin 2B$ (3)

2.4.2 $\cos B$ (4)

2.4.3 $\sin B$ (3)

2.4.4 $\cos (B + 45°)$ (4)

2.5. If $sinA = \frac{3}{5}$ and $cosA < 0$,determine, WITHOUT using a calculator, the value of :

2.5.1 $\sin(-A)$ (2)

2.5.2 $tanA$ (3)

2.6. Simplify the following expression without a calculator:

2.6.1 $\dfrac{\cos 225^0 . \sin(-135^0) - \sin 330^0}{\tan 225^0}$ (6)

2.6.2 $\dfrac{\tan 480°. \sin 300°. \cos 14°. \sin(-135°)}{\sin 104°. \cos 220°}$ (6)

2.6.3 $\dfrac{\cos 300^0 . \sin 140^0}{\sin(-160^0). tan405^0 . \sin 290^0}$ (7)

2.7. Simplify the following expression to a single trigonometric function:

2.7.1 $\dfrac{4\cos(-x).\cos(90°+x)}{\sin(30°-x).\cos x+\cos(30°-x).\sin x}$ (6)

2.7.2 $\dfrac{\sin(A-360^0).\cos(90^0+A)}{\cos(90^0-A).\tan(-A)}$ (6)

2.7.3 $\dfrac{\cos 140^0-\sin(90^0-\theta)}{sin130^0+\cos(-\theta)}$ (5)

2.7.4 $\dfrac{\tan(180^0-\theta).\sin(90^0+\theta)}{cos300^0.\sin(\theta-360^0)}$ (6)

3. TRIGONOMETRIC IDENTITIES

- Work with the LHS and RHS separately.
- Choose most difficult side and use identities to simplify it.
- Look for square identities
- If there are fractions: Get LCM.
- Factorisation or simplify if necessary

NB: ***"hence"*** means that the previous answer/previous working must be used or be revisited

- **Identities:**

  $\tan x = \dfrac{\sin x}{\cos x}$; $\sin^2 x + \cos^2 x = 1$

- **Compound angles**:
  - $\cos(\alpha+\beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$
  - $\cos(\alpha-\beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$
  - $\sin(\alpha+\beta) = \sin\alpha.\cos\beta + \sin\beta.\cos\alpha$
  - $\sin(\alpha-\beta) = \sin\alpha.\cos\beta - \sin\beta.\cos\alpha$
- **Doubles angles**
  - $\sin 2A = 2\sin A.\cos A$
  - $\cos 2A = 2\cos^2 A - 1$
  - $\cos 2A = 1 - 2\sin^2 A$
  - $\cos 2A = \cos^2 A - \sin^2 A$

Example 5

Prove the identity: $\dfrac{cosx}{1+sinx} + \dfrac{1+sinx}{cosx} = \dfrac{2}{cosx}$

Solution: LHS=$\dfrac{cosx}{1+sinx} + \dfrac{1+sinx}{cosx}$

$$= \frac{(cosx.)(cosx)+(1+sinx)(1+sinx)}{(1+sinx)cosx}$$

$$= \frac{cos^2 x+(1+2sinx+sin^2 x)}{(1+sinx)cosx}$$

$$= \frac{(2+2sinx)}{(1+sinx)cosx}$$

$$= \frac{2(1+sinx)}{(1+sinx)cosx}$$

$$= \frac{2}{cosx} \qquad \text{(LHS = RHS)}$$

ACTIVITIES

3.1 Prove the following identities:

3.1.1 $\frac{\cos A - cos2A + 2}{3sinA - sin2A} = \frac{1 + cosA}{sinA}$ (6)

3.1.2 $\frac{2sin^2 x}{2tanx - sin2x} = \frac{1}{\tan x}$ (6)

3.1.3 $\frac{sinx + \sin 2x}{1 + \cos x + cos2x} = tanx$ (4)

3.1.4 $\frac{\cos 2x - 1}{sin2x} = -tanx$ (3)

3.1.5 $\frac{\sin 2x}{\cos 2x + sin^2 x} = 2tanx$ (4)

3.1.6 $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{2}{\sin x}$ (5)

3.1.7 $\frac{1 + \sin x}{1 - \sin x} - \frac{1 - \sin x}{1 + \sin x} = \frac{4\tan x}{\cos x}$ (NB simplify both sides) (6)

4. GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

- Write the equation on its own on one side of the equation.
- Start by simplifying an equation as far as possible. Use identities, double and compound angle formulae, and factorisation where possible. You want to have *one* trig ratio and *one* angle equal to a *constant*, for example $\cos x = \frac{1}{2}$
- Find the reference angle.
- Identify the possible quadrants in which the terminal rays of the angles could be, based on the sign of the function.
- Because trigonometric functions are periodic, there will be a number of possible solutions to an equation. You will need to write down the ***general solution*** of the equation.
- Once the solution has been solved, write down the general solution by adding the following:
  - $+ k360^0$, $k \in Z$ for cosine and sine, because they repeat every 360^0
  - $+ k180^0$, $k \in Z$ for tangent, because it repeats every 180^0
- If an equation contain double angles, for example, $\sin 3\theta$, $\cos 2x$ and $\tan 5y$, find the general solutions for 3θ, $2x$ and $5y$ first. Only then divide by 3, 2 or 5 to find the final solutions. If you divide first, you will lose valid solutions.
- Apply restrictions.

Example 6

Determine the general solution of the following equation: $2sinx.cosx = cosx$

Solution: $2sinx.cosx = cosx$

$2sinx = 1$

$sinx = \frac{1}{2}$

$x = sin^{-1}(\frac{1}{2})$

$ref\ x = 30^0$

$x = 0 + 30^0 + k, 360^0, k \in Z$	$x = 180^0 - 30^0 + k, 360^0, k \in Z$
$x = 30^0 + k, 360^0, k \in Z$	$x = 150^0 + k, 360^0, k \in Z$

ACTIVITIES

1.1 Find the general solutions of the following:

4.1.1 $7\cos 2x + 2 = 0$ (4)

4.1.2 $\sin 2x = 4\cos 2x$ (5)

4.1.3 $2\cos x.\sin x - \cos x = 0$ (4)

4.1.4 $8cos^2x - 2\cos x - 1 = 0$ (6)

4.1.5 $cos2x - 7cosx - 3 = 0$ (5)

4.2 Consider the identity: $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$

4.2.1 Prove the identity (5)

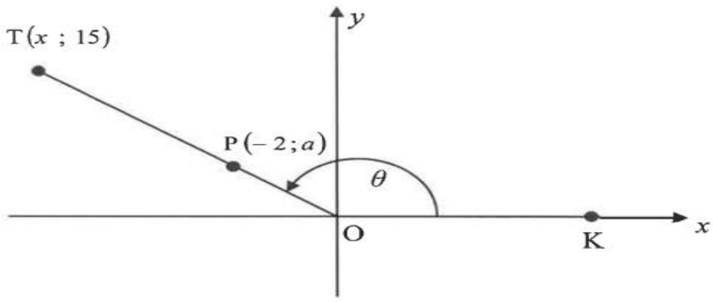
4.2.2. Determine the values of *x*,where $x \in [-180^0; 180^0]$ for which this identity is undefined.

(3)

PAST EXAM PAPERS QUESTIONS

QUESTION 5

5.1 In the diagram below, $T(x\,;15)$ is a point in the Cartesian plane such that $OT = 17$ units. $P(-2\,;a)$ lies on OT. K is a point on the positive x-axis and $T\hat{O}K = \theta$.



Determine, with the aid of the diagram, the following:

5.1.1 The value of x (2)

5.1.2 $\tan\theta$ (1)

5.1.3 $\cos(180° - \theta)$ (2)

5.1.4 $\sin^2\theta$ (2)

5.1.5 The value of a (3)

5.2 Simplify WITHOUT using a calculator:

$$\frac{\sin 120°.\cos 210°.\tan 315°.\cos 27°}{\sin 63°.\cos 540°}$$ (7)

5.3 Prove the identity:

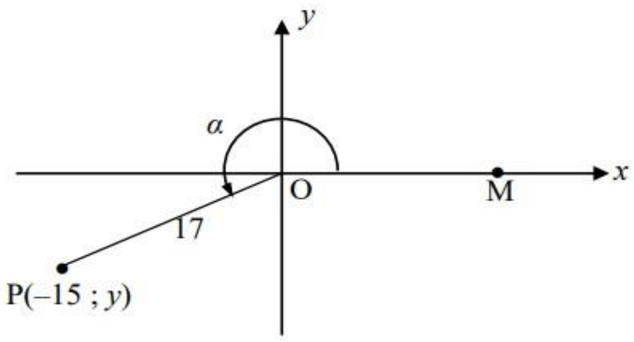
$$\frac{1}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta} = \tan\theta$$ (5)

5.4 Determine the general solution of $3\sin x = 2\tan x$ (6)

[28]

QUESTION 5

5.1 In the diagram, $P(-15\,;y)$ is a point in the Cartesian plane.
$OP = 17$ units and reflex $M\hat{O}P = \alpha$.



Determine the value of the following **without using a calculator**:

5.1.1 y (2)

5.1.2 $\sin(90° + \alpha)$ (2)

5.1.3 $\tan\beta$, if $\alpha + \beta = 540°$ (3)

5.2 Consider: $\dfrac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)}$

5.2.1 Simplify $\dfrac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)}$ to a single trigonometric ratio. (5)

5.2.2 Hence, or otherwise, **without using a calculator,** solve for θ if $0° \le \theta \le 360°$:

$$\frac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)} = 0{,}5$$ (3)

5.3 5.3.1 Prove that $\dfrac{8}{\sin^2 A} - \dfrac{4}{1 + \cos A} = \dfrac{4}{1 - \cos A}$. (5)

5.3.2 For which value(s) of A in the interval $0° \le A \le 360°$ is the identity in QUESTION 5.3.1 undefined? (3)

5.4 Determine the general solution of $8\cos^2 x - 2\cos x - 1 = 0$. (6)
[26]

QUESTION 5

5.1 Simplify, without using a calculator, the following expressions: (Show ALL the calculations.)

5.1.1 $\dfrac{\cos 150°.\tan 225°}{\sin(-60°).\cos 480°}$ (Leave the answer in simplified surd form.) (5)

5.1.2 $\dfrac{\cos(90° + x)}{\cos(360° - x).\tan(180° - x)}$ (5)

5.1.3 $\cos^2 x\left[\dfrac{1}{\sin x - 1} + \dfrac{1}{\sin x + 1}\right]$ (6)

5.2 Determine, without using a calculator, the value of the following in terms of t, if $\sin 34° = t$:

5.2.1 $\cos 56°$ (2)

5.2.2 $\tan(-34°)$ (3)

5.3 5.3.1 Solve for x if $7\cos 2x + 2 = 0$ and $x \in [0°;\ 360°]$. (6)

5.3.2 Determine the general solution of $\cos x\ (\sin x - 1) = 0$. (5)

[32]

QUESTION 4

4.1 If $13\sin\theta - 5 = 0$ and $\cos\theta < 0$ find the value of:

$24\tan\theta + 26\cos\theta$ with the aid of a sketch. [4]

4.2 If $\tan 72^\circ = k$ find each of the following in terms of k:

4.2.1 $\tan 252^\circ$ [2]

4.2.2 $\cos(-72^\circ)$ [2]

4.2.3 $\dfrac{\sin 108^\circ}{\sin 18^\circ}$ [3]

4.3 Simplify each of the following fully:

4.3.1 $\dfrac{\sin(180^\circ - x).\cos^2(x-180^\circ).\tan x}{\cos(90^\circ - x).\sin(360^\circ + x)}$ [5]

4.3.2 $\dfrac{\sin 210^\circ.\cos 150^\circ.\tan 25^\circ}{\tan 205^\circ.\cos 315^\circ \sin 135^\circ}$ [7]

4.4 Consider the identity:

$$\frac{\sin(180^\circ + \theta)\cos(90^\circ + \theta) + \cos(-\theta)\cos(720 - \theta)}{\sin\theta} + \frac{1}{\tan\theta} = \frac{1 + \cos\theta}{\sin\theta}$$

4.4.1 Prove the identity [6]

4.4.2 Write down one value of θ for which this identity is undefined [1]

/30/

QUESTION 3

3.1 If $13\sin\alpha = -5$ and $\tan\alpha > 0$, use a diagram to evaluate: $3\cos\alpha$. (5)

3.2 Simplify the following expressions **without** the use of a calculator.

3.2.1 $\dfrac{\sin(\theta - 180^\circ).\tan(360^\circ - \theta).\sin(90^\circ - \theta)}{\cos^2(\theta + 180^\circ)}$ (6)

3.2.2 $\dfrac{\sin 210^\circ.\cos 400^\circ}{\sin(-50^\circ).\cos 120^\circ}$ (6)

3.3 If $(4\theta - 8)\sin 30^\circ = (\theta^3 - 8)$ and $(\theta^2 + 2\theta + 4) = 2$, determine the value of $\tan 240^\circ$ **without** the use of a calculator. (5)

3.4 3.4.1 Prove that : $\dfrac{1}{\tan\alpha}(\sin\alpha\tan\alpha + \cos\alpha) = \dfrac{1}{\sin\alpha}$ (4)

3.4.2 Determine for which value(s) of α is:

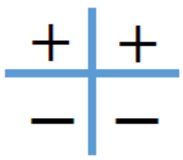
$\dfrac{1}{\tan\alpha}(\sin\alpha\tan\alpha + \cos\alpha)$ undefined for $\alpha \in [0^\circ ; 360^\circ]$. (2)

[28]

TRIGONOMETRIC GRAPHS

Complete the table and draw the following graphs on the same set of axis:

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
$y=\sin x$													
$y=2\sin x$													
$y=3\sin x$													
$y=\frac{1}{2}\sin x$													
$y=\cos x$													
$y=-2\cos x$													
$y=3\cos x$													
$y=\frac{1}{2}\cos x$													



Signs in 4 quadrants

+ means graph is above *x*- axis

- means graph is above *x*- axis

Shape: Wave-like shape, starting at the origin

Intercepts: y-intercept = 0

x-intercept **(zeros)** = $0°;180°;360°$ (every $180°$ starting at $0°$)

Domain: The domain is usually limited to the interval $[0°;360°]$ or $[-360°;360°]$

Infinite angles are possible as a line centred at the origin on the Cartesian plane can be rotated many times. Rotating the line anti-clockwise gives positive angles, and rotating clockwise gives negative angles.

Period: The period of the function is the number of degrees the function needs to complete one cycle. This corresponds to one rotation. The sine function repeats itself every $360°$.

Trigonometric Functions are periodic functions and the graph form $0°$ to $360°$ is exactly the same as the graph from $360°$ to $720°$

Range: Minimum value: -1 when the angle x is 270° or -270°
Maximum value: 1 when the angle x is 90° or -90°
$[-1; 1]$

Amplitude: The amplitude of a trigonometric graph is the greatest distance the function moves above or below the x-axis.
(This only applies for sin or cos graphs). For the basic graph (also known as parent function) the amplitude is 1. It is half the range $(i.e., \frac{1+1}{2}=1)$.

Strategies for Graphical Interpretations: Typical interpretation questions

Type of question	Interpretation
$f(x) < 0$	$f(x)$ must be less than zero $f(x)$ lies below the x-axis value is excluded
$f(x) < g(x)$	$f(x)$ must be less than $g(x)$ $f(x)$ lies below $g(x)$ e.g. $\sin(x)$ lies below $\cos(x)$
$f(x) \geq 0$	$f(x)$ must be greater than or equal to zero $f(x)$ lies above x-axis value is included
$f(x) > g(x)$	$f(x)$ must be greater than $g(x)$ $f(x)$ lies above $g(x)$ value is excluded
$f(x).\ g(x) < 0$	One of the two graphs must be above x-axis while the other must be below x-axis. Value is excluded
$f(x).g(x) \geq 0$	Both the two graphs are above the x-axis or both the two graphs are below the x-axis. Value is included.
$f(x)\ .\ g(x) = 0$	Either $f(x)=0$ or $g(x)=0$ x-intercepts of f(x) and g(x)
$f(x) - g(x) = 0$	$f(x) = g(x)$ Points of intersection
(	Excluded because of restriction
]	Included because of restriction
(;]	*first*-value excluded and *last*-value is included because of restriction
(;) or < > or ○	Values excluded
[;] or ≥ ≤ or •	Values included

Summary of the Basic characteristics of trigonometric graphs

Finding of	$y=a\sin bx$	$y=a\cos bx$	$y=a\tan bx$
Domain	$[-360^\circ;360^\circ]$	$[-360^\circ;360^\circ]$	$(-90^\circ;0^\circ],(-90^\circ;-270^\circ)$, $(-270^\circ;360^\circ],[0^\circ;90^\circ)$, $(90^\circ;270^\circ),(270^\circ;360^\circ]$
Range	$[-1;1]$	$[-1;1]$	$[-\infty;\infty]$
Period	$\frac{360^\circ}{\lvert b\rvert}=\frac{360^\circ}{\lvert 1\rvert}=360^\circ$	$\frac{360^\circ}{\lvert b\rvert}=\frac{360^\circ}{\lvert 1\rvert}=360^\circ$	$\frac{180^\circ}{\lvert b\rvert}=\frac{180^\circ}{\lvert 1\rvert}=180^\circ$
Amplitude	$a=1$	$a=1$	No
Minimum value	-1	-1	No
Maximum value	1	1	No
Asymptote	NO	NO	$x=-90^\circ,x=-270^\circ$ $x=90^\circ,x=270^\circ$
NOTE: the value of *b* affects the period only and the amplitude (maximum & minimum) remains the same.			

1. The Worked Out Examples/Activities

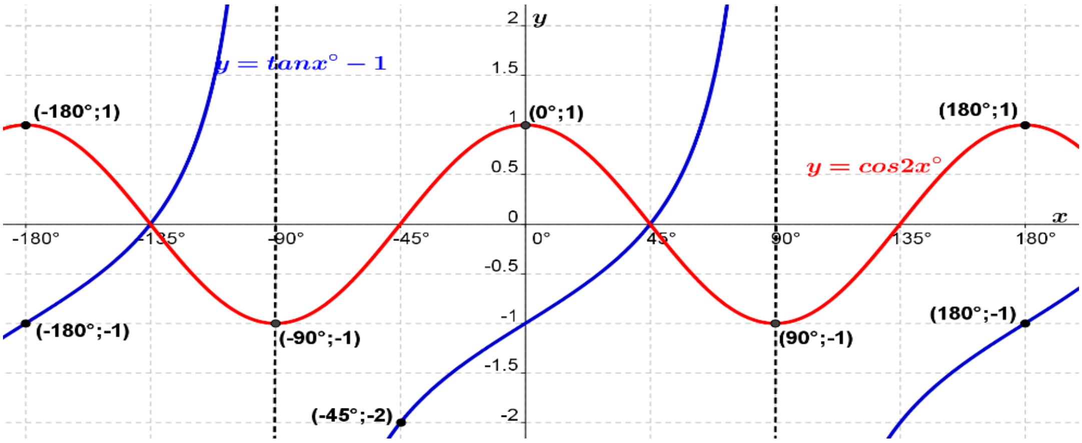
Example 1.

On the same system of axes, draw the sketch graphs of:

$y = tanx° - 1$ and $y = cos2x°$, for the interval[-180°; 0°]. Show all the intercepts with the axes and the co-ordinates of the turning points. Show the asymptotes of $y = tanx° - 1$.

(a) Use the sketch graphs to the value of x if:$cos2x° + 1 \leq tanx°$, in the interval [-180°; 0°].
(b) If the curve of $y = tanx° - 1$is moved upwards by 3 units, what will the new equation be?
(c) Write down the period of $y = cos2x°$

Solution

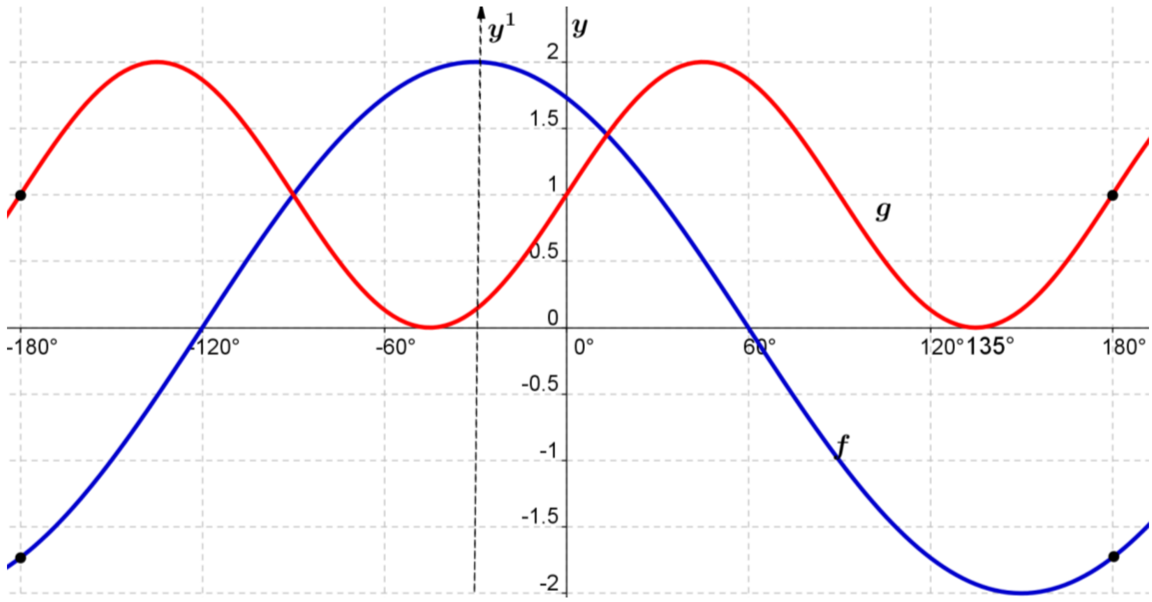


(a) $cos2x° + 1 \leq tanx°$
$cos2x° \leq tanx° - 1$
$-135° \leq x < 90°$
(b) $y = tanx° + 2$
(c) Period of $y = cos2x°$: 180°

Example 2

In the given figure the graphs of:

$\boldsymbol{f} = \{(x; y)|y = a\cos(x° + b)\}$ and $\boldsymbol{g} = \{(x; y)|y = c + sindx°\}$ if $x \in [-180°; 180°]$ are given.



(a) the values of a, b, c and d by using the graphs, and write down the two equations in the form of: $y = \cdots$

(b) $f(0°)$ without using a calculator.

(c) x from the graphs if:

(i) $g(x) = 2$ and $x \in [-180°; 180°]$

(ii) $f(x) < g(x)$ and $x \in [-180°; 0°]$

(d) The $\boldsymbol{y}$ −axis is moved to the $\boldsymbol{y^1}$ position in order to pass through a turning point of $\boldsymbol{f}$**.** the equation of $\boldsymbol{f}$ in the form: $\boldsymbol{y} = \cdots$ with respect to the new system of axes.

Solution

(a) $a = 2$; $b = 30°$, $c = 1$ and $d = 2$

$\therefore f(x) = 2\cos(x° + 30°)$ and $g(x) = 1 + sin2x°$

(b) $f(0°) = 2cos30°$

$= 2\frac{\sqrt{3}}{2}$

$= \sqrt{3}$

(c) (i) $x = 45°;\ -135°$

(iii) $-180° \leq x < -90°$

(d) $y = 2\cos(x° + 30° - 30°)$

$y = 2cosx°$

2. Examples/Activities

(Sketching and Interpreting Graphs)

QUESTION 1

Given the functions $f(x) = 2sinx$ and $g(x) = tanx - 1$.

1.1 Sketch the graphs of f and g on the same system of axes **on the diagram sheet**, for $x \in [-45° ; 180°]$, clearly labelling endpoints, turning points, intercepts with the axes and asymptotes. (7)

1.2 Use your graphs to determine the value(s) of x for which $f(x).\ g(x) < 0$ (2)

1.3 If $h(x) = f(x) + 2$, write down the range of h. (2)

QUESTION 2

2.1 Draw the graphs of $y = \sin 4x$ and $y = \sin 2x$ on the same system of axes for $x \in [0°;180°]$ (4)

2.2 For which values of x ε [0°; 180°] is $\sin 4x \geq \sin 2x$? (2)

QUESTION 3

3.1 Use the system of axes on **DIAGRAM SHEET 3** to sketch the graphs of:

$$f(x) = -\frac{1}{2}\sin(x + 30°) \text{ and } g(x) = \cos 2x \text{ if } -180° \leq x \leq 180°$$

(6)

3.2 Write down the period of g. (1)

3.3 Graph h is obtained when the y-axis for f is moved 120° to the left.

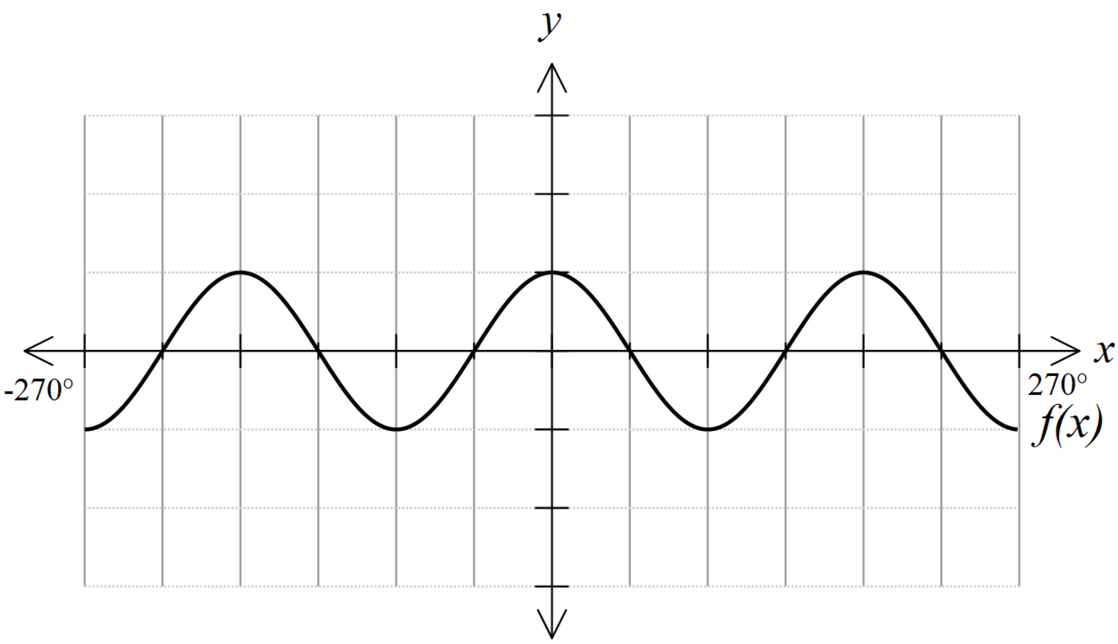
Give the equation of h in the form $h(x) = ...$ (2)

3.4 Determine the general solution of: $\cos 2x = 1$ **(2)**

(Sketching and Interpreting trig graphs/general solution)

QUESTION 4

The graph of $f(x) = \cos 2x$ for $-270° \leq x \leq 270°$ is drawn below



4.1 Write down the period of f. (1)

4.2 Write down the range of $\frac{f(x)}{2}$. (2)

4.3 Draw the graph of $g(x) = 1 + \sin x$ on the same set of axes as $f(x)$.
Show all turning points and intercepts with the axes. (3)

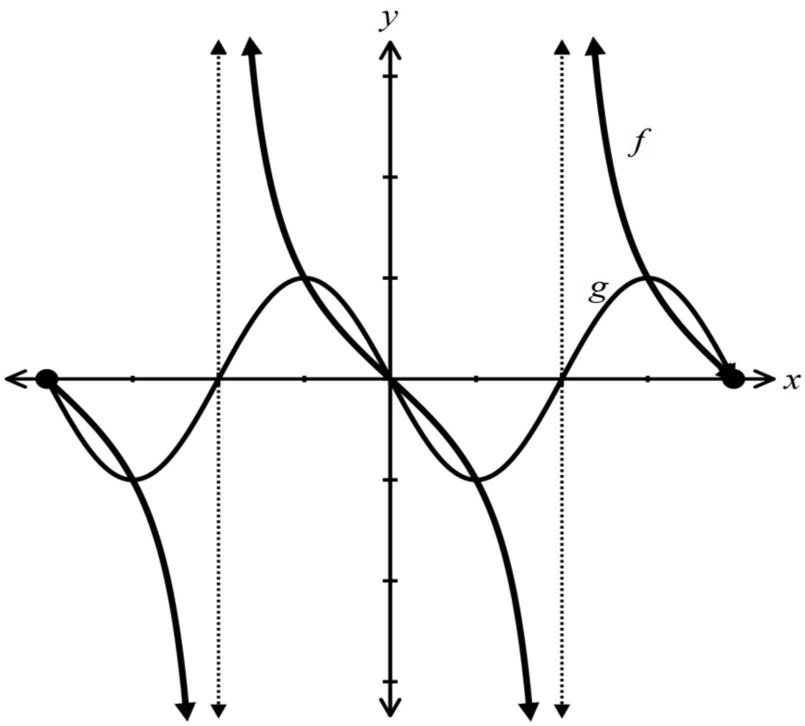
4.4 Use the graphs to determine the value(s) of x for which:

4.4.1 $g(x) - f(x) = 3$ in the interval $0° < x < 180°$ (1)

4.4.2 $f(x).g(x) \leq 0$ in the interval $0° < x < 180°$ (2)

QUESTION 5

The graphs of $f(x) = a\tan x$ and $g(x) = -\sin bx$ for $x \in [-180°; 180°]$ are drawn below



Use the graphs to answer the following:

5.1 Determine the values of a and b. (2)

5.2 Solve for $x \in [0°; 180°]$ if $2\sin x\cos x - \tan x = 0$ (3)

5.3 For which values of $x \in [-90°; 180°]$ is $g(x) \geq f(x)$? (4)

5.4 If $g(x)$ is shifted 90° to the right to form a new graph $h(x)$, write down the equation of $h(x)$ in its simplest form (2)

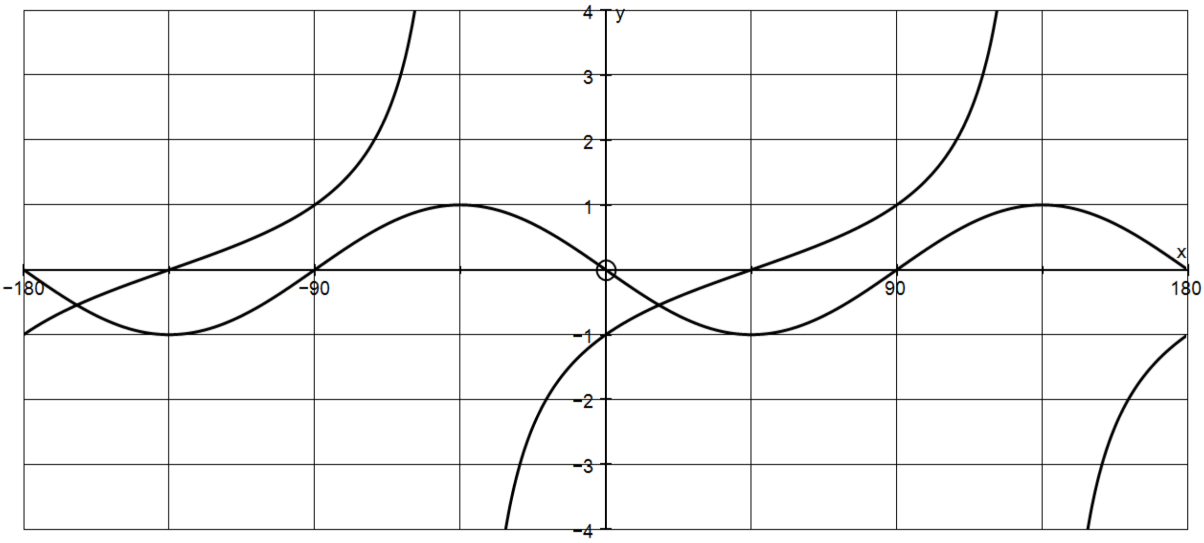
QUESTION 6

A function is defined as $f(x) = a \sin bx + c$

The function satisfies the following conditions:

- The period is $120°$
- The range is $y \in [-2; 6]$
- The co-ordinates of a maximum point are $(210°; 6)$

Write down the values of a, b and c [4]



Question 7

The diagram shows the graphs of $f(x) = \tan(x + d)$ and $g(x) = e\sin qx$ for the interval $-180° \leq x \leq 180°$.

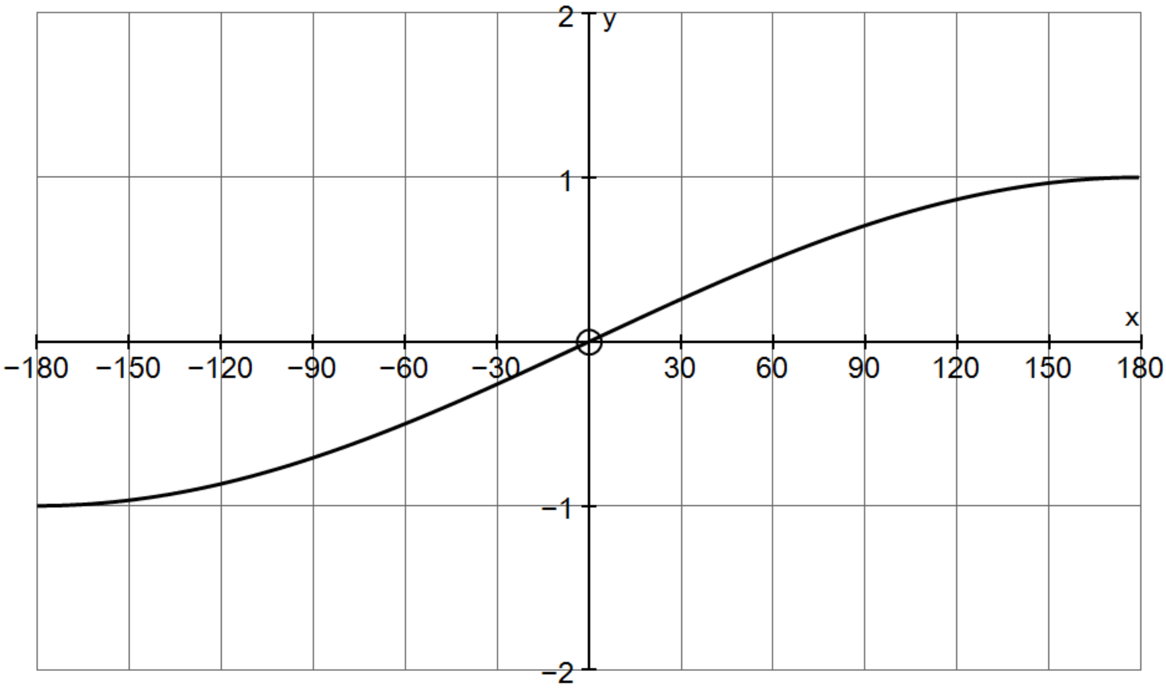
The graphs intersect at A$(x_a; y_a)$ and B$(x_b; y_b)$.

7.1 Write down the values of d, e and q. [3]

7.2 Use the graphs to solve for x if

7.2.1 $g(x) < 0$ [3]

7.2.2 $f'(x)$ and $g'(x)$ are both greater than or equal to zero but $x < 0$. [3]



QUESTION 8

Sketched below is $f(x) = \sin\frac{x}{2}$

8.1 For $f(x)$, write down the

8.1.1 range (1)

8.1.2 period (1)

8.2 Draw $g(x) = \cos(x + 60°)$ for $x \in [-180°; 180°]$.Clearly draw all intercepts with axes, turning points and starting and ending points. (3)

8.3 For which values of $x \in [-180°; 180°]$ is $f(x).g(x) \geq 0$. (4)

<u>FORMULAE AND THE SOLUTION OFTRIANGLES</u>

<u>HIGHTS AND DISTANCES</u>

The Prior Knowledge The Background Knowledge

The Assumed Knowledge The Previous Knowledge

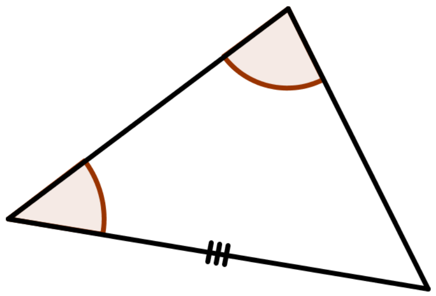
The Perceived Knowledge

1. THE SINE FORMULA

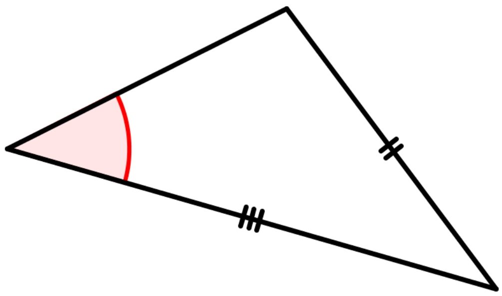
$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \quad \textbf{or} \quad \frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

Remember: The sine formula is used if **two angles and a side** are given in a triangle, or if **two sides and a non-included angle** are given.

TWO ANGLES AND A SIDE



TWO SIDES AND A NON INCLUDED ANGLE



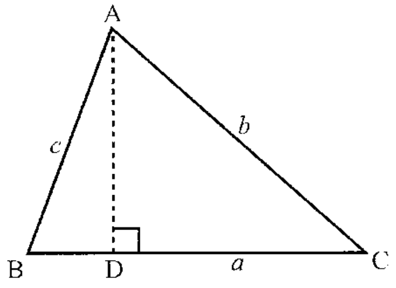
Remember: Opposite the longest side is the largest angle.

∴ When the triangle is obtuse-angled, the longest side is opposite the obtuse angle.

Given: An acute-angled ΔABC

Required to prove: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Construction: AD ⊥ BC



Proof:

$$\sin B = \frac{AD}{c}$$

$$\therefore AD = c.\sin B$$

$$\sin C = \frac{AD}{b}$$

$$\therefore AD = b.\sin C$$

$$\therefore c.\sin B = b.\sin C$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly by drawing CE $\perp$ AB, it can be proved that:

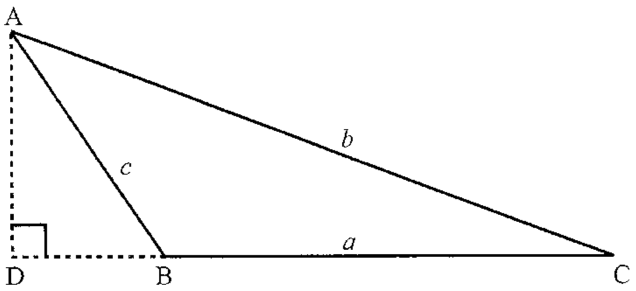
$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Given: ΔABC with B > 90°

Required to prove: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Construction: AD $\perp$ BC with CB produced.



Proof:

$$\sin A\hat{B}C = \sin A\hat{B}D = \frac{AD}{c}$$

$$\therefore AD = c.\sin B$$

$$\sin C = \frac{AD}{b}$$

$$\therefore AD = b.\sin C$$

$$\therefore c.\sin B = b.\sin C$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly by drawing CE $\perp$ AB with AB produced, it can be proved that:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

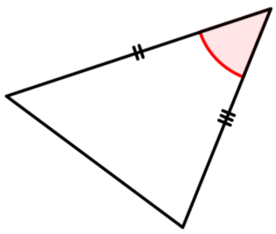
$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

2. THE COSINE FORMULA

$$\boldsymbol{a^2 = b^2 + c^2 - 2bcCosA}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2acCosB}$$

$$c^2 = a^2 + b^2 - 2abCosC$$



Remember: We use the cosine formula in this form to the third side of a triangle when two sides and the included angle are given.

$CosA = \frac{b^2+c^2-a^2}{2bc}$ $CosB = \frac{a^2+c^2-b^2}{2ac}$ $CosC = \frac{a^2+b^2-c^2}{2ab}$

Remember: We use this formula in this form if **three sides** of a triangle are given and an angle must be calculated.

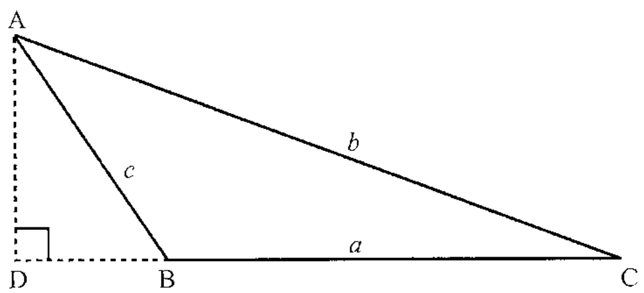
NB:

The Cosine-formula is used when the information in the triangle entails: S, S, S or S, A, S.

Given: ΔABC with $B > 90°$

Required to prove: $b^2 = a^2 + c^2 - 2ac \cos B$

Construction: $AD \perp BC$ with CB produced.



Proof:

$$\begin{aligned} b^2 &= AD^2 + DC^2 \text{ (Pythagoras in } \Delta ADC) \\ &= AD^2 + (a + BD)^2 \\ &= AD^2 + a^2 + 2a \cdot BD + BD^2 \\ &= (AD^2 + BD^2) + a^2 + 2a \cdot BD \\ &= c^2 + a^2 + 2a \cdot BD \end{aligned}$$

$$\begin{aligned} \cos A\hat{B}C &= -\cos A\hat{B}D \\ &= -\frac{BD}{c} \\ \therefore BD &= -c \cos B \end{aligned}$$

$$\therefore\ b^2 \quad = \quad a^2 + c^2 - 2ac\cos B$$

3. THE AREA FORMULA

Area of a ΔABC= $\frac{1}{2}ab\,Sin\hat{C}$

Area of a ΔABC = $\frac{1}{2}bc\,Sin\hat{A}$

Area of a ΔABC = $\frac{1}{2}ac\,Sin\hat{B}$

Remember: The area formula is used to calculate the area of a triangle.

An unknown side can also be d if the area, a side and an angle are given. Note that this formula actually means: The area of a triangle= $\frac{1}{2}$(product of two adjacent sides) multiplied by the sine of an included angle.

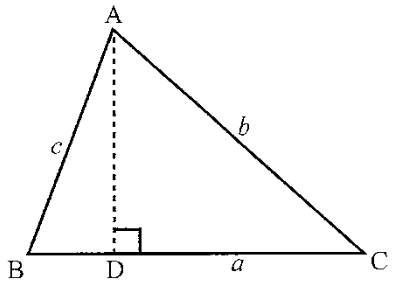
∴ In order to apply this formula, you only need: S, A, S in the triangle.

Proof :

Given: Acute-angled ΔABC.

Required to prove: Area $\Delta ABC = \frac{1}{2}ab\sin C$

Construction: $AD \perp BC$



Proof:

$$\text{Area of } \Delta ABC = \frac{1}{2}a \cdot AD$$

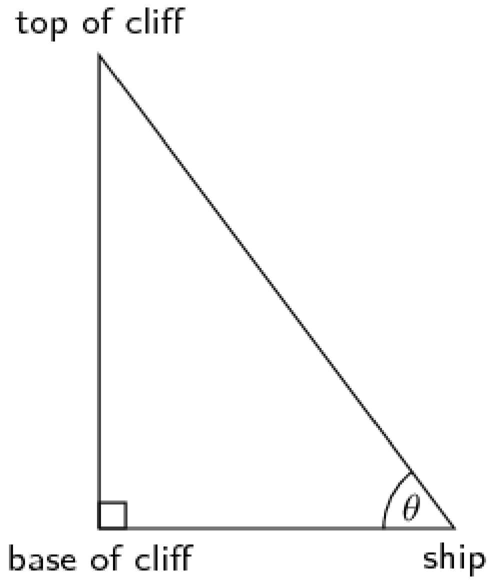
$$\sin C = \frac{AD}{b}$$

$$AD = b.\sin C$$

$$\therefore \text{Area } \Delta ABC = \frac{1}{2}ab\sin C$$

Two-dimensional

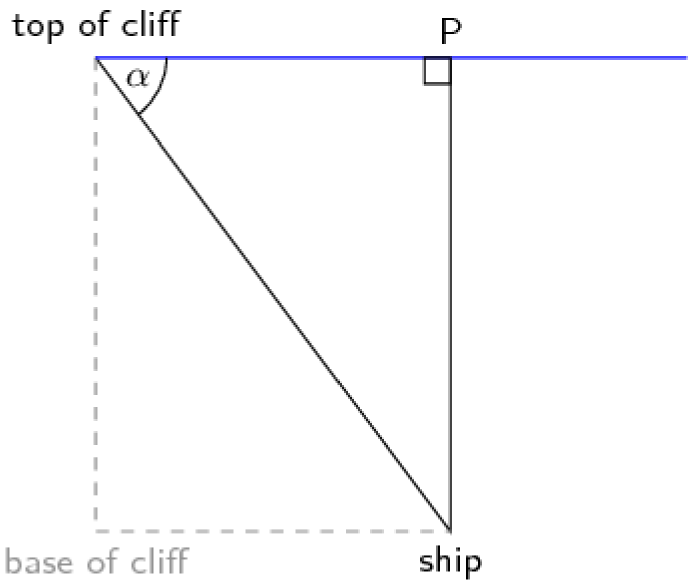
In two-dimensional problems we will often refer to the angle of elevation and the angle of depression. To understand these two angles let us consider a person sailing alongside some cliffs. The person looks up and sees the top of the cliffs as shown below:



In this diagram θ is the angle of elevation.

Angle of elevation

The angle of elevation is the angle formed by the line of sight and the horizontal plane for an object above the horizontal plane.



In this diagram α is the angle of depression.

Angle of depression

> The angle of depression is the angle formed by the line of sight and the horizontal plane for an object below the horizontal plane.

4. SOLVING PROBLEMS IN TWO DIMENSIONS

Always in problems involving two or more triangles, the same method as for a single triangle is used.

HINTS, CLUES

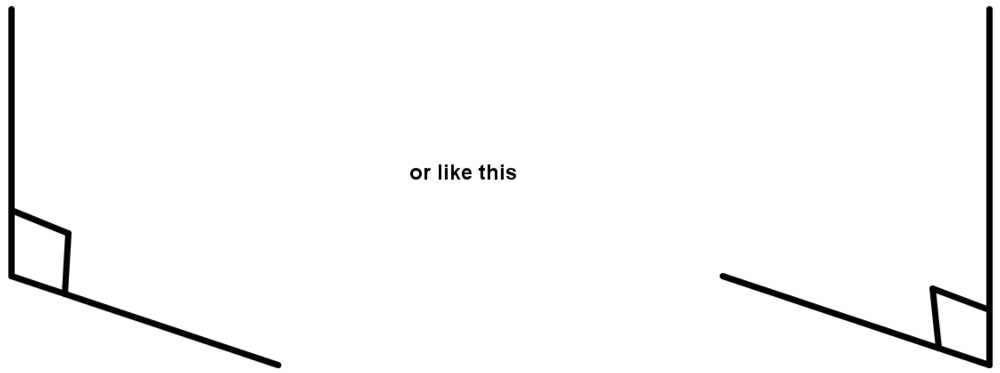
- The problem usually involves two triangles with a common side
- Often, one of the triangles is right-angled
- Use Geometry to obtain additional information, e.g. exterior angle of a triangle, corresponding and alternate angles
- Decide in which triangle the required side occurs. Start with the other triangle and calculate the common side using the sine or cosine formula
- Then use the sine formula or the cosine formula or trigonometrical ratios to solve the problem.

5. PROVING A FORMULA

- Sometimes we are required to prove some sort of a formula, before calculating a side or an angle.
- We use the same procedure as in solving a problem
- Trigonometrical identities such as: $Sin(90° - \theta) = Cos\theta$, $Cos(90° - \theta) = Sin\theta$ $Sin[180° - (x + y)] = Sin(x + y)$, etc. are used.

PROBLEMS IN THREE DIMENSIONS

- In three-dimensional problems right angles often don't look like right angles.
- Draw all vertical lines, vertical, so that a right angle may look like this:



- Always shade the horizontal plane roughly.
- Where you encounter problems with three triangles, you must work from the one with the most information via the second to the third.
- The cosine formula is used more often than in problems in two dimensions.

8. Examples/Activities

QUESTION 7

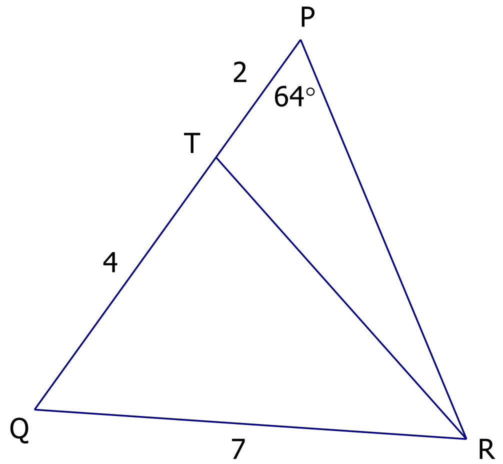
In the diagram, ΔPQR is drawn with T on PQ.

$\angle P = 64°$

QR = 7 units

PT = 2 units

QT = 4 units



1.1.1 Calculate the size of $\angle Q$, correct to the nearest degree. (4)

1.1.2 If $\angle Q = 66°$, determine the following:

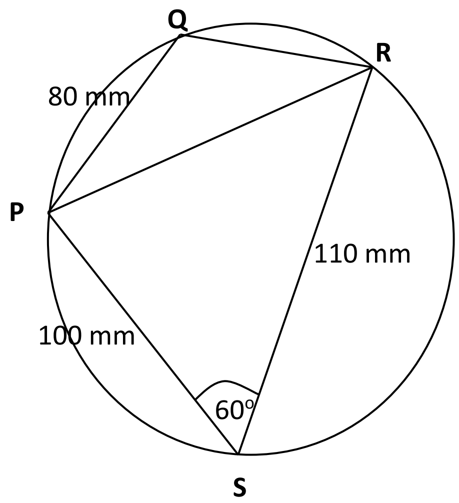
1.1.2.1 the area of ΔTQR. (3)

1.1.2.2 the length of TR. (3)

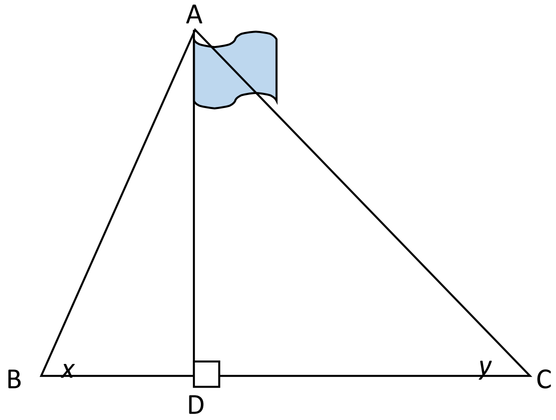
1.2 In the figure below PQ = 80 mm, PS = 100 mm, SR = 110 mm. $P\hat{S}R = 60^{\circ}$.

1.2.1 Show, by calculation that PR = 105,36 m (2)

1.2.2 Find the area of ΔPRS. (2)



1.2.1 B and C are in the same horizontal plane as D, the foot of flagpole AD. The angles of elevation to A (the top of the flagpole) are x and y degrees from B and C respectively. The sketch below illustrates the situation.

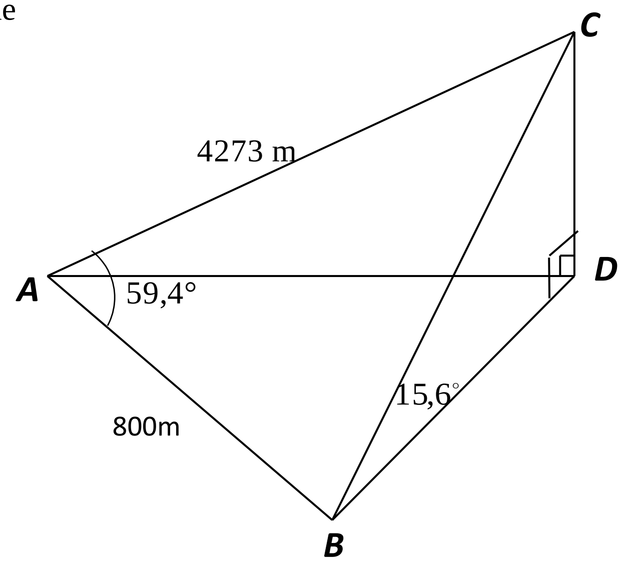


1.4.1 Show that: $AD = \frac{BC.\sin x .\sin y}{\sin(x+y)}$ (Hint: First find the length of AB) (5)

QUESTION 2

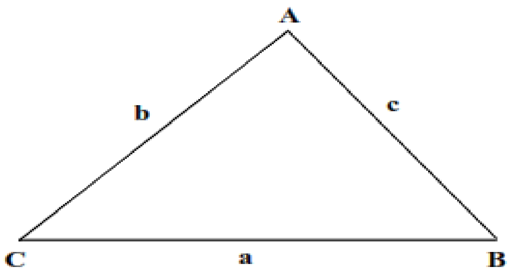
2.1 C is the top of a tower CD. A, B and D are in the same horizontal plane. The distance between A and B is 800m. CA is 4273m, $C\hat{A}B = 59,4°$ and the angle of elevation of C from B is $15,6°$.

Calculate the height of the tower, CD.



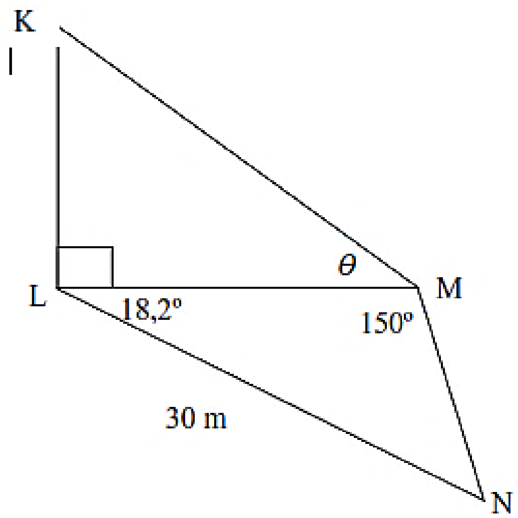
QUESTION 3

3.1 Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove that: $c^2 = a^2 + b^2 - 2ab.\cos C$



3.2 In the diagram below, KL is a building. Points L, M and N are in the same horizontal plane. The angle of elevation from M to the top of the building is .

$L\hat{M}N = 150°$ and $M\hat{L}N = 18{,}2°$. $LN = 30\ metres$.

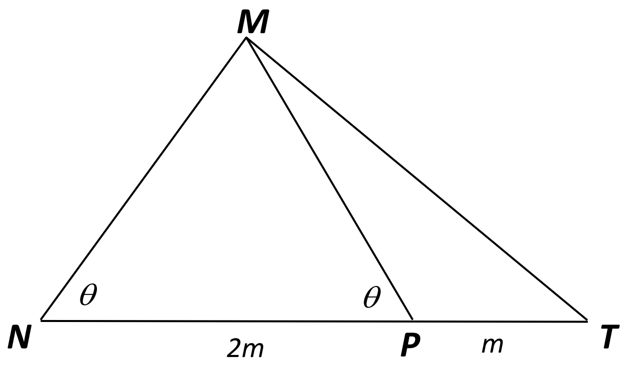


3.2.1 Show that $KL = 60\tan\theta\ .\sin 11{,}8°$. (5)

3.2.2 Calculate the height of the building, KL, if $\theta = 52{,}7°$. (2)

3.2.3 Calculate the area of ΔLMN. (2)

3.4 MNP is a triangle and P is a point on NT. MP is joined. PT = m, NP = $2m$ and $M\hat{P}N = \hat{N} = \theta$.



Prove :

Area $\Delta MTP = \frac{1}{2} m^2 \tan\theta$.

(4)

EUCLIDEAN GEOMETRY

GRADE 10

- Revise basic results established in earlier grades.
- Investigate line segments joining the mid-points of two sides of a triangle.
- Define the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and the trapezium. Investigate and make conjectures about properties of the sides, angles, diagonals and areas of these quadrilaterals. Prove these quadrilaterals.
- Solve problems and prove sides using the properties of parallel lines, triangles and quadrilaterals.

GRADE 11

- Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles.
    - The line drawn from the centre of a circle perpendicular to a chord bisects the chord;
    - The perpendicular bisector of a chord passes through the centre of the circle;
    - The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);
    - Angles subtended by a chord of the circle, on the same side of the chord, are equal;
    - The opposite angles of a cyclic quadrilateral are supplementary;
    - Two tangents drawn to a circle from the same point outside the circle are equal in length;
    - The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.
- Solve circle geometry problems, providing reasons for statements when required.
- Prove riders.

GRADE 12

1. Revise earlier (Grade 9) work on the necessary and sufficient conditions for polygons to be similar.
2. Prove (accepting results established in earlier grades):

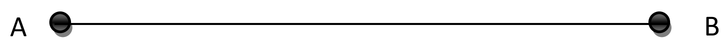
- that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Mid-point Theorem as a special case of this theorem);
- that equiangular triangles are similar;
- that triangles with sides in proportion are similar;
- the Pythagorean Theorem by similar triangles; and
- Riders

TERMINOLOGY

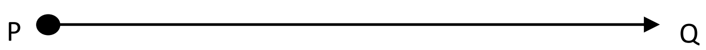
1.1 **POINT:** It is a location that can be described by giving its coordinates. It has no length or width. It is usually represented by a DOT (.). A capital letter (alphabet) is used to denote a point .e.g. P.

1.2 **LINE:** A set of points joined together. It can be straight or curved.

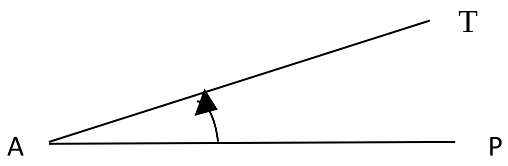
1.2.1 Line Segment: is obtained when two points are joined from one point **directly** to another. It has a fixed length. e.g.



1.2.2 **RAY**: a portion of a line which starts at a point and continue infinitely. It has no measurable length because it goes forever.



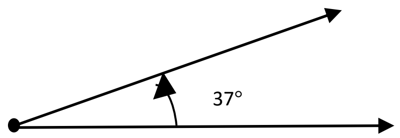
ANGLE: It is formed when two line segments meet at a point called a vertex. A shape formed by two lines or rays diverging from a common point (vertex).



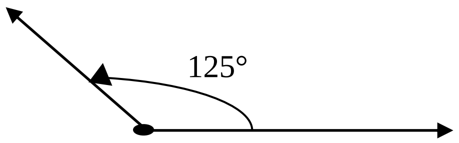
AP is a fixed arm and AT is a rotating arm. A protractor is an instrument used to measure an angle. The unit of measurement is degrees and is denoted by

<u>TYPES OF ANGLES</u>

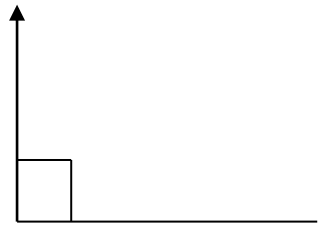
2.1 **ACUTE** ANGLE: angle between 0° and 90°



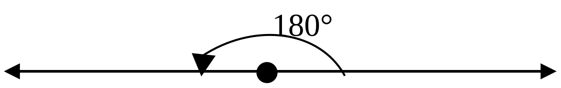
2.2 **OBTUSE** ANGLE: angle between 90° and 180°



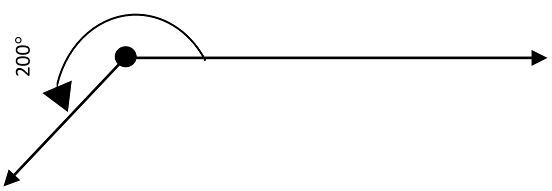
2.3 **RIGHT** ANGLE: angle which is of the size 90°



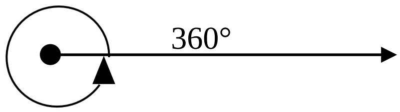
2.4 **STRAIGHT** ANGLE: angle of which the size is 180°



2.5 **REFLEX** ANGLE: angle between 180° and 360°

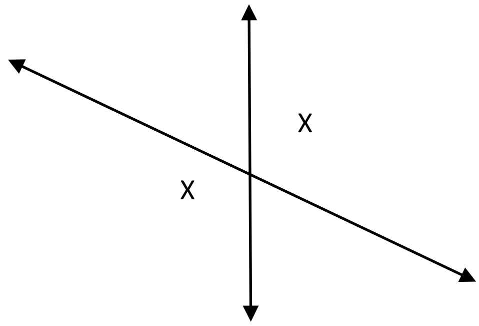


2.6 **REVOLUTION**: (FULL ANGLE) angle of which its magnitude is 360°

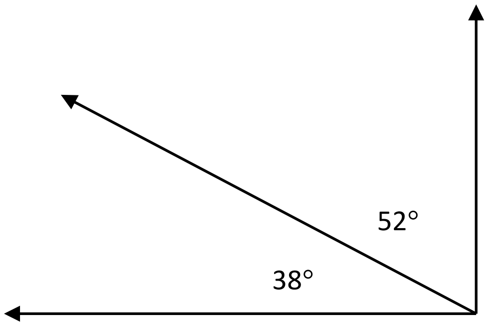


ANGLE RELATIONSHIP

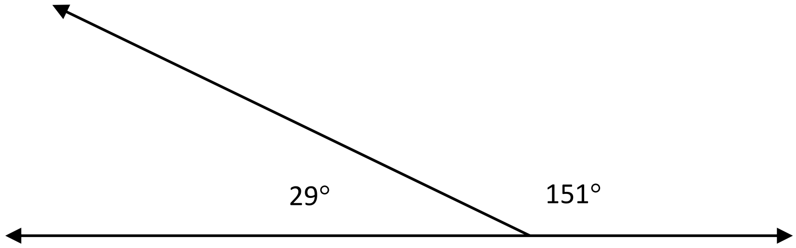
- Vertically Opposite: formed by intersection of two straight lines. Its "vertical" because they share the same vertex not that they are upright. They are equal.



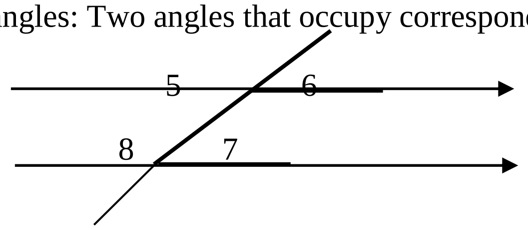
- Complementary angles: they add up to 90°



- Supplementary: they add up to 180°



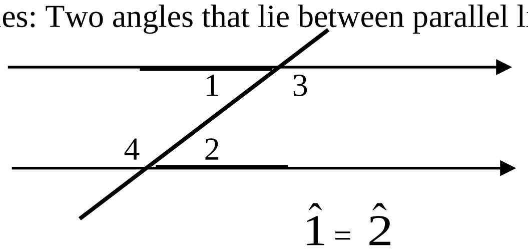
- Corresponding angles: Two angles that occupy corresponding positions (They form **F shape**).



$$\hat{6} = \hat{7}$$

$$\hat{5} = \hat{8}$$

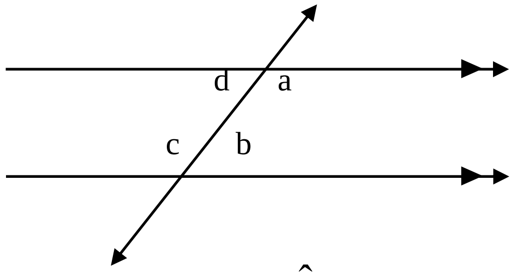
- Alternate angles: Two angles that lie between parallel lines on opposite sides of the transversal.



$$\hat{1} = \hat{2}$$

$$\hat{3} = \hat{4}$$

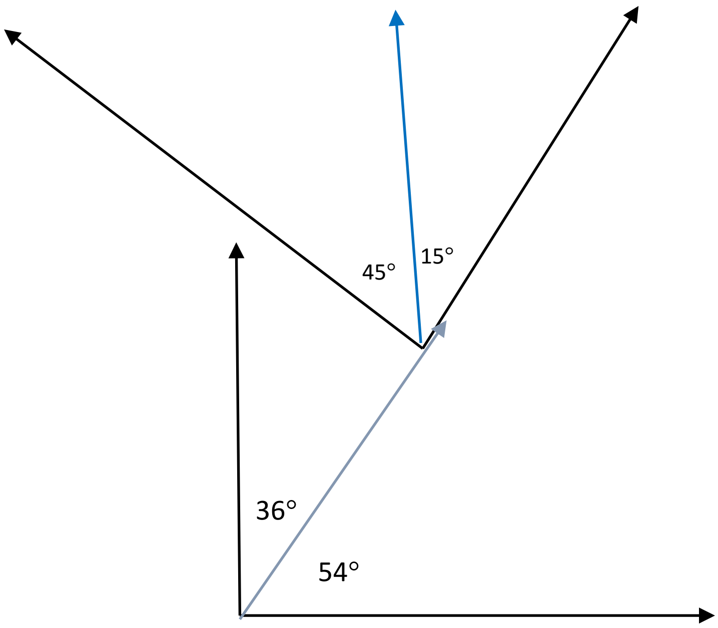
- Co-interior angles: Two angles that lie between parallel lines on the same side of the transversal. They add up to 180°.

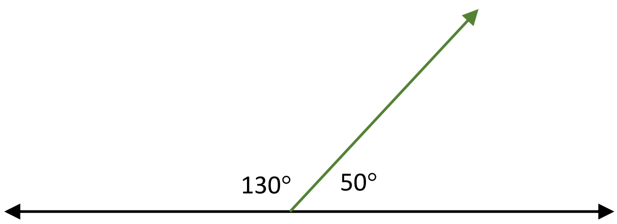


$$\hat{a} + \hat{b} = 180°$$

$$\hat{d} + \hat{c} = 180°$$

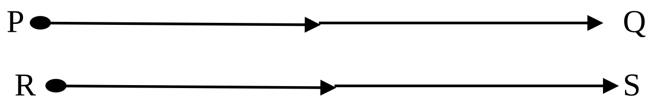
- Adjacent angles: Are "side by side" and share a common ray.





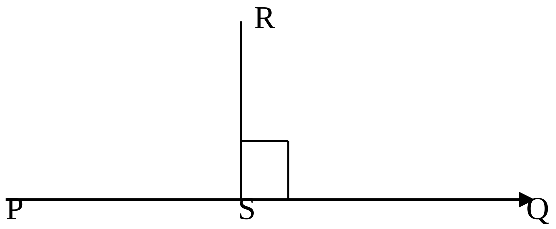
TYPES OF LINES

3.1 **PARALLEL** LINES: Lines which will never meet. They are denoted by sign **//.** They are always the same distance apart.

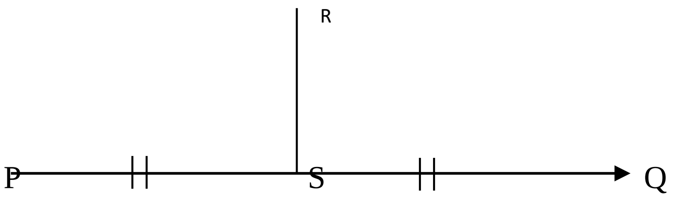


PQ // RS

3.2 **PERPENDICULAR** LINES**:** Lines that form an angle of 90 at their point of contact.

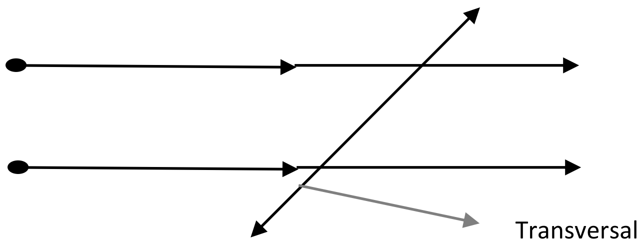


3.3 **BISECTOR**: A line, ray or line segment which cuts another line into two equal parts.



PS = SQ

3.3 **TRANSVERSAL** LINE: A line that cuts across the parallel lines.

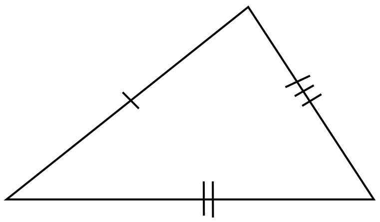


TRIANGLES

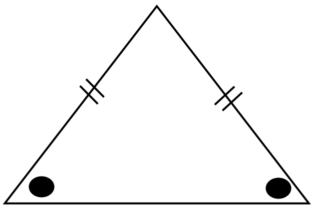
It is a closed geometrical figure with three sides and three interior angles. The three angles always add up to 180°.

TYPES OF TRIANGLES

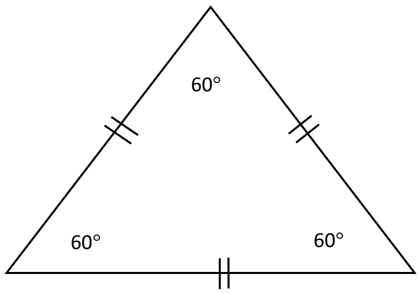
a) Scalene: no sides are equal and no angles are equal.



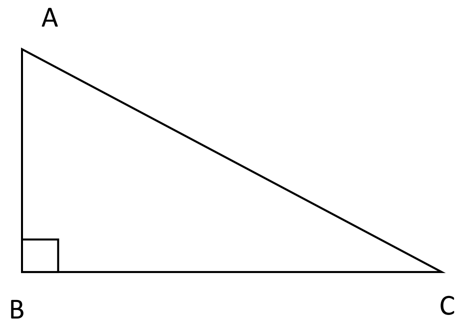
b) Isosceles: two sides are equal and two angles opposite equal sides.



c) Equilateral: three sides are equal and three angles are equal, each equal to 60°.



d) Right-angle: one angle is equal to 90°.

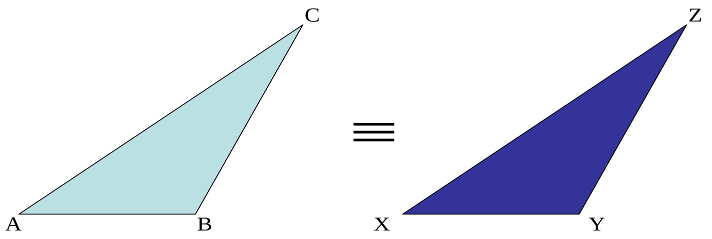


CONGRUENCY

CONGRUENT TRIANGLES

Congruent triangles are triangles that have the same shape and size. .i.e. corresponding sides are equal and corresponding angles are equal.

ΔABC is congruent to ΔXYZ

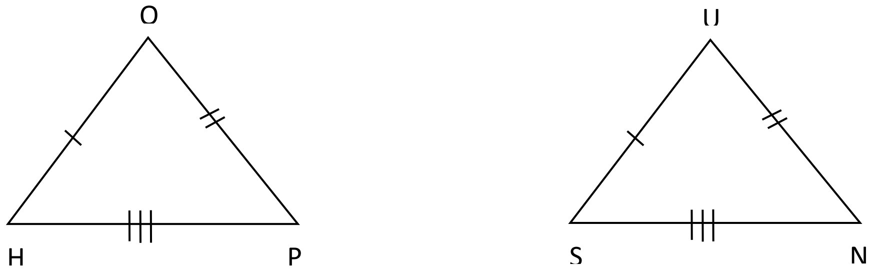


Corresponding parts of these triangles are equal.

Corresponding parts are angles and sides that "match."

CONDITIONS FOR CONGRUENCY

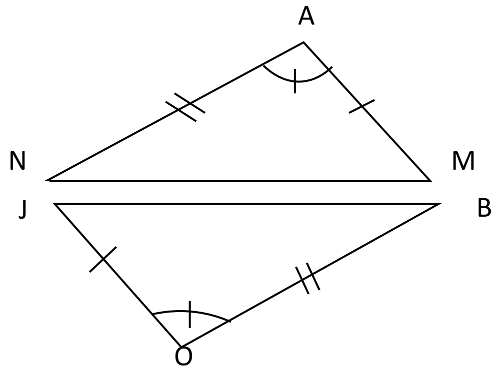
1



Δ HOP $\equiv$ Δ SUN (SSS)

i.e. corresponding sides (SSS) of the two triangles are equal.

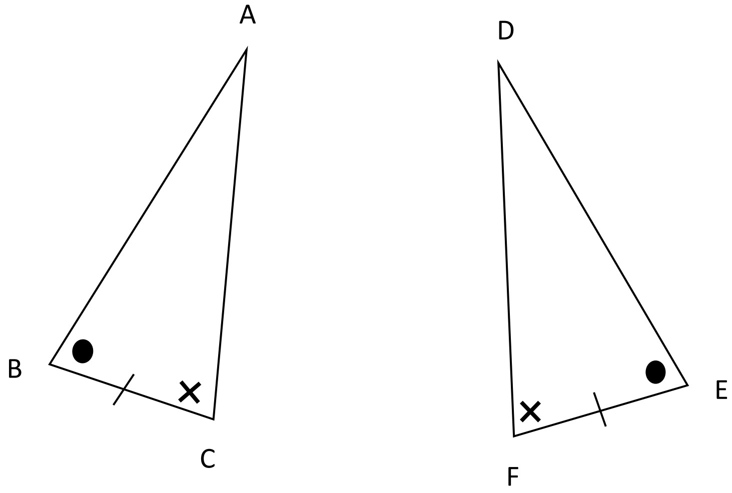
2



Δ MAN $\equiv$ Δ JOB (SAS)

i.e. if two sides and the angle between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

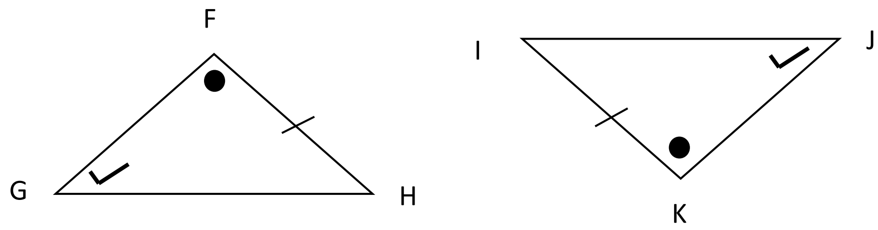
3



$\triangle$ ABC $\equiv$ $\triangle$ DEF (ASA)

i.e. If two angles and a side between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

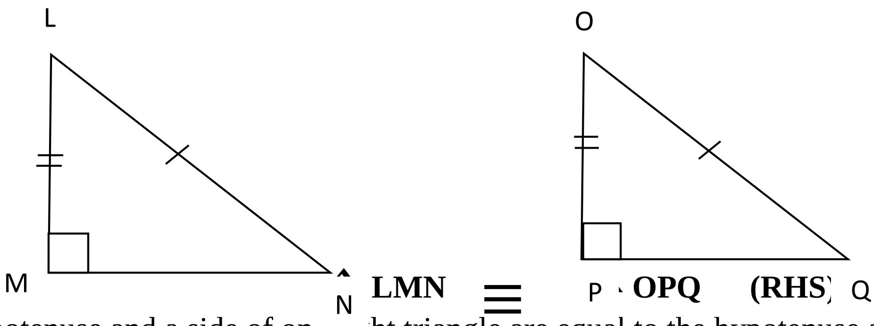
4



$\triangle$ FGH $\equiv$ $\triangle$ KJI (AAS)

i.e. If two angles and a side NOT between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

5



$\triangle$ LMN $\equiv$ $\triangle$ OPQ (RHS)

i.e. If the hypotenuse and a side of one right triangle are equal to the hypotenuse and side of another triangle, then the triangles are congruent.

<u>SUMMARY</u>

- For Grade 10, congruency is limited to triangles only.
- Conditions for triangles to be congruent:
  - $\rightarrow$ S,S,S
  - $\rightarrow$ S,A,S (Included Angle)
  - $\rightarrow$ A,A,S

→ A,S,A(Included Side)
→ R,H,S

QUADRILATERALS

A **quadrilateral** is a plane figure bounded by four sides.

Definitions of quadrilaterals

A **trapezium** is a quadrilateral with only one pair of sides parallel.

A **kite** is a quadrilateral with two pairs of adjacent sides equal but with no side common to both pairs.

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

A **rhombus** is a parallelogram with a pair of adjacent sides equal.

A **rectangle** is a parallelogram with a right angle.

A **square** is a rectangle with a pair of adjacent sides equal or a **square** is a rhombus with a right angle.

Five ways of proving that a quadrilateral is a parallelogram

Try to prove that:
Both pairs of opposite sides are parallel
Or both pairs of opposite sides are equal
Or both pairs of opposite angles are equal
Or the diagonals bisect each other
Or one pair of opposite sides is parallel **and** equal.

GRADE 11 EUCLIDEAN GEOMETRY

Note:

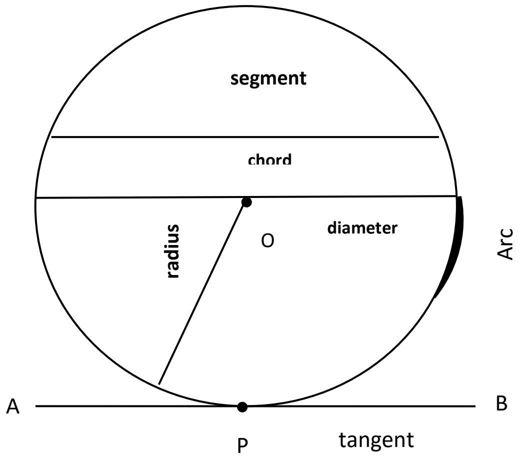
Proofs of theorems can be asked in examinations, but their converses (where they hold) cannot be asked.

CIRCLE GEOMETRY

Definitions

- A **circle** is a set of points that are equidistant from a fixed point called the **center.**
- The **circumference** of the circle is the distance around the edge of a circle.
- The **radius** is a line from the centre to any point on the circumference of the circle.
- A **chord** divides the circle into two **segments**.
- A **diameter** is a chord that passes through the centre. It is the longest chord and is equal to twice the radius.
- An **arc** is part of the circumference.
- A **semi-circle** is half the circle.
- A **tangent** is a line touching the circle at a point.
- **Cyclic quadrilaterals** have all their vertices on the circumference of a circle.

PARTS OF A CIRCLE



<u>Ways Of Presenting A Proof Argument</u>

1. **<u>Classical method:</u>** This is done by writing two columns (HOLY-CROSS), the first being a list of statements and the second column a <u>matching</u> list of legal justifications. i.e. theorems and axioms referred to as REASONS.

e.g.

Statement	Reason

<u>TYPES OF QUESTIONS FOR EUCLIDEAN GEOMETRY</u>

1. Calculations
2. Expressing an angle in terms of:
3. Proof type

1. Most of PROOF type questions would require that we first prove that angles are equal. How to Prove that angles are equal:

Suppose we are to prove that $\hat{A} = \hat{B}$

Scenario 1: (1) Write $\hat{A}$ and below it write $\hat{B}$

Statement	Reason
$\hat{A}$	
$\hat{B}$	

(2) Look for any angle equal to $\hat{A}$ (with reason)

Statement	Reason
$\hat{A} = \hat{P}$	Vertically opposite
$\hat{B}$	

(3) Compare $\hat{P}$ with $\hat{B}$. In most cases the two angles will be equal, then:

Statement	Reason
$\hat{A} = \hat{P}$	Vertically opposite
$\hat{B} = \hat{P}$	tan-chord theorem

(4) Then conclude:

Statement	Reason
$\hat{A} = \hat{P}$	vertically opposite
$\hat{B} = \hat{P}$	tan-chord theorem
$\therefore \hat{A} = \hat{B}$	transitivity of equality

2. For any PROOF type questions, there is always 3 steps to follow.

Step 1: Ask a question of WHEN

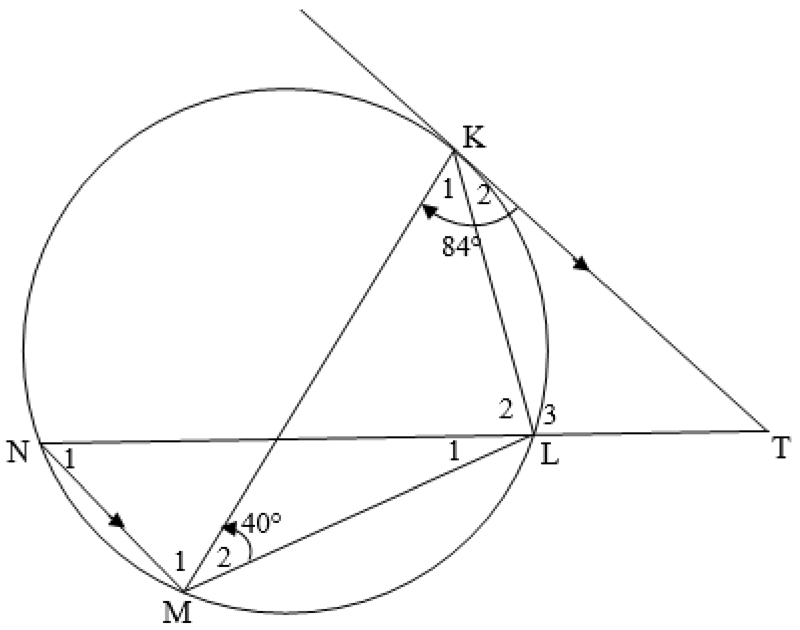
Step 2: Give possible answers to the question above.

Step 3: Prove one of the answers in Step 2 above.

EXAMPLES

Examples 1

In the diagram below, tangent KT to the circle at K is parallel to the chord NM. NT cuts the circle at L. ΔKML is drawn. $M_2 = 40^0$ and $M\hat{K}T = 84^0$



Determine, giving reasons, the size of:

1.1 $\hat{K}_2$

1.2 $\hat{N}_1$

1.3 $\hat{T}$

1.4 $\hat{L}_2$

1.5 $\hat{L}_1$

Solutions:

From the given information, key words are tangent parallel lines and chords. Therefore statements and reasons will be based on theorems which have these words. Use those theorems to determine the sizes of angles which their sizes are not given. When you put the size of an angle, write a short hand reason.

i.e. $M_1 = 84^0$ Alternate angles, NM // KT , $M_1 = L_2$ subtended by the same arc KN, $K_2 = 40^0$ tan chord theorem, $K_1 = N_1 = 44^0$ subtended by ML, $L_1 = 180^0 - (124 + 40^0) = 16^0$ sum of angles of a triangle etc.

1.1 $\hat{K}_2 = \hat{M}_2 = 40^0$ tan chord theorem

1.2 $\hat{N}_1 = \hat{K}_1 = 84^0 - 40^0 = 44^0$ subtended by chord ML

1.3 $\hat{T} = N_1 = 44^0$ Alternate angles NM // KT

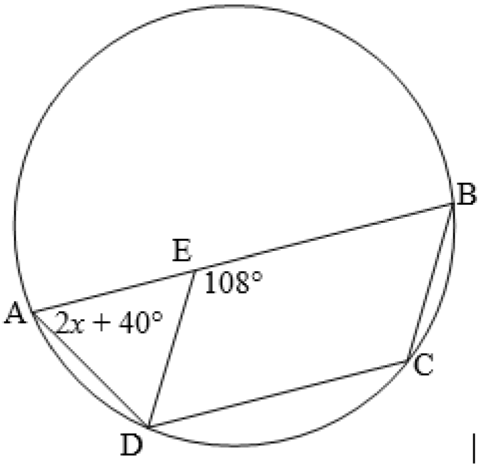
1.4 $\hat{M}_1 = 84^0$ Alternate angles, NM // KT

$L_2 = M_1 = 84^0$ subtended by the same arc KN

1.5 $\hat{L}_1 = 180^0 - (\hat{M} + \hat{N}_1)$ = $180^0 - (124 + 40^0) = 16^0$ sum of angles of a triangle

Example 2

In the diagram below, AB and DC are chords of a circle. E is a point on AB such that BCDE is a parallelogram. $D\hat{E}B = 108°$ and $D\hat{A}E = 2x + 40°$.



Calculate, giving reasons, the value of x.

Solution:

Properties of parallel gram and cyclic quadrilateral.

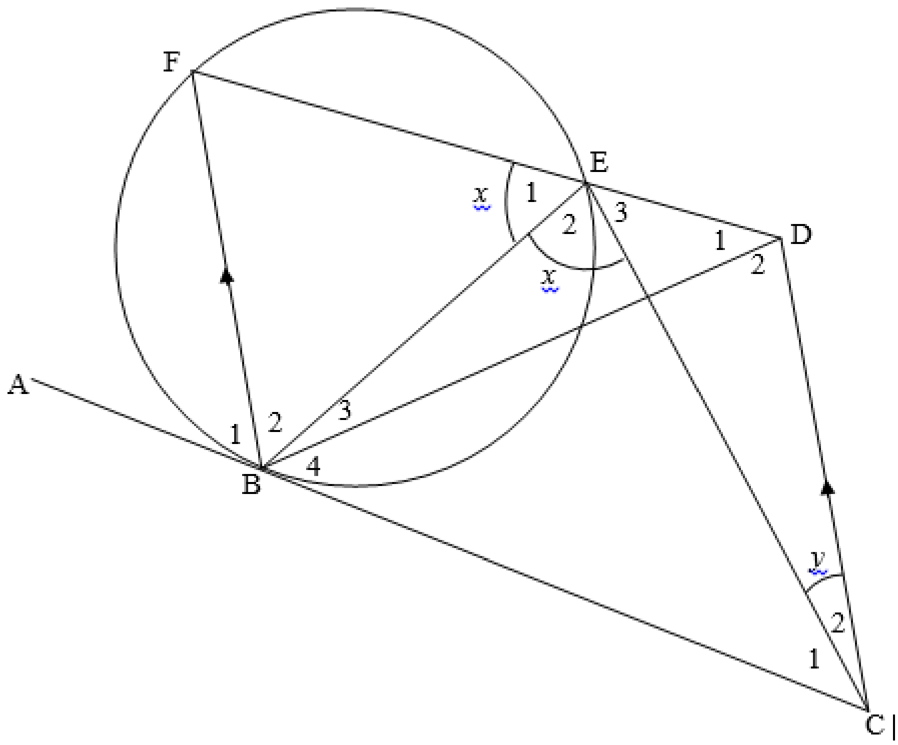
$\hat{C} = 108^0$ Opposite angles of a parallelogram are equal

$\hat{C} + D\hat{A}E = 180^0$; $108^0 + 2x + 40^0 = 180^0$ Opposite angles of cyclic quadrilateral ABCD

$\therefore 2x = 180^0 - 148^0 \Rightarrow 2x = 32^0$

$x = 16^0$

Example 3



3.1 Give reasons for the following statements

3.1.1 $\hat{B}_1 = x$

3.1.2 $B\hat{C}D = \hat{B}_1$

3.2 Prove that BCDE is a cyclic quadrilateral.

3.3 Which TWO other angles are each equal to *x*?

3.4 Prove that $\hat{B}_2 = \hat{C}_1$.

Solutions

3.1.1 tangent chord theorem

3.1.2 corresponding; FB || DC

3.2 $\hat{E}_1 = B\hat{C}D$

$\therefore$ BCDE = cyclic quad [converse ext $\angle$ cyc quad]

3.3 $\hat{D}_2 = \hat{E}_2$ [$\angle s$ in the same segment/$\angle e$ *in dies segment*]

$\hat{D}_2 = F\hat{B}D$ [alt $\angle s$, BF $\|$ CD/*verwiss* $\angle e$,*BF*||*CD*]

3.4 $\hat{B}_3 = y$ *OR* $\hat{B}_3 = \hat{C}_2$ [$\angle$s in the same segment]

$\hat{B}_2 = x - y$ *OR* $\hat{B}_3 + \hat{B}_2 = x$

$\hat{C}_1 = x - y$

$\therefore \hat{B}_2 = \hat{C}_1$

OR/*OF*

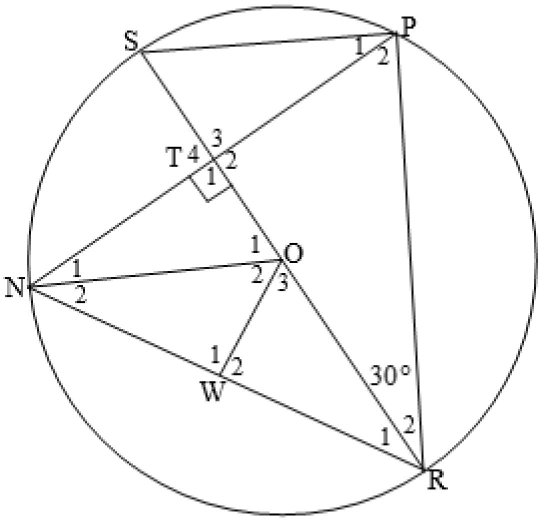
In ΔBFE and ΔBEC

$\hat{E}_1 = \hat{E}_2 \qquad [= x]$
$\hat{F} = \hat{B}_3 + \hat{B}_4 \qquad$ [tan - chord theorem]
$\therefore \Delta BFE /// \Delta CBE \quad [\angle, \angle, \angle]$
$\therefore \hat{B}_2 = \hat{C}_1$

EXERCISE

QUESTION 1

In the diagram, the vertices of ΔPNR lie on the circle with centre O. Diameter SR and chord NP intersect at T. Point W lies on NR. OT $\perp$ NP. $\hat{R}_2 = 30°$.



1.1 $\hat{S}$

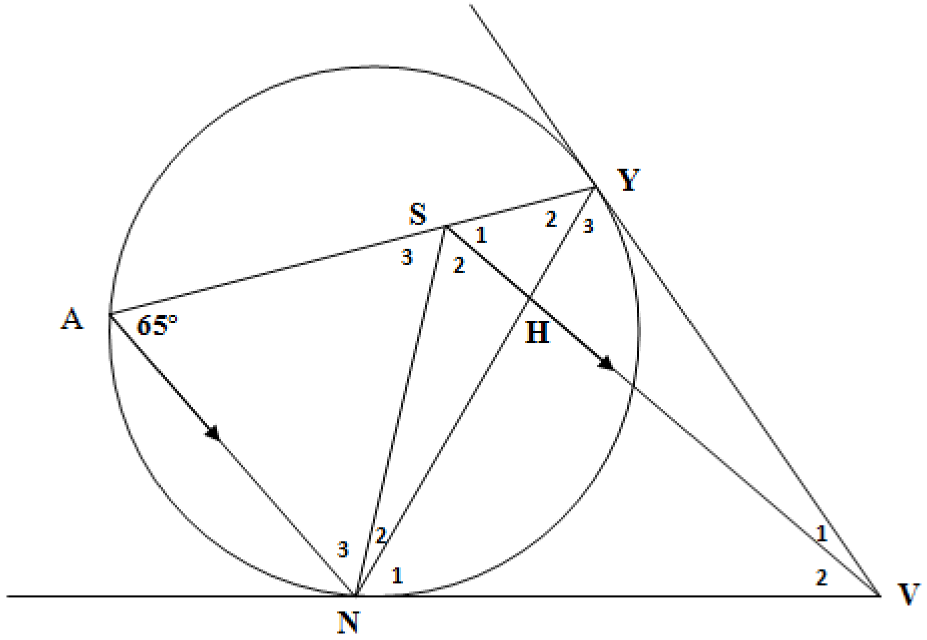
1.2 $\hat{R}_1$

1.3 $\hat{N}_1$

1.4 If it is further given that NW = WR, prove that TNWO is a cyclic quadrilateral.

QUESTION 2

VN and VY are tangents to the circle at N and Y. A is a point on the circle, and AN, AY and NY are chords so that $\hat{A} = 65°$. S is a point on AY so that AN || SV. S and N are joined



2.1 Write down, with reasons, THREE other angles each equal to 65°

2.2 Prove that VYSN is a cyclic quadrilateral.

2.3 Prove that ΔASN is isosceles.

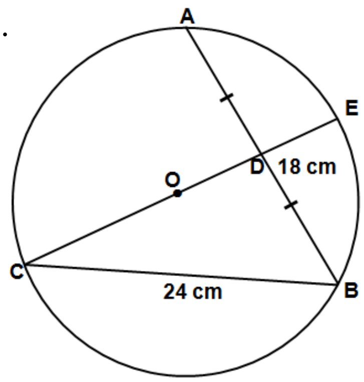
QUESTION 3

3.1 Complete the following so that the Euclidean Geometry statement is true:

A line drawn from the centre of a circle to the midpoint of the chord is ....to the chord (1)

3.2 In the circle with centre O, chord AB = 18cm and AD = DB.

Chord CB = 24 cm.



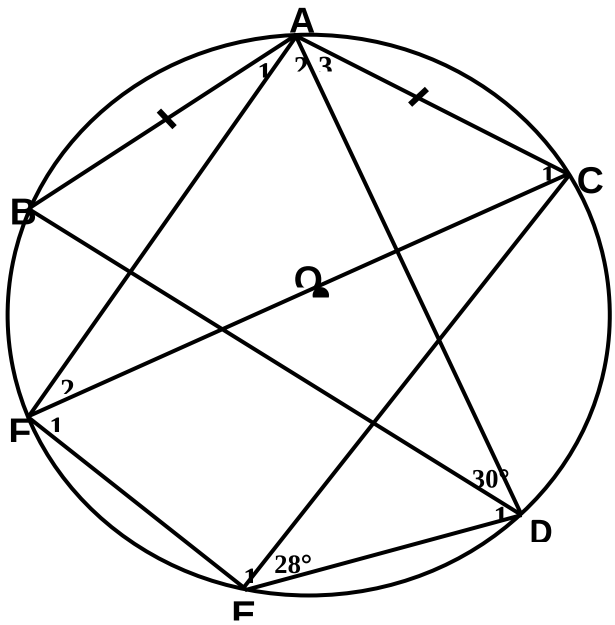
3.2.1 Calculate the length of CD. Leave the answer in simplest surd form. (4)

3.2.2 If $\frac{CD}{DE} = 3$, calculate the length of DE. (2)

[7]

QUESTION 4

In the diagram, O is the centre of the circle. Chords AB = AC. $C\hat{E}D = 28°$ and $A\hat{D}B = 30°$



Calculate, with reasons, the sizes of the following angles:

4.1 $\hat{E}_1$ (2)

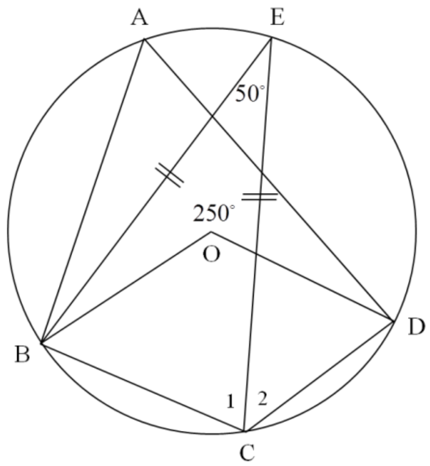
4.2 $\hat{A}_2$ (3)

4.3 $\hat{F}_2$ (2)

[7]

QUESTION 5

Refer to the figure below:



The circle, centred at O, has points A, B, C, D and E on the circumference of the circle. Reflex angle $B\hat{O}D = 250°$ and $B\hat{E}C = 50°$. Chord BE = EC. Determine the following, **stating all necessary reasons**:

5.1 $\hat{A}$ (2)

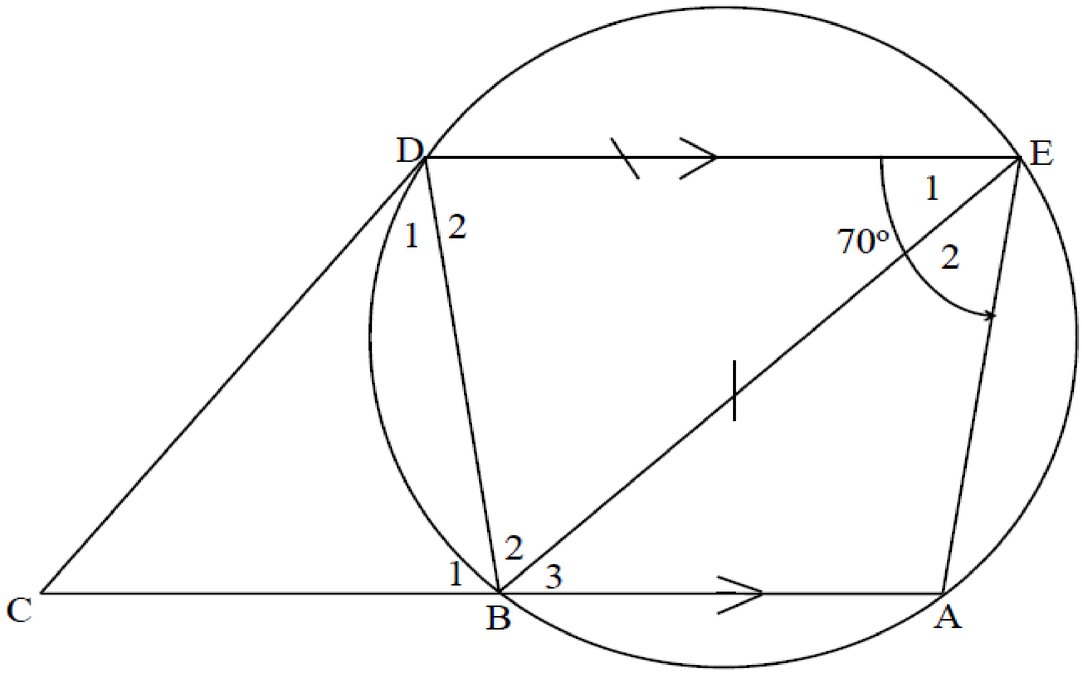
5.2 $B\hat{C}D$ (1)

5.3 $\hat{C}_2$ (2)

[5]

QUESTION 6

6.1 In the diagram below, BAED is a cyclic quadrilateral with BA || DE. BE = DE and $A\hat{E}D = 70°$. The tangent to the circle at D meets AB produced at C.



Calculate, with reasons the sizes of the following.

6.1.1 $\hat{A}$ (2)

6.1.2 $\hat{B}_1$ (2)

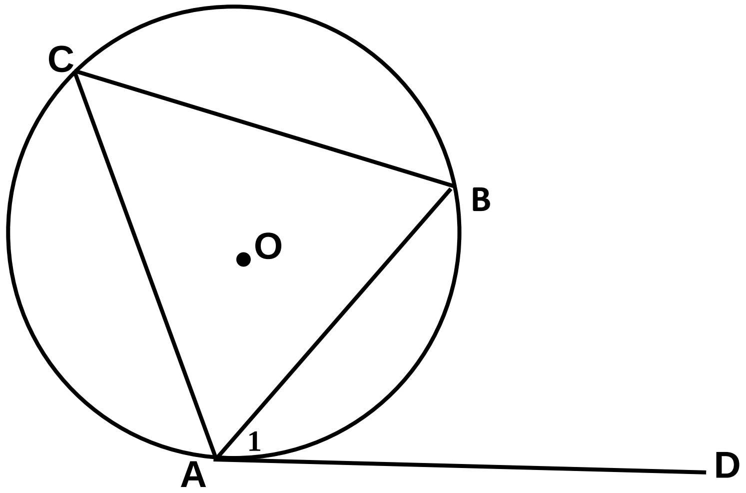
6.1.3 $\hat{D}_2$ (2)

6.1.4 $\hat{B}_2$ (2)

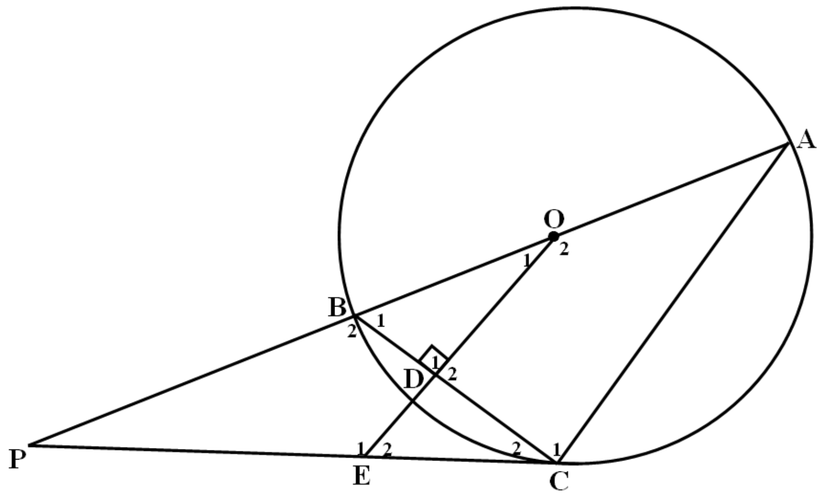
6.1.5 $\hat{D}_1$ (3)

QUESTION 7

7.1 Use the diagram to prove the theorem that states that $\hat{A}_1 = \hat{C}$ (5)



7.2 In the diagram, AB is a diameter of circle, centre O. AB is produced to P. PC is a tangent to the circle at C. $OE \perp BC$ at D.



7.2.1 Prove, with reasons, that EO || CA. (4)

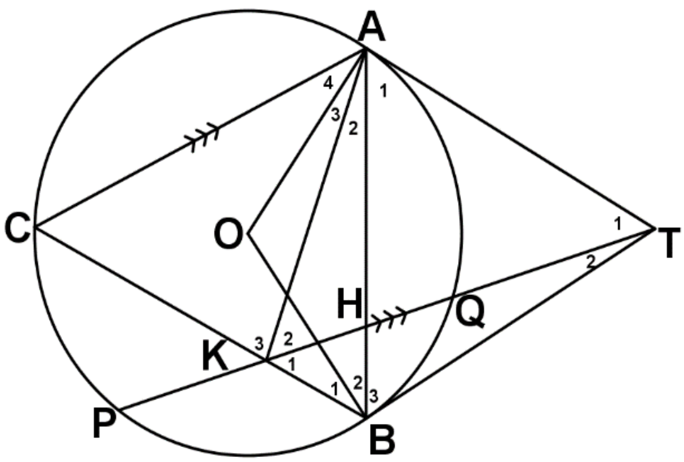
7.2.2 If $\hat{C}_2 = x$, name with reasons, two other angles each equal to x. (3)

7.2.3 Calculate the size of $\hat{P}$ in terms of x. (2)

[14]

QUESTION 8

In the diagram O is the centre of the circle passing through C,A and B.
TA and TB are two tangents to the circle at A and B.TQP cuts the circle at Q and P.
CA ‖PT. QP cuts AB and BC at H and K respectively.



Prove that:

8.3.1 AOBT is a cyclic quadrilateral. (4)

8.3.2 $H\hat{K}B = \hat{A}_1$. (4)

8.3.3 TA is a tangent to the circle through A, H and K. (4)

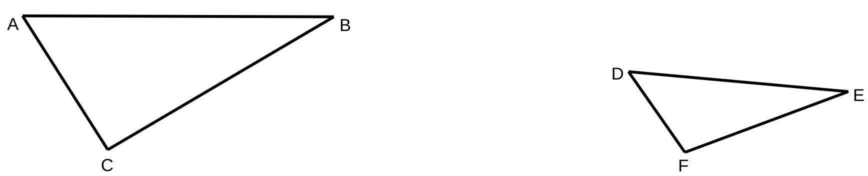
[20]

GRADE 12 EUCLIDEAN GEOMETRY

Similar triangles are triangles that have the same shape but not necessarily the same size. The symbol ||| stands for the phrase "is similar to".

Two triangles are similar if:

i. Their corresponding angles are equal, and
ii. their corresponding sides are proportional.



$$\Delta ABC \;|||\; \Delta DEF$$

When we say that triangles are similar there are several repercussions that come from it.

$\angle A = \angle D$

$\angle B = \angle E$ $\qquad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

$\angle C = \angle F$

SUMMARY

Conditions for similarity

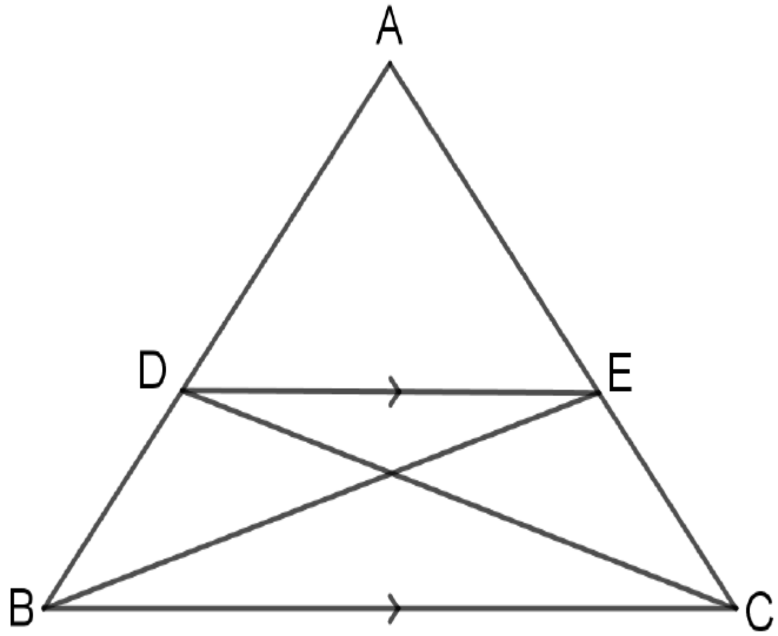
- If in two triangles, the corresponding angles are equal, the triangles are similar. (AAA)
- If the corresponding sides of two triangles are proportional, the triangles are similar.

EXERCISES

QUESTION 9

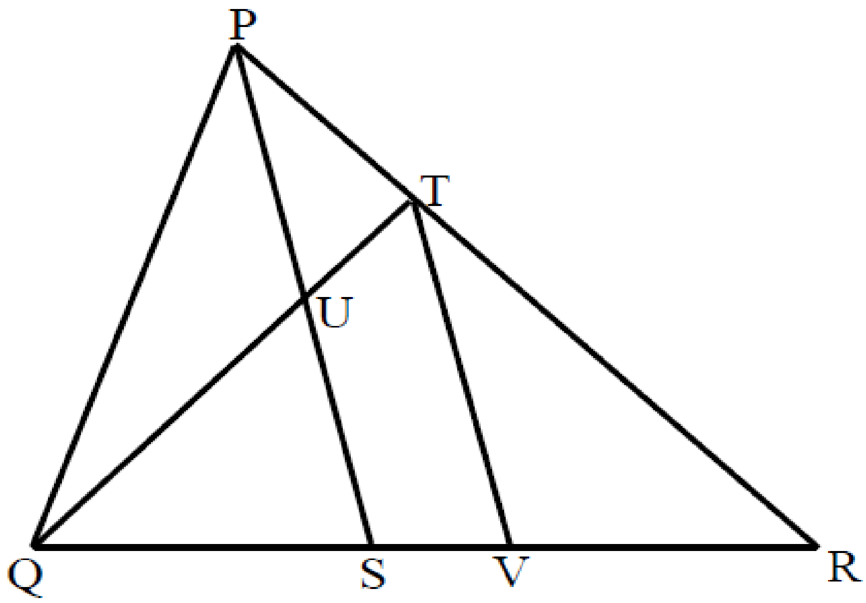
9.1 In the diagram below, ΔABC has $DE \parallel BC$. Prove the theorem that states

$$\frac{AD}{DB} = \frac{AE}{EC}$$



(7)

9.2 In ΔPQR, PS ∥ VT and QS : SR = $2:3$. T is a point on PR such that PT : TR = $2:7$.

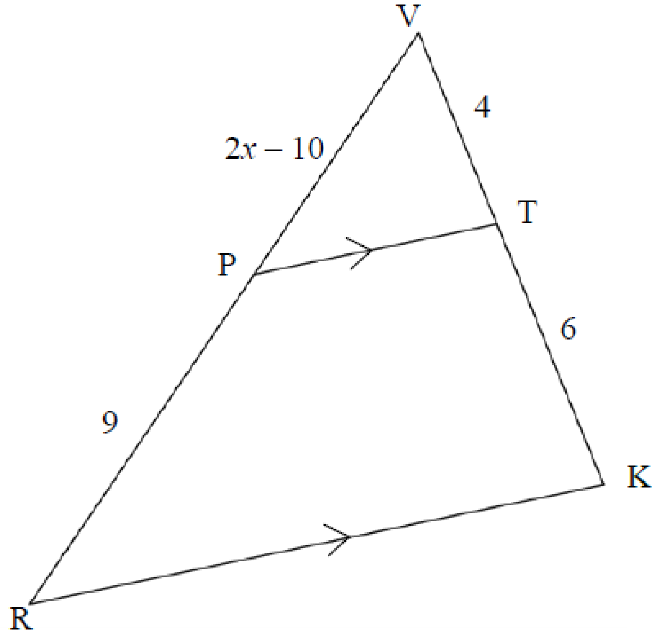


Calculate with reasons SV and then prove that QU : UT = $3:1$. (5)

QUESTION 10

In the diagram below, ΔVRK has P on VR and T on VK such that PT || RK. VT = 4 units, PR = 9 units, TK = 6 units and VP = $2x - 10$ units.

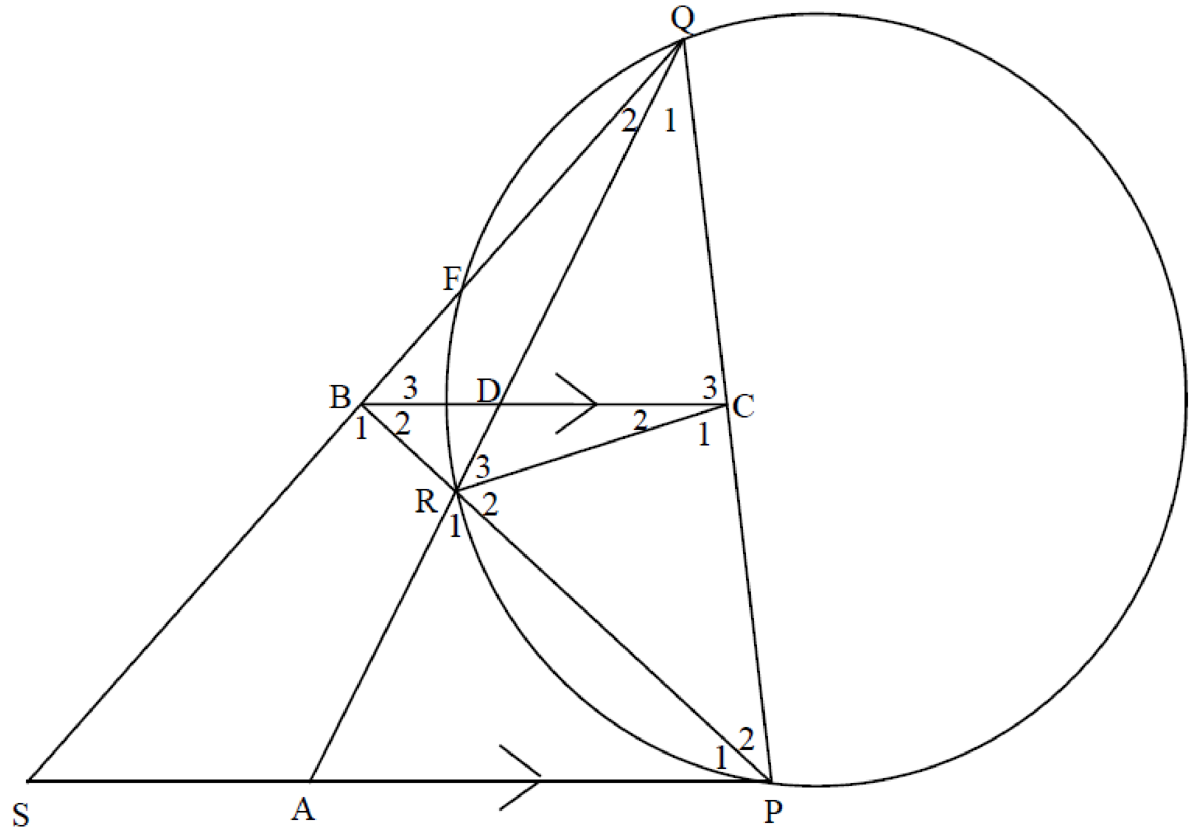
Calculate the value of x.



(4)

QUESTION 11

In the diagram below, SP is a tangent to the circle at P and PQ is a chord. Chord QF produced meets SP at S and chord RP **bisects** $\boldsymbol{Q\hat{P}S}$. PR produced meets QS at B. BC || SP and cuts the chord QR at D. QR produced meets SP at A. Let $\hat{B}_2 = x$



11.1 Name, with reasons, 3 angles equal to x. (6)

11.2 Prove that PC = BC (2)

11.3 Prove that RCQB is a cyclic quadrilateral. (2)

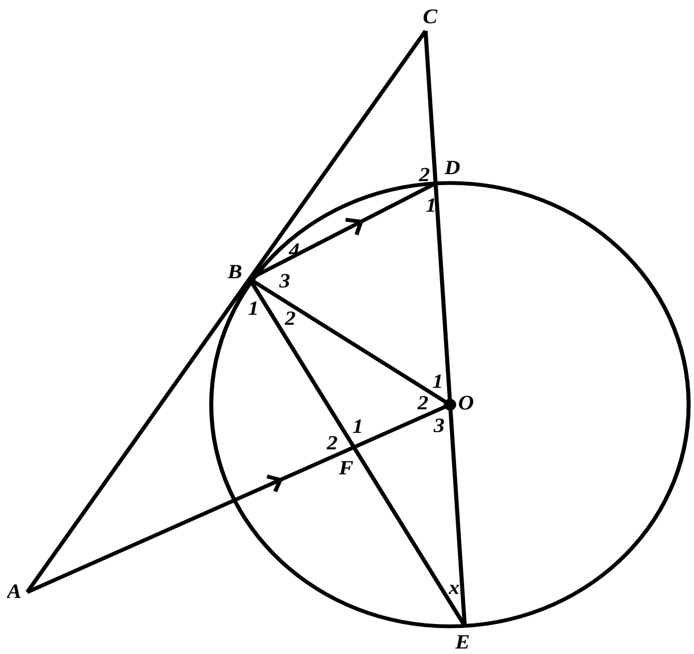
11.4 Prove that ΔPBS ||| ΔQCR. (3)

11.5 Show that PB.CR = QB.CP (3)

[16]

QUESTION 12

ED is a diameter of the circle, with center O. ED is extended to C. CA is a tangent to the circle at B. AO intersects BE at F. BD || AO. $\hat{E} = x$.



12.2.1 Write down, with reasons, THREE other angles equal to *x*. (4)

12.2.2 Determine, with reasons, $C\hat{B}E$ in terms of *x*. (2)

12.2.3 Prove that F is the midpoint of BE. (3)

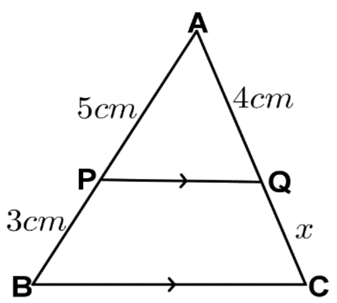
12.2.4 Prove that ΔCBD ||| Δ CEB. (3)

12.2.5 Prove that 2 EF. CB = CE. BD. (3)

[15]

QUESTION 13

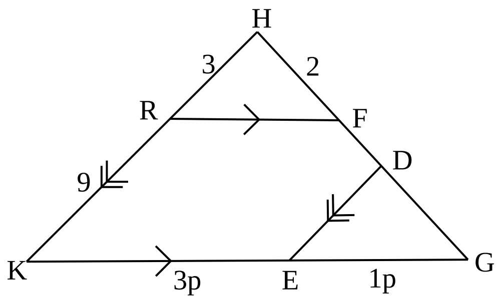
Determine the value of x if PQ $\|$ BC



QUESTION 14

In the diagram, RF ∥ KG , ED ∥ KH,

RH = 3 units, RK = 9 units, HF = 2 units.   GE: EK = 1:3



Calculate (stating reasons) the lengths of:

14.1 FG

14.2 FD