SCIENCE CLINIC

MATHS ESSENTIALS

GRADE 12

Content Acknowledgement

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INFORMATION SHEET: MATHEMATICS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$A = P(1 + ni)$ $\quad A = P(1 - ni)$ $\quad A = P(1 - i)^n$ $\quad A = P(1 + i)^n$

$T_n = a + (n-1)d$ $\quad S_n = \frac{n}{2}[2a + (n-1)d]$

$T_n = ar^{n-1}$ $\quad S_n = \frac{a(r^n - 1)}{r - 1}\ ;\ r \neq 1$ $\quad S_\infty = \frac{a}{1 - r};\ -1 < r < 1$

$F = \frac{x[(1+i)^n - 1]}{i}$ $\quad P = \frac{x[1 - (1+i)^{-n}]}{i}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\quad M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$y = mx + c$ $\quad y - y_1 = m(x - x_1)$ $\quad m = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad m = \tan\theta$

$(x - a)^2 + (y - b)^2 = r^2$

InΔABC: $\quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc.\cos A$

$area\,\Delta ABC = \frac{1}{2}ab.\sin C$

$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$ $\quad \sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$ $\quad \cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}$$

$\sin 2\alpha = 2\sin\alpha.\cos\alpha$

$\bar{x} = \frac{\sum x}{n}$ $\quad \sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$

$P(A) = \frac{n(A)}{n(S)}$ $\quad$ P(A or B) = P(A) + P(B) – P(A and B)

$\hat{y} = a + bx$ $\quad b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}$

NATURE OF ROOTS

NUMBERS → Non-real (R′)
(Sometimes referred to as imaginary)

Real (R)

- Rational (Q)
  Integers and Fractions
  - Integers (Z)
    Positive and negative whole numbers
    - Whole numbers (N_0)
      0 + N
      - Natural numbers (N)
        1, 2, 3...
- Irrational (Q′)
  EXAMPLE
  π
  $\sqrt{2}$
  All decimals that cannot be written as a fraction.

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a = the coefficient of x^2
b = the coefficient of x
c = the constant term

Used to factorise quadratic equations.

EXAMPLE

$$3x^2 + 2x - 4 = 0$$
$$a = 3$$
$$b = 2$$
$$c = -4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-4)}}{2(3)}$$

Then solve for x

THE DISCRIMINANT

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Indicated by Δ.

$$\therefore \Delta = b^2 - 4ac$$

EXAMPLE

$$3x^2 + 2x - 4 = 0$$
$$a = 3$$
$$b = 2$$
$$c = -4$$

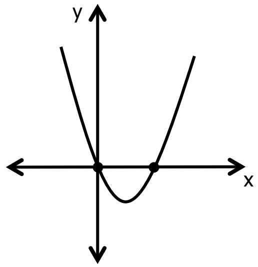
$$\Delta = b^2 - 4ac$$
$$\Delta = (2)^2 - 4(3)(-4)$$

Then solve for Δ

NATURE: Refers to the type of numbers that the roots are.

ROOTS: The x-intercepts/solutions/zeros of a quadratic equation.

Two real roots



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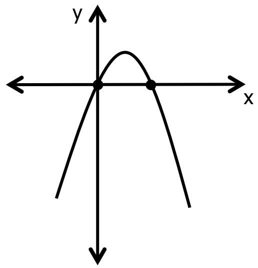
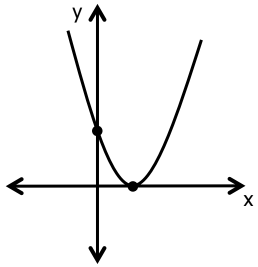


One real root



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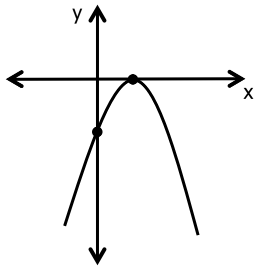
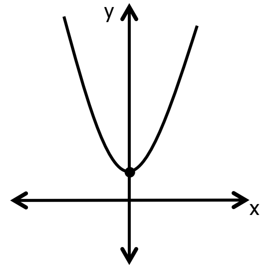


No (0) real roots



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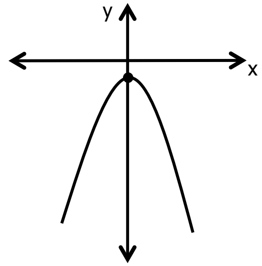


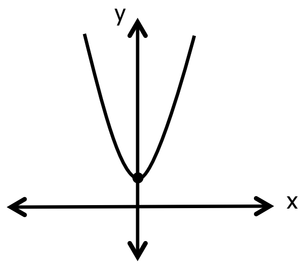
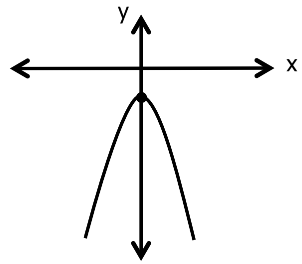
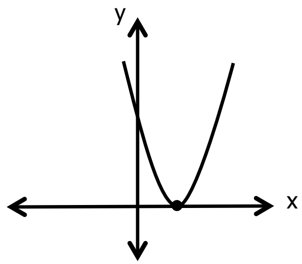
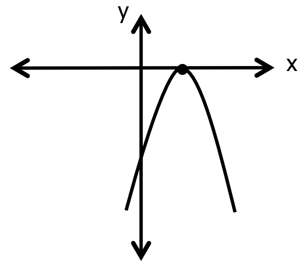
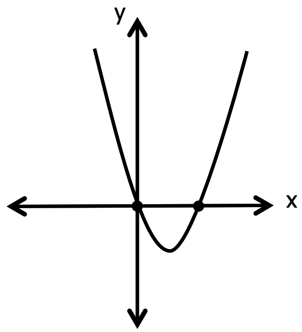
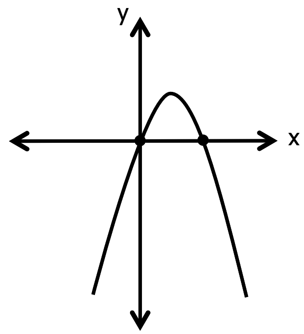
DETERMINING THE NATURE OF THE ROOTS

The **DISCRIMINANT** is used to determine the nature of the roots.

Δ

- $\Delta < 0$
  NO Real roots
- $\Delta \geq 0$
  Real roots
  - $\Delta > 0$
    2 *unequal* roots
    - Δ = perfect square
      rational roots
    - $\Delta \neq$ perfect square
      irrational roots
  - $\Delta = 0$
    2 *equal*, real roots

NATURE OF ROOTS

<table>
<tr><th rowspan="2">EXAMPLES</th><th rowspan="2">DISCRIMINANT
($\Delta = b^2 - 4ac$)</th><th rowspan="2">NATURE OF ROOTS</th><th rowspan="2">NUMBER OF REAL ROOTS</th><th colspan="2">$b^2 - 4ac$</th></tr>
<tr><th>a > 0</th><th>a < 0</th></tr>
<tr><td>$x^2 + x + 1 = 0$
a b c</td><td>$\Delta = b^2 - 4ac$
$= (1)^2 - 4(1)(1)$
$= 1 - 4$
$= -3$
$\Delta < 0$</td><td>Non real</td><td>0</td><td></td><td></td></tr>
<tr><td>$x^2 - 6x + 9 = 0$
a b c</td><td>$\Delta = b^2 - 4ac$
$= (-6)^2 - 4(1)(9)$
$= 36 - 36$
$= 0$
$\Delta = 0$</td><td>Real ($\Delta = +$)
Rational ($\Delta =$ perfect square)
Equal ($\Delta = 0$)</td><td>1 (2 of the same)</td><td></td><td></td></tr>
<tr><td>$x^2 - 5x - 6 = 0$
a b c</td><td>$\Delta = b^2 - 4ac$
$= (-5)^2 - 4(1)(-6)$
$= 25 + 24$
$= 49$
$\Delta > 0$ (perfect square)</td><td>Real ($\Delta = +$)
Rational ($\Delta =$ perfect square)
Unequal ($\Delta \neq 0$)</td><td>2</td><td rowspan="2"></td><td rowspan="2"></td></tr>
<tr><td>$2x^2 + 3x - 7 = 0$
a b c</td><td>$\Delta = b^2 - 4ac$
$= (3)^2 - 4(2)(-7)$
$= 9 + 56$
$= 65$
$\Delta > 0$ (not perfect square)</td><td>Real ($\Delta = +$)
Irrational ($\Delta \neq$ perfect square)
Unequal ($\Delta \neq 0$)</td><td>2</td></tr>
</table>

Nature of Roots

DETERMINING THE NATURE OF ROOTS WITHOUT SOLVING THE EQUATION

The roots of an equation can be determined by calculating the value of the discriminant (Δ).

Steps to determine the roots using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the nature of the roots of the equation

EXAMPLE

Determine the nature of the roots of $x^2 = 2x + 1$ without solving the equation

1. Standard form

$x^2 = 2x + 1$
$x^2 - 2x - 1 = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-1)$
$\Delta = 4 + 4$
$\Delta = 8$

3. Determine the nature of the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

FOR WHICH VALUES OF k WILL THE EQUATION HAVE EQUAL ROOTS?

The discriminant (Δ) can be used to calculate the unknown value of k. (e.g. Ask yourself, for which values of k will the discriminant be 0?)

Steps to determine the values of k using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Equate the discriminant to 0 and solve for k (quadratic equation)

EXAMPLE

For which values of k the equation will have equal roots?

REMEMBER: $\Delta = 0$ for equal roots

1. Standard form

$x^2 + 2kx = -4x - 9k$
$x^2 + \underline{2kx + 4x} + 9k = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (2k + 4)^2 - 4(1)(9k)$
$\Delta = 4k^2 + 16k + 16 - 36k$
$\Delta = 4k^2 - 20k + 16$

3. Equate to zero (0) and solve for k

$0 = 4k^2 - 20k + 16 \quad (\div 4)$
$0 = k^2 - 5k + 4$
$0 = (k - 1)(k - 4)$
Therefore $k = 1$ **or** $k = 4$
k needs to either be **1** or **4** to ensure that the discriminant of the equation is 0 (the discriminant **must** be 0 in order for equal roots)

PROVE THE NATURE OF THE ROOTS

The nature of the roots will be supplied and the discriminant can be used to prove the nature, with either one, or no, unknown value.

Steps to prove the nature of roots (NO unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

Prove the equation has two, unequal, irrational roots: $x^2 = 2x + 9$

1. Standard form

$x^2 - 2x - 9 = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-9)$
$\Delta = 4 + 36$
$\Delta = 40$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

Steps to prove the nature of roots (ONE unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

For the equation $x(6x - 7m) = 5m^2$, prove that the roots are real, rational and unequal if $m > 0$

1. Standard form

$6x^2 - 7mx - 5m^2 = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-7m)^2 - 4(6)(-5m^2)$
$\Delta = 49m^2 + 120m^2$
$\Delta = 169m^2$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Rational ($\Delta =$ perfect square)

QUADRATIC EQUATIONS

Quadratic Equations are equations of the second degree (i.e. the highest exponent of the variable is 2). The degree of the equation determines the maximum number of real roots/solutions/x-intercepts/zeros. The standard form of a quadratic equation is:

$$ax^2 + bx + c = 0 \quad where\ a \neq 0$$

SOLVING QUADRATIC EQUATIONS

FACTORING	QUADRATIC FORMULA	DIFFERENCE OF TWO SQUARES	COMPLETE THE SQUARE	ONE ROOT	TWO ROOTS	FRACTIONS AND RESTRICTIONS
1. Put in standard form 2. Apply the zero factor law 3. List possible solutions	Substitute into the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a = coefficient of x^2, b = coefficient of x, c = constant term		1. Write in standard form 2. Move C across 3. Divide both sides by A 4. Add (½ x b)^2 to both sides 5. Factorise and solve	1. Substitute the known root 2. Solve for the variable 3. Substitute the value of the variable and solve for the root	1. Substitute the roots into the equation 2. Use "FOIL" for the quadratic equation	1. Find the LCD and list restrictions 2. Solve for x 3. Check your answers against your restrictions REMEMBER: $\frac{0}{x} = 0$ BUT $\frac{x}{0} = undefined$
Eg. $x^2 = -2x + 63$ $x^2 + 2x - 63 = 0$ Find factors of 63 so that F1 x F2 = –63 and F1 + F2 = 2 $(x + 9)(x - 7) = 0$ $x + 9 = 0$ or $x - 7 = 0$ $x = -9$ $\quad$ $x = 7$	Eg. $-3x^2 = -12 + 7x$ $-3x^2 - 7x + 12 = 0$ a = –3; b = –7; c = 12 $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(12)}}{2(-3)}$ $x = \frac{7 \pm \sqrt{49 + 144}}{-6}$ $x = \frac{7 \pm \sqrt{193}}{-6}$ $x = \frac{7 + \sqrt{193}}{-6} \quad or \quad x = \frac{7 - \sqrt{193}}{-6}$ Answer in surd form or can be calculated/rounded off to 2 decimals x = –3,48 OR x = 1,15	Either method may be used Eg. $x^2 = 25$ $x^2 - 25 = 0$ $(x - 5)(x + 5) = 0$ x = 5 or x = –5 Therefore x = ±5 $\sqrt{x^2} = \pm\sqrt{25}$ x = ±5	Eg. $x^2 + 2x = 1$ (½ . 2)^2 (1) $x^2 + 2x + 1 = 1 + 1$ $\sqrt{(x+1)^2} = \pm\sqrt{2}$ $x + 1 = \pm\sqrt{2}$ $x + 1 = -\sqrt{2} \quad or \quad x + 1 = \sqrt{2}$ $x = -1 - \sqrt{2} \quad\quad x = -1 + \sqrt{2}$	Eg. $x^2 + mx - 15 = 0$, where 5 is a root. $(5)^2 + (5)m - 15 = 0$ $25 + 5m - 15 = 0$ $5m = -10$ $m = -2$ $x^2 - 2x - 15 = 0$ $(x - 5)(x + 3) = 0$ x = 5 or x = –3 (given)	Eg. –9 and 7 are the roots of a quadratic equation x = –9 or x = 7 x + 9 = 0 $\quad$ x – 7 = 0 $(x + 9)(x - 7) = 0$ $x^2 - 7x + 9x - 63 = 0$ $x^2 + 2x - 63 = 0$	Eg. $\frac{x}{x-2} = \frac{1}{x-3} - \frac{2}{2-x}$ $\frac{x}{x-2} = \frac{1}{x-3} + \frac{2}{x-2}$ LCD: $(x-2)(x-3)$ Restrictions: $x - 2 \neq 0;\ x - 3 \neq 0$ $\frac{x\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} = \frac{1(x-2)\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{2\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$ $x(x - 3) = 1(x - 2) + 2(x - 3)$ $x^2 - 3x = x - 2 + 2x - 6$ $x^2 - 3x - x - 2x + 2 + 6 = 0$ $x^2 - 6x + 8 = 0$ $(x - 4)(x - 2) = 0$ x – 4 = 0 or x – 2 = 0 x = 4 $\quad$ x = 2 Check restrictions: x ≠ 2, x ≠ 3 Thus, x = 4 is the only solution.

Remember:

* LHS = Left Hand Side * RHS = Right Hand Side
* Fractions and their restrictions

Finding factors:

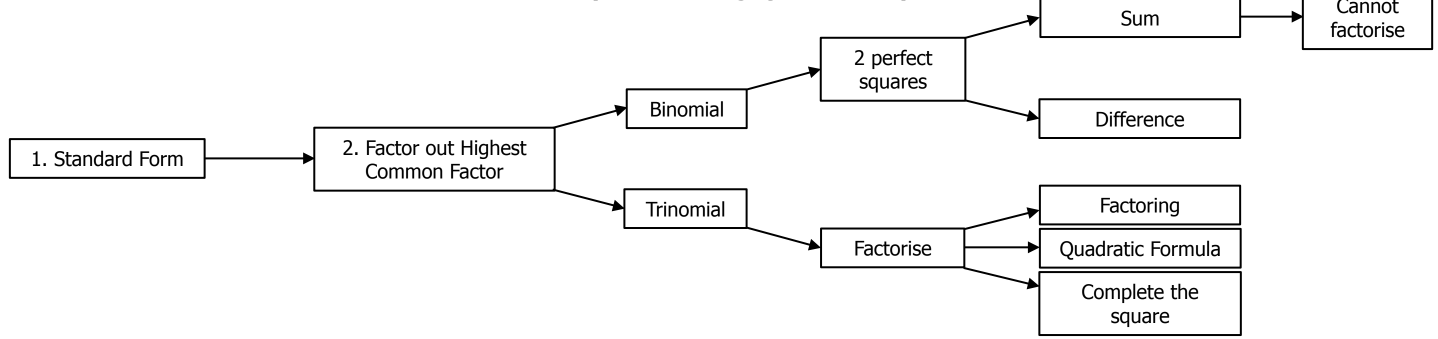
* Factor 1 **+** Factor 2 = **b** * Factor 1 **x** Factor 2 = **c**

Zero factor law:
If A x B = 0 then A, B or both = 0

EXAMPLE

x . y = 0
x = 0 **or** y = 0 **or** (x + 1)(x – 3) = 0
x + 1 = 0 **or** x – 3 = 0

General Steps for solving Quadratic Equations



EXPONENTS AND SURDS

WHAT ARE:

Exponents: Exponents occur when multiplying or dividing expressions/ bases/variables numerous times by similar expressions/bases/variables

Surds: A surd is the Mathematical terminology for irrational roots, when numbers are left in "root-form" as opposed to rounding them off to a decimal place.

HELPFUL HINTS FOR EQUATIONS\EXPRESSIONS

1. Express larger numbers in exponential form by prime factorising
2. Remove a common factor if two unlike terms are separated by a +/-
3. Ensure your surds are always expressed in their simplest form
4. Express surds in exponential form for simplification
5. Take note of the following:

A common error, when solving for an unknown base with a fraction as an exponent, is to multiply the exponents on both sides by the unknown exponent's inverse (so that the exponent will be 1). However, if you express these fractions as surds, you will notice the following:

a. An even power will always produce a positive AND negative solution

$x^{\frac{4}{3}} = 3$

$\sqrt[3]{x^4} = 3$

$x^4 = 27$

$x = \pm \sqrt[4]{27}$

b. A negative number inside an even root cannot solve for a real solution

$-2^{\frac{1}{2}} = x$

$\sqrt{-2} = x$

No real solution

c. An unknown inside an even root cannot solve for a negative solution

$x^{\frac{3}{4}} = -2$

$\sqrt[4]{x^3} = -2$

No real solution

ADDING AND SUBTRACTING LIKE-TERMS

Like terms are terms in an equation/expression that have **identical variables and exponents.** To add/subtract these, simply add/subtract their coefficients. Exponents **never** change when the operator is +/-

EXAMPLE

1. $3x^2y^4 - 5x^3y + 2x^2y^4 + x^3y = \mathbf{5x^2y^4 - 4x^3y}$

2. $3\sqrt{2} + 5\sqrt{3} - 8\sqrt{2} + \sqrt{3} = \mathbf{-5\sqrt{2} + 6\sqrt{3}}$

LAWS OF EXPONENTS

Laws of exponents only apply to multiplication, division, brackets and roots. NEVER adding or subtracting

	Algebraic Notation	Exponential Notation	Exponential Law in operation
1	$16 = 2 \times 2 \times 2 \times 2$	$16 = 2^4$	When we MULTIPLY the SAME bases we ADD the exponents.
2	$\frac{64}{16} = 4$	$\frac{2^6}{2^4} = 2^2$	When we DIVIDE the SAME bases we MINUS the exponents (always top minus bottom).
3	$4^3 = 64$	$(2^2)^3 = 2^6$	When we have the exponents outside the BRACKETS we DISTRIBUTE them into the brackets.
4	$\frac{64}{64} = 1$	$\frac{2^6}{2^6} = 2^0 = 1$	Any base to the POWER OF ZERO is equal to one. (But 0^0 is undefined).
5	$\sqrt[3]{64} = 4$	$\sqrt[3]{2^6} = 2^2$	The POWER inside the root is DIVIDED by the size of the root.
6	$4 \times 9 = 36$	$2^2 \times 3^2 = 6^2$	When we have non-identical bases, but identical exponents, we keep the exponents and multiply the bases. (This same rule will also apply for division).
7	$\sqrt{2} \times \sqrt{3} = \sqrt{6}$	$2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$	
8	$\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$	$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^1$	Any square root multiplied by itself will equal the term inside the root.

CONVERTING SURDS INTO EXPONENTIAL FORM (AND VICE VERSA)

The power inside the root becomes the NUMERATOR and the size of the root becomes the DENOMINATOR.

$$\sqrt[D]{x^N} = x^{\frac{N}{D}}$$

EXAMPLE

1. $\sqrt[5]{x^2}$

$= x^{\frac{2}{5}}$

2. $x^{\frac{3}{4}}$

$= \sqrt[4]{x^3}$

OPERATIONS WITH SURDS

Steps for working with surds:
1. Express the surd in its simplest surd form
2. Identify like terms (+ and −) or use Laws of Exponents (× and ÷)

Note: If you use your calculator, make sure to show the changes you made i.e. $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

EXAMPLE 1

Simplify

$\sqrt{50} + 3\sqrt{18} - \sqrt{98}$

$= 5\sqrt{2} + 9\sqrt{2} - 7\sqrt{2}$

$= 7\sqrt{2}$

EXAMPLE 2

Simplify

$\frac{\sqrt[5]{81} \times \sqrt[4]{27}}{\sqrt[5]{9} \times \sqrt{3}}$

$= \frac{\sqrt[5]{3^4} \times \sqrt[4]{3^3}}{\sqrt[5]{3^2} \times \sqrt{3}}$

$= \frac{3^{\frac{4}{5}} \times 3^{\frac{3}{4}}}{3^{\frac{2}{5}} \times 3^{\frac{1}{2}}}$

$= \frac{3^{\frac{31}{20}}}{3^{\frac{9}{10}}}$

$= 3^{\frac{13}{20}}$

$= \sqrt[20]{3^{13}}$

EXAMPLE 3

A rectangle has a length of $\sqrt{7} + 1$ and a width of $\sqrt{7} - 1$. Determine the length of the diagonal.

$(\sqrt{7} + 1)^2 + (\sqrt{7} - 1)^2 = r^2$

$7 + 2\sqrt{7} + 1 + 7 - 2\sqrt{7} + 1 = r^2$

$16 = r^2$

$4 = r$

RATIONALISING THE DENOMINATOR

The process of finding an equivalent fraction that can be expressed **without a surd in the denominator**

Steps for rationalising monomial denominators:

1. Multiply the numerator and denominator by the denominator's surd
2. Simplify

EXAMPLE 1

Express the following with rational denominators:

1. $\frac{3}{\sqrt{7}}$

$= \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{3\sqrt{7}}{7}$

2. $\frac{6+3\sqrt{2}}{2\sqrt{3}}$

$= \frac{6+3\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{2 \times 3}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{6}$

$= \frac{2\sqrt{3}+\sqrt{6}}{2}$

EXAMPLE 2

If $x = \sqrt{3}+2$, simplify: $\frac{x^2+2}{x-2}$ and express the answer with a rational denominator

1. $\frac{x^2+2}{x-2}$

$= \frac{(\sqrt{3}+2)^2+2}{(\sqrt{3}+2)-2}$

$= \frac{3+4\sqrt{3}+4+2}{\sqrt{3}}$

$= \frac{9+4\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{9\sqrt{3}+4 \cdot 3}{3}$

$= 3\sqrt{3}+4$

Steps for rationalising binomial denominators:

1. Multiply numerator and denominator by the binomial in the denominator with the opposite sign (conjugate)
2. Simplify

Why do we do this?
Multiplying the binomial by itself will give us a trinomial with an irrational middle term. To avoid this, we multiply the binomial by its **conjugate** (same binomial with the opposite sign) to create a difference of two squares.

EXAMPLE 1

Express the following fractions with rational denominators:

1. $\frac{3}{5-\sqrt{7}}$

$= \frac{3}{5-\sqrt{7}} \times \frac{5+\sqrt{7}}{5+\sqrt{7}}$

$= \frac{15+3\sqrt{7}}{25-7}$

$= \frac{15+3\sqrt{7}}{18}$

$= \frac{5+\sqrt{7}}{6}$

2. $\frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}}$

$= \frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}} \times \frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{\sqrt{x}+\frac{1}{\sqrt{x}}}$

$= \frac{7\sqrt{x}+\frac{7}{\sqrt{x}}}{x-\frac{1}{x}}$

$= \frac{\frac{7x+7}{\sqrt{x}}}{\frac{x^2-1}{x}}$

$= \frac{7x+7}{\sqrt{x}} \div \frac{x^2-1}{x}$

$= \frac{7\cancel{(x+1)}}{\sqrt{x}} \times \frac{x}{\cancel{(x+1)}(x-1)}$

$= \frac{7x}{\sqrt{x}(x-1)} \times \frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{7\cancel{x}\sqrt{x}}{\cancel{x}(x-1)}$

$= \frac{7\sqrt{x}}{(x-1)}$

EXPONENTS AND SURDS

FACTORISING

Factorising is the **opposite** of distribution, which means that you will subtract the exponents when "taking out" factors. There are 6 different types of factorisation.

1. Common Factor:

Remove the highest common factor from the coefficients and common variables.

EXAMPLES

Factorise the following:

1. $3x^5y^4 + 9x^3y^5 - 12x^2y^4$

$= 3x^2y^4(x^3 + 3xy - 4)$

2. $\frac{4x^5}{9y^3} - \frac{8x^3}{27y^2} + \frac{16x^2}{3y}$

$= \frac{4x^2}{3y}\left(\frac{x^3}{3y^2} + \frac{2x}{9y} + 4\right)$

4. Exponential Factorising:

Similar to common factorising (1). Remove the highest common factor, in this case, a base with its exponent(s). Exponents are subtracted from the same bases.

EXAMPLES

Factorise the following:

1. $2^{x+3} - 2^{x+1}$

$= 2^x(2^3 - 2)$

$= 2^x \cdot 6$

2. $\frac{9^{x+2} - 3^{2x}}{3^x \cdot 2^3 \times 3^x \cdot 5}$

$= \frac{(3^2)^{x+2} - 3^{2x}}{3^{2x} \cdot 8 \cdot 5}$

$= \frac{3^{2x+4} - 3^{2x}}{3^{2x} \cdot 40}$

$= \frac{\cancel{3^{2x}}(3^4 - 1)}{\cancel{3^{2x}} \cdot 40}$

$= \frac{80}{40}$

$= 2$

3. $\frac{5^x - 5^{x-2}}{2 \cdot 5^x - 5^x}$

$= \frac{\cancel{5^x}(1 - 5^{-2})}{\cancel{5^x}(2 - 1)}$

$= \frac{1 - \frac{1}{25}}{1}$

$= \frac{24}{25}$

2. Difference of two squares:

Applied when there are two perfect squares separated by a '–' sign. The square root of both terms will be in both pairs of brackets, one with a + and the other with a –

A perfect square is a term whose number will not leave an irrational solution once square-rooted, and whose exponents are divisible by 2.

EXAMPLES

Factorise the following:

1. $9x^2 - 4y^6$

$= (3x + 2y^3)(3x - 2y^3)$

2. $x^4 - 16$

$= (x^2 + 4)(x^2 - 4)$

$= (x^2 + 4)(x + 2)(x - 2)$

3. $\frac{x^2 - 7}{x + \sqrt{7}}$

$= \frac{\cancel{(x + \sqrt{7})}(x - \sqrt{7})}{\cancel{x + \sqrt{7}}}$

$= x - \sqrt{7}$

4. $a^2 + 2ab + b^2 - x^2$

$= (a + b)^2 - x^2$

$= (a + b + x)(a + b - x)$

5. Grouping:

Remove the common binomial factor from the expression

EXAMPLES

Factorise the following:

1. $x(y - 4) + 3(y - 4)$

$= (y - 4)(x + 3)$

2. $a^2 + 2ab + b^2 - 3a - 3b$

$= (a + b)^2 - 3(a + b)$

$= (a + b)(a + b - 3)$

3. $5x - 15y + 9ay - 3ax$

$= 5(x - 3y) + 3a(3y - x)$

$= 5(x - 3y) - 3a(x - 3y)$

$= (x - 3y)(5 - 3a)$

3. Sum or difference of two cubes:

Applied when there are two perfect cubes separated by a +/–. The final answer will be a binomial in the one bracket and a trinomial in the other.

A perfect cube is a term whose number will not leave an irrational solution once cube-rooted, and whose exponents are divisible by 3.

EXAMPLES

Factorise the following:

1. $x^3 - 8$

$= (x - 2)(x^2 + 2x + 4)$

2. $27x^6 + 64y^9$

$= (3x^2 + 4y^3)(9x^4 - 12x^2y^3 + 16y^6)$

6. Trinomials:

Note: Ratio of exponents of term 1 to term 2 is 2:1. A combination of factors of term 1 and term 3 must give you term 2.

EXAMPLES

Factorise the following: (Q2 - Q6 are conceptually the same)

1. $3x^2 - 5x - 2$

$= (3x + 1)(x - 2)$

2. $x^2 + 3x - 10$

$= (x + 5)(x - 2)$

3. $x^4 + 3x^2 - 10$

$= (x^2 + 5)(x^2 - 2)$

4. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 10$

$= (x^{\frac{1}{3}} + 5)(x^{\frac{1}{3}} - 2)$

5. $5^{2x} + 3 \cdot 5^x - 10$

$= (5^x + 5)(5^x - 2)$

6. $3^{2x} + 3^{x+1} - 10$

$= 3^{2x} + 3 \cdot 3^x - 10$

$= (3^x + 5)(3^x - 2)$

EXPONENTS AND SURDS

EQUATIONS

1. Linear Equations:

Move all the variables to the one side, and the constants to the other to solve. Linear equations have only **one** solution.

EXAMPLES

Solve:

1. $3(x-2)+10=5-(x+9)$

$3x-6+10=5-x-9$

$3x+4=-x-4$

$4x=-8$

$x=-2$

2. $(x-2)^2-1=(x+3)(x-3)$

$\cancel{x^2}-4x+4-1=\cancel{x^2}-9$

$-4x+3=-9$

$-4x=-12$

$x=3$

2. Quadratic Equations:

Move everything to one side and equate to zero. By factorising the trinomial, you should find **two** solutions.

EXAMPLES

Solve: (Q3 - Q6 are the most likely exam-type questions)

1. $x^2+5=6x$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=5$ or $x=1$

2. $(3x-4)(5x+2)=0$

$3x=4$ or $5x=-2$

$x=\frac{4}{3}$ or $x=\frac{-2}{5}$

3. $x^4+3x^2-10=0$

$(x^2+5)(x^2-2)=0$

$x^2=-5$ or $x^2=2$

No sol. or $x=\pm\sqrt{2}$

4. $x^{\frac{2}{3}}+3x^{\frac{1}{3}}-10=0$

$(x^{\frac{1}{3}}+5)(x^{\frac{1}{3}}-2)=0$

$x^{\frac{1}{3}}=-5$ or $x^{\frac{1}{3}}=2$

$x=-125$ or $x=8$

5. $x+3\sqrt{x}-10=0$

$x+3x^{\frac{1}{2}}-10=0$

$(x^{\frac{1}{2}}+5)(x^{\frac{1}{2}}-2)=0$

$x^{\frac{1}{2}}=-5$ or $x^{\frac{1}{2}}=2$

$\sqrt{x}=-5$ or $\sqrt{x}=2$

No sol. or $x=4$

6. $2^{2x}-6\cdot2^x-16=0$

$(2^x+2)(2^x-8)=0$

$2^x=-2$ or $2^x=8$

No sol. or $2^x=2^3$

$x=3$

3. Simultaneous Equations:

Solve for two unknowns in two different equations using the substitution method. Remember to solve for both unknowns by substituting them back into the original equation.

EXAMPLES

Solve:

1. Equation 1: $2x+3y=18$

Equation 2: $-3x+5y=11$

From 1: $2x+3y=18$

$2x=-3y+18$

$x=\frac{-3y+18}{2}$....1a

Sub 1a into 2: $-3x+5y=11$

$-3\left(\frac{-3y+18}{2}\right)+5y=11$

$\frac{9y-54}{2}+5y=11$

$9y-54+10y=22$

$19y=76$

$y=4$....3

Sub 3 into 1: $2x+3(4)=18$

$2x=6$

$x=3$

$(3;4)$

2. Equation 1: $y+3x=2$

Equation 2: $y^2-9x^2=16$

From 1: $y+3x=2$

$y=-3x+2$...1a

Sub 1a into 2: $y^2-9x^2=16$

$(-3x+2)^2-9x^2=16$

$\cancel{9x^2}-12x+4-\cancel{9x^2}=16$

$-12x=12$

$x=-1$...3

Sub 3 into 1: $y+3(-1)=2$

$y=5$

$(-1;5)$

4. Surd Equations:

Isolate the surd on the one side of the equation. Power both sides of the equation by the root. Ensure that you check your solutions by substituting your answers back into the original equation.

EXAMPLES

Solve:

1. $\sqrt{x-2}=3$

$x-2=9$

$x=9+2$

$x=11$

2. $\sqrt{x+5}-x=3$

$\sqrt{x+5}=x+3$

$x+5=x^2+6x+9$

$0=x^2+5x+4$

$0=(x+1)(x+4)$

$x=-1$ or $x\neq-4$

Check:

$LHS=\sqrt{(-1)+5}-(-1)$

$LHS=3 \quad RHS=3$

$\therefore x=-1$

$LHS=\sqrt{(-4)+5}-(-4)$

$LHS=5 \quad RHS=3$

$\therefore x\neq-4$

5. Exponential Equations:

Make sure that you get a term on the one side of the equation that has a base that is equal to the base with the unknown exponent. Then, drop the bases, equate the exponents and solve.

Hints:

- NEVER drop the base if the terms are separated by a + or –
  - Remove common factors until the equation is in its simplest form and then solve
- Always convert decimals to fractions and then to bases with negative exponents

EXAMPLES

1. $4^x=8$

$2^{2x}=2^3$

$2x=3$

$x=\frac{3}{2}$

2. $0{,}0625^x=64$

$\left(\frac{1}{16}\right)^x=2^6$

$\left(\frac{1}{2^4}\right)^x=2^6$

$2^{-4x}=2^6$

$-4x=6$

$x=\frac{-3}{2}$

3. $2\cdot3^{x+1}+5\cdot3^x=33$

$3^x(2\cdot3+5)=33$

$3^x(11)=33$

$3^x=3^1$

$x=1$

4. $27^{3x+1}=81^{2x+5}$

$(3^3)^{3x+1}=(3^4)^{2x+5}$

$3^{9x+3}=3^{8x+20}$

$9x+3=8x+20$

$x=17$

5. $0{,}5^x\cdot\sqrt{1+\frac{9}{16}}=10$

$\left(\frac{1}{2}\right)^x\cdot\sqrt{\frac{25}{16}}=10$

$2^{-x}\cdot\frac{5}{4}=10$

$2^{-x}=8$

$2^{-x}=2^3$

$-x=3$

$x=-3$

SEQUENCES AND SERIES

REMINDERS:

1. Consecutive: directly follow one another
2. Common/constant difference: difference between two consecutive terms in a pattern
$$d = T_2 - T_1$$
3. General term T_n:
Also referred to as the nth term.
   - General term for linear patterns: $T_n = dn + c$
   - General term for quadratic patterns: $T_n = an^2 + bn + c$
4. $T_1;\ T_2; \dots . T_{100}$: Terms indicated by T and the number of the term as a sub-script.
5. Objective:
   a. Find the values of the variables.
   b. Use the values to find the general term
   c. Use the general term to calculate specific term values
   d. Use specific term values to find the term number

Patterns/ Sequences: ordered set of numbers

Linear:

Constant **first** difference between consecutive terms.

$$\boxed{T_n = dn + c}$$

T_n = general term
d = constant difference

Notice how this is similar to a linear function $y = mx + c$

Steps to determine the nth term:

1. Find the constant difference
2. Substitute the constant difference (d) and the term value, along with the term number
3. Substitute the c- and d-values to define the nth term.

EXAMPLE

1. Determine the nth term of the following sequence:

T_1	T_2	T_3	T_4
2;	**7;**	**12;**	**17**

First differences: $7-2 = \mathbf{5}$; $12-7 = \mathbf{5}$; $17-12 = \mathbf{5}$

Using term 3 where $T_3 = 12$

$T_n = 5n + c$

$12 = 5(3) + c$

$12 = 15 + c$

$12 - 15 = c$

$-3 = c$

$$\therefore\ T_n = 5n - 3$$

2. Determine the 100th term

$T_{100} = 5(100) - 3$

$= 500 - 3$

$= 497$

$\therefore\ T_{100} = 497$

Quadratic:

Constant **second** difference between consecutive terms.

$$\boxed{T_n = an^2 + bn + c}$$

T_n = general term
n = number of the term

Notice how this is similar to the quadratic equation and formula for the parabola

Steps to determine the nth term:

1. Find the constant difference
2. Use the value of the second difference to find "a"
3. Use the "a" value and first difference to find "b"
4. Use "a" and "b" to find "c"

EXAMPLE

Determine the nth term of the following sequence:

T_1	T_2	T_3	T_4
6;	**17;**	**34;**	**57**

Term 1 (a + b + c) → **6**

First difference (3a + b) → $17-6 = \mathbf{11}$; $34-17 = \mathbf{17}$; $57-34 = \mathbf{23}$

Second difference (2a) → $17-11 = \mathbf{6}$; $23-17 = \mathbf{6}$

Second difference $= 2a$

$6 = 2a$

$3 = a$

First difference $= 3a + b$

$11 = 3a + b$

$11 = 3(3) + b$

$2 = b$

Term 1 $= a + b + c$

$6 = (3) + (2) + c$

$6 - 3 - 2 = c$

$1 = c$

$$\therefore\ T_n = 3n^2 + 2n + 1$$

ARITHMETIC SEQUENCE

A sequence formed by adding a common difference (d) to the previous term

$$\therefore d = T_2 - T_1 = T_3 - T_2$$

$$\boxed{T_n = a + (n-1)d}$$

T_n = general term
n = number of the term
a = T_1
d = first difference

Steps to determine the nth term:

1. Find the constant difference (d)
2. Use T_1 as "a"

EXAMPLE 1

Given: –6; –2; 2; 6; 10; ...; 110. Determine:

a) the general term.
b) which term in the sequence exceeds 84.
c) the value of the 14^{th} term.
d) How many terms there are in the sequence.

–6; –2; 2; 6; 10; ...; 110

–2–(–6) 2–(–2) 6–2 10–6

4 4 4 4 ← First difference (d)

a) $T_n = a + (n-1)d$
$T_n = -6 + (n-1)4$
$T_n = 4n - 10$

b)
$$\begin{aligned} 4n - 10 &> 84 \\ 4n &> 94 \\ n &> 23{,}5 \end{aligned}$$
$\therefore 24^{th}$ term

c) $T_n = 4n - 10$
$T_{14} = 4(14) - 10$
$T_{14} = 46$

d)
$$\begin{aligned} T_n &= 4n - 10 \\ 110 &= 4n - 10 \\ 4n &= 120 \\ n &= 30 \end{aligned}$$

EXAMPLE 2

m; $2m + 2$ and $5m + 3$ are three consecutive terms of an arithmetic sequence.

a) Determine the value of m.
b) If the 12th term is 28, determine the sequence.

a) Arithmetic series, thus;
$$\begin{aligned} T_2 - T_1 &= T_3 - T_2 \\ (2m+2) - m &= (5m+3) - (2m+2) \\ m + 2 &= 3m + 1 \\ -2m &= -1 \\ m &= \frac{1}{2} \end{aligned}$$

b)
$$\begin{aligned} d &= T_2 - T_1 \\ &= (2m+2) - m \\ &= 2\left(\tfrac{1}{2}\right) + 2 - \tfrac{1}{2} \\ &= \tfrac{5}{2} \end{aligned}$$

$$\begin{aligned} T_n &= a + (n-1)d \\ 28 &= a + (12-1)\left(\tfrac{5}{2}\right) \\ 28 &= a + \left(\tfrac{55}{2}\right) \\ a &= \tfrac{1}{2} \end{aligned}$$

$$\therefore T_n = \frac{1}{2} + (n-1)\left(\frac{5}{2}\right)$$

NOTE: If asked for sequence, it means the first three terms

$$\begin{aligned} T_1 &= m \\ T_1 &= \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} T_2 &= 2m + 2 \\ T_2 &= 2\left(\tfrac{1}{2}\right) + 2 \\ T_2 &= 3 \end{aligned}$$

$$\begin{aligned} T_3 &= 5m + 3 \\ &= 5\left(\tfrac{1}{2}\right) + 3 \\ &= \tfrac{11}{2} \end{aligned}$$

$$\therefore \mathbf{\frac{1}{2};\ 3;\ \frac{11}{2}}$$

EXAMPLE 3

Find the arithmetic sequence if the 6^{th} term is 10 and the 14^{th} term is 58.

For n = 6;
$$\begin{aligned} T_n &= a + (n-1)d \\ 10 &= a + 5d \ldots\ldots\ldots\ldots eq1 \end{aligned}$$

For n = 14;
$$\begin{aligned} T_n &= a + (n-1)d \\ 58 &= a + 13d \ldots\ldots\ldots\ldots eq2 \end{aligned}$$

eq2 – eq1;
$$\begin{aligned} a + 13d &= 58 \\ -a + 5d &= 10 \\ 8d &= 48 \\ \therefore d &= 6 \end{aligned}$$

b into eq1;
$$\begin{aligned} 10 &= a + 5d \\ 10 &= a + 5(6) \\ a &= -20 \end{aligned}$$

NOTE: If asked for sequence, it means the first three terms

$$\begin{aligned} T_n &= a + (n-1)d \\ T_1 &= -20 + (1-1)(6) \\ T_1 &= -20 \end{aligned}$$

$$\begin{aligned} T_n &= a + (n-1)d \\ T_2 &= -20 + (2-1)(6) \\ T_1 &= -14 \end{aligned}$$

$$\begin{aligned} T_n &= a + (n-1)d \\ T_1 &= -20 + (3-1)(6) \\ T_1 &= -8 \end{aligned}$$

$$\therefore \mathbf{-20;\ -14;\ -8}$$

GEOMETRIC SEQUENCE

A sequence formed by multiplying the previous term by a common ratio (r).

$$\therefore r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

$$\boxed{T_n = a \cdot r^{n-1}}$$

T_n = general term
n = number of the term
a = T_1
r = common ratio

Steps to determine the nth term:

1. Find the common ratio
2. Use T_1 as "a"

EXAMPLE 1

Given sequence 6; 18; 54; ....; 118 098. Determine:

a) the next 2 terms

b) the n^th^ term

c) how many terms there are in the sequence.

6; $\frac{6}{18}$ **18;** $\frac{18}{54}$ **54; ....; 118 098**

3 **3**

a) $r = 3$

$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ T_4 &= 6 \cdot 3^{4-1} \\ T_4 &= 162 \end{aligned} \qquad \begin{aligned} T_n &= a \cdot r^{n-1} \\ T_5 &= 6 \cdot 3^{5-1} \\ T_5 &= 486 \end{aligned}$$

b)
$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ &= 6 \cdot 3^{n-1} \\ &= 2 \times 3 \cdot 3^{n-1} \\ &= 2 \cdot 3^n \end{aligned}$$

c)
$$\begin{aligned} T_n &= 2 \cdot 3^n \\ 118\,098 &= 2 \cdot 3^n \\ 59\,049 &= 3^n \\ 3^{1}0 &= 3^n \\ \therefore n &= 10 \end{aligned} \quad \text{or} \quad \begin{aligned} n &= \log_3 59\,049 \\ \therefore n &= 10 \end{aligned}$$

EXAMPLE 2

4g–2 ; g+1 ; g–3 are the first three terms of a geometric sequence.

a) Determine the value of g if g is an integer and hence show that the first three terms are 18; 6; 2.

b) Write down the n^th^ term of the sequence

a)
$$\begin{aligned} r = \frac{T_2}{T_1} &= \frac{T_3}{T_2} \\ \frac{g+1}{4g-2} &= \frac{g-3}{g+1} \\ (g+1)(g+1) &= (g-3)(4g-2) \\ g^2+g+1 &= 4g^2-14g+6 \\ 3g^2-16g+5 &= 0 \\ (3g-1)(g-5) &= 0 \end{aligned}$$

$$\therefore g \neq \frac{1}{3} \text{ (not an integer)} \quad \text{or} \quad g = 5$$

$$\begin{aligned} T_1 &= 4g-2 \\ &= 4(5)-2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} T_2 &= g+1 \\ &= (5)+1 \\ &= 6 \end{aligned}$$

$$\begin{aligned} T_3 &= g-3 \\ &= (5)-3 \\ &= 2 \end{aligned}$$

$$\therefore \mathbf{18; 6; 2}$$

b)
$$\begin{aligned} r &= \frac{T_2}{T_1} \\ &= \frac{6}{18} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ &= 18 \cdot \left(\tfrac{1}{3}\right)^{n-1} \\ &= 2 \times 3^2 \times 3^{-n+1} \\ &= 2 \cdot 3^{3-n} \end{aligned}$$

EXAMPLE 3

The 6^th^ term of a geometric sequence is $\sqrt{3}$, and the 11^th^ term is 27.

Determine the sequence.

For n = 6;
$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ \sqrt{3} &= a \cdot r^5 \ldots\ldots\ldots\ldots eq1 \end{aligned}$$

For n = 11;
$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ 27 &= a \cdot r^{10} \ldots\ldots\ldots\ldots eq2 \end{aligned}$$

eq2 ÷ eq1;
$$\begin{aligned} \frac{ar^{10}}{ar^5} &= \frac{27}{\sqrt{3}} \\ r^5 &= \frac{3^3}{3^{\frac{1}{2}}} \\ r^5 &= 3^{3-\frac{1}{2}} \\ r^5 &= 3^{\frac{5}{2}} \\ r &= 3^{\frac{1}{2}} = \sqrt{3} \end{aligned}$$

r into eq1;
$$\begin{aligned} \sqrt{3} &= a \cdot r^5 \\ \sqrt{3} &= a \cdot \left(\sqrt{3}\right)^5 \\ a &= \frac{1}{\left(\sqrt{3}\right)^4} \\ a &= \frac{1}{9} \end{aligned}$$

$$\begin{aligned} T_n &= a \cdot r^{n-1} \\ T_1 &= \left(\tfrac{1}{9}\right)\left(\sqrt{3}\right)^{1-1} \\ T_1 &= \frac{1}{9} \end{aligned} \qquad \begin{aligned} T_n &= a \cdot r^{n-1} \\ T_1 &= \left(\tfrac{1}{9}\right)\left(\sqrt{3}\right)^{2-1} \\ T_1 &= \frac{\sqrt{3}}{9} \end{aligned} \qquad \begin{aligned} T_n &= a \cdot r^{n-1} \\ T_1 &= \left(\tfrac{1}{9}\right)\left(\sqrt{3}\right)^{3-1} \\ T_1 &= \frac{1}{3} \end{aligned}$$

$$\therefore \mathbf{\frac{1}{9}; \frac{\sqrt{3}}{9}; \frac{1}{3}}$$

SEQUENCES AND SERIES - SERIES

A series is the sum of a number of terms in a sequence and is represented by S_n.

ARITHMETIC SERIES

General Term for an arithmetic series:

$T_n = a + (n-1)d$

Sum of the arithmetic series:

$S_n = \frac{n}{2}[2a + (n-1)d]$

Last term:

$l = a + (n-1)d$

$\therefore S_n = \frac{n}{2}[a + l]$

PROOF:

$$\begin{array}{rcccccccccccccc}
S_n & = & a & + & [a+d] & + & [a+2d] & + & \ldots & + & [a+(n-3)d] & + & [a+(n-2)d] & + & [a+n-1)d] \\
+S_n & = & [a+(n-1)d] & + & [a+(n-2)d] & + & [a+(n-3)d] & + & \ldots & + & [a+2d] & + & [a+d] & + & a \\
\hline
2S_n & = & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] & + & [2a+(n-1)d] \\
\therefore 2S_n & = & n[2a+(n-1)d] \\
\therefore S_n & = & \frac{n}{2}[2a+(n-1)d]
\end{array}$$

EXAMPLE 1

Determine the sum of the following:

a) The first 25 terms of 18 + 13 + 8 + ...

$d = -5$

$$\begin{aligned}
S_n &= \tfrac{n}{2}[2a + (n-1)d] \\
S_{25} &= \tfrac{25}{2}[2(18) + (25-1)(-5)] \\
&= -1050
\end{aligned}$$

b) The series 5 + 8 + 11 + ... + 35

$a = 5; d = 3; l = 35$

Find n :

$$\begin{aligned}
35 &= 5 + (n-1)(3) \\
30 &= 3(n-1) \\
10 &= n-1 \\
\therefore n &= 11
\end{aligned}$$

$$\begin{aligned}
S_n &= \tfrac{n}{2}[a + l] \\
S_{11} &= \tfrac{11}{2}[5 + 35] \\
&= 220
\end{aligned}$$

EXAMPLE 2

The sixth term of an arithmetic sequence is 23 and the sum of the first six terms is 78.
Determine the sum of the first twenty-one terms.

1.
$$\begin{aligned}
T_n &= a + (n-1)d \\
23 &= a + (6-1)d \\
23 &= a + 5d \ldots\ldots\ldots\ldots\ldots\ldots eq1
\end{aligned}$$

2.
$$\begin{aligned}
S_n &= \tfrac{n}{2}[2a + (n-1)d] \\
78 &= \tfrac{6}{2}[2a + (6-1)d] \\
26 &= 2a + 5d \ldots\ldots\ldots\ldots\ldots\ldots eq2
\end{aligned}$$

3. q2 − q1 :
$$\begin{aligned}
23 &= a + 5d \\
-\ 26 &= 2a + 5d \\
3 &= a
\end{aligned}$$

4. a into eq1 :
$$\begin{aligned}
23 &= a + 5d \\
23 - (3) &= 5d \\
d &= 4
\end{aligned}$$

5.
$$\begin{aligned}
S_n &= \tfrac{n}{2}[2a + (n-1)d] \\
S_{21} &= \tfrac{21}{2}[2(3) + (21-1)(4)] \\
&= 903
\end{aligned}$$

GEOMETRIC SERIES

General Term for a geometric series:

$T_n = a \cdot r^{n-1}$

Sum of the geometric series:

$S_n = \dfrac{a(1-r^n)}{(1-r)}$ if $r < 1$ OR $S_n = \dfrac{a(r^n-1)}{(r-1)}$ if $r > 1$

PROOF:

$$\begin{array}{rcccccccccccccc}
S_n & = & a & + & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} \\
rS_n & = & & & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \\
\hline
S_n - rS_n & = & a - ar^n \\
S_n(1-r) & = & a(1-r^n) \\
\therefore S_n & = & \frac{a(1-r^n)}{(1-r)}
\end{array}$$

EXAMPLE 1

Given the series −2 +6 −18 +54 +.... Determine:

a) the sum to nine terms

$r = \dfrac{T_2}{T_1} = \dfrac{6}{-2} = -3$

$$\begin{aligned}
S_n &= \frac{a(1-r^n)}{(1-r)} \\
S_9 &= \frac{-2\left(1-(-3)^9\right)}{(1-(-3))} \\
&= -9\ 842
\end{aligned}$$

b) the value of n if the sum of the series is −797 162.

$$\begin{aligned}
S_n &= \frac{a(1-r^n)}{(1-r)} \\
-797\ 162 &= \frac{-2\left(1-(-3)^n\right)}{(1-(-3))} \\
-3\ 188\ 648 &= -2\left(1-(-3)^n\right) \\
1\ 594\ 324 &= 1-(-3)^n \\
(-3)^n &= -1\ 594\ 323 \\
(-3)^n &= (-3)^{13} \\
\therefore n &= 13
\end{aligned}$$

EXAMPLE 2

The sum of the geometric series $6 + \cdots + \dfrac{3}{12}$ is $\dfrac{1533}{128}$.

Determine the common ratio and the number of terms in the sequence.

$$\begin{aligned}
T_n &= ar^{n-1} \\
\frac{3}{128} &= 6r^{n-1} \\
\frac{1}{256} &= r^{n-1} \\
r^n \cdot r^{-1} &= \frac{1}{256} \\
r^n \cdot \frac{1}{r} &= \frac{1}{256} \\
r^n &= \frac{r}{256}
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{a(1-r^n)}{(1-r)} \\
\frac{1533}{128} &= \frac{6\left(1-\frac{r}{256}\right)}{1-r} \\
15339(1-r) &= 768\left(1-\frac{r}{256}\right) \\
1533 - 1533r &= 768 - 3r \\
1530r &= 765 \\
r &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
r^n &= \frac{r}{256} \\
\frac{1}{2}^n &= \frac{\frac{1}{2}}{256} \\
\frac{1}{2}^n &= \frac{1}{512} \\
2^{-n} &= 2^{-9} \\
\therefore n &= 9
\end{aligned}$$

Sequences and Series – Series

SUM TO INFINITY (S_∞)

Consider the following values of Sn as n →∞, in the following series:

1. $2+4+6+8+\cdots=\infty$ (Arithmetic and $d=2$)
2. $2+0-2-4-6+\cdots=\infty$ (Arithmetic and $d=-2$)
3. $2+4+8+16+\cdots=\infty$ (Geometric and $r=2$)
4. $2-4+8-16+32+\cdots=\infty$ (Geometric and $r=-2$)
5. $8+4+2+1+\frac{1}{2}+\cdots=\infty$ $\left(\text{Geometric and } r=\frac{1}{2}\right)$

Note for 5:

$$S_{20}=\frac{8(1-(\frac{1}{2})^{20})}{1-\frac{1}{2}}=15{,}9999 \qquad S_{30}=\frac{8(1-(\frac{1}{2})^{30})}{1-\frac{1}{2}}=15{,}9999 \qquad S_{40}=\frac{8(1-(\frac{1}{2})^{40})}{1-\frac{1}{2}}=16 \qquad S_{80}=\frac{8(1-(\frac{1}{2})^{80})}{1-\frac{1}{2}}=16$$

$$\therefore S_\infty = 16$$

Thus only a geometric series with $-1<r<1,\ r\neq 0,$, will have a sum to infinity, as S_∞ will approach a set value. This is known as a convergent series, or a series that converges.

If $r=\frac{1}{p},\ p\in N$, then as $\lim_{p\to\infty}\frac{1}{p}=0$, therefore $1-r^\infty=1-0=1$, and therefore;

$$S_\infty=\frac{a}{1-r}$$

EXAMPLE 1

For which value(s) of k will the series $4(k-2)+8(k-2)^2+16(k-2)^3+\cdots$ converge?

$$\begin{aligned} r &= \frac{T_2}{T_1}\\ r &= \frac{8(k-2)^2}{4(k-2)}\\ r &= 2(k-2) \end{aligned}$$

$$\begin{array}{llll} -1< & r & <1 & ,\quad r\neq 0\\ -1< & 2(k-2) & <1 & ,\quad 2(k-2)\neq 0\\ -\frac{1}{2}< & k-2 & <\frac{1}{2} & ,\quad k-2\neq 0\\ 1\frac{1}{2}< & k & <2\frac{1}{2} & ,\quad k\neq 2 \end{array}$$

EXAMPLE 2

Use the sum to infinity to write $0{,}\dot{4}$ as a proper fraction.

$$0{,}\dot{4}=0{,}4+0{,}04+0{,}004+0{,}0004+\cdots$$

$$\begin{aligned} r &= \frac{T_2}{T_1}\\ r &= \frac{0{,}04}{0{,}4}\\ r &= 0{,}1 \end{aligned}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r}\\ &= \frac{0{,}4}{1-0{,}1}\\ &= \frac{4}{9} \end{aligned}$$

SIGMA NOTATION

Sigma notation is denoted by **Σ**, which means 'the sum of'.

end term
$\sum$ Formula for the series (T_n)
start term

$$\sum_{n=p}^{r}(T_n)$$

$T_1=a$
$n=r-p+1$

Find the first three terms to check if arithmetic or geometric. Then use formula for S_n.

EXAMPLE

$$\sum_{n=3}^{7}2n$$

Start term : 3
End term : 7
Series formula : $T_n=2n$

$$\begin{aligned} \sum_{n=3}^{7}2n &= 2(3)+2(4)+2(5)+2(6)+2(7)\\ &= 6+8+10+12+14\\ &= 50 \end{aligned}$$

$$\begin{aligned} S_n &= \tfrac{n}{2}[a+l]\\ &= \tfrac{5}{3}[6+14]\\ &= 50 \end{aligned}$$

a = 6
d = 2
n = r − p + 1
= 7 − 3 + 1

EXAMPLE 1

Write the following in sigma notation:

$2+10+50+\cdots+781\ 250$

$r=5\ ;\ a=2$

$\therefore T_n=2\cdot 5^{n-1}$

$$\begin{aligned} T_n &= 2\cdot 5^{n-1}\\ 781\ 250 &= 2\cdot 5^{n-1}\\ 390\ 625 &= 5^{n-1}\\ 5^8 &= 5^{n-1}\\ \therefore n &= 9 \end{aligned}$$

$$\therefore\sum_{n=1}^{9}2\cdot 5^{n-1}$$

EXAMPLE 2

Determine the following:

a) $\sum_{n=1}^{\infty}10^{2-n}$ b) $\sum_{n=4}^{18}2-5n$

First find the first three terms to check if arithmetic or geometric.

a) $10+1+0{,}1+\cdots$

$\therefore n=0{,}1$

$$\begin{aligned} S_\infty &= \frac{a}{1-r}\\ S_\infty &= \frac{10}{1-0{,}1}\\ S_\infty &= 11{,}11 \end{aligned}$$

b) $-18-23-28$

$\therefore d=-5$

$$\begin{aligned} n &= r-p+1\\ &= 18-4+1\\ &= 15 \end{aligned}$$

$$\begin{aligned} S_n &= \tfrac{n}{2}[3a+(n-1)d]\\ &= \tfrac{15}{2}[2(-18)+(15-1)(-5)]\\ &= -795 \end{aligned}$$

Grade 11 Functions: Quadratic Functions

$$y = a(x-p)^2 + q$$

Shape:
- a > 0
- a < 0

Horizontal shift:
- x-value of turning pt
- axis of symmetry

Vertical shift:
- y-value of turning pt

Steps for sketching $y = a(x-p)^2 + q$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point
4. Plot points and sketch graph

EXAMPLE 1

Sketch $f(x) = (x+1)^2 - 9$

- Shape: a > 0 ∴
- x-intercept (y = 0)

  $0 = (x+1)^2 - 9$

  $9 = (x+1)^2$

  $\pm\sqrt{9} = x + 1$

  $+3 = x + 1$ OR $-3 = x + 1$

  $2 = x$ OR $-4 = x$
- y-intercept (x = 0)

  $y = (0+1)^2 - 9$

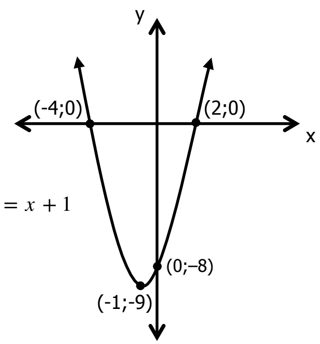
  $y = -8$
- Turning point (p;q)

  $(-1; -9)$
- Axis of symmetry

  $x = -1$
- Domain

  $x \in R$
- Range

  $y \geq -9$



Remember:
$(x-(-1))^2 - 9$
$(x-p)^2 + q$

NOTE:
$f(x) = x^2$
→ moved 1 unit to the left
→ moved 9 units down

Steps for sketching $y = ax^2 + bx + c$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point $(\frac{-b}{2a})$
4. Plot points and sketch graph

EXAMPLE 2

Sketch $f(x) = x^2 - 4x + 3$

- Shape: a > 0 ∴
- x-intercept (y = 0)

Option 1: $0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 3$ OR $x = 1$

Option 2: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{4 \pm 2}{2}$

$x = 3$ OR $x = 1$

- y-intercept (x = 0)

  $y = 3$
- Turning point (p;q)
  1) x-value of TP = $\frac{-b}{2a} = \frac{-(-4)}{2(1)}$

     $x = 2$
  2) Subst. into original eq:

     $y = (2)^2 - 4(2) + 3$

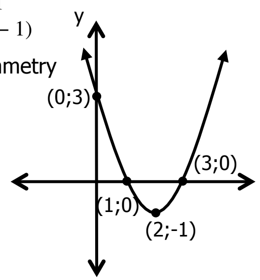
     $y = -1$

     TP $(2; -1)$
- Axis of symmetry

  $x = 2$
- Domain

  $x \in R$
- Range

  $y \geq -1$



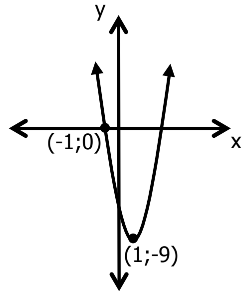
Finding the equation in the form $y = a(x-p)^2 + q$

Given the **turning point** and **another point**

1. Substitute the turning point into $y = a(x-p)^2 + q$
2. Substitute the other point into the equation to find 'a'
3. Determine the equation of the graph

EXAMPLE 1

Find the equation of the following graph:



- Turning point (p;q)

  $p = 1$ and $q = -9$

  $y = a(x-1)^2 - 9$
- Other point

  $(-1; 0)$

  $0 = a(-1-1)^2 - 9$

  $9 = 4a$

  $a = \frac{9}{4}$
- Equation

  $y = \frac{9}{4}(x-1)^2 - 9$

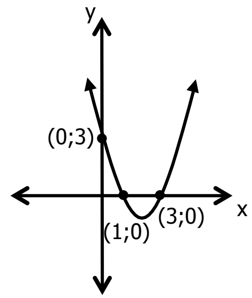
Finding the equation in the form $y = ax^2 + bx + c$

Given the **x-intercepts** and **another point**

1. Substitute the x-intercepts into $y = a(x-x_1)(x-x_2)$
2. Substitute the other point in to find 'a'
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- x-intercepts

  $x = 1$ OR $x = 3$

  Formula: $y = a(x-x_1)(x-x_2)$

  $y = a(x-1)(x-3)$
- Other point

  $(0; 3)$

  $3 = a(-1)(-3)$

  $1 = a$
- Equation

  $y = 1(x-1)(x-3)$

  $y = x^2 - 4x + 3$

NOTE:
If you need to write this equation in the form $y = a(x-p)^2 + q$ complete the square

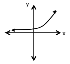
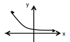
$y = (x-2)^2 + 3 - 4$

$y = (x-2)^2 - 1$

Exponential Graphs

$$y = ab^{x-p} + q$$

Shape:

- b > 1 
- 0 < b < 1 

Vertical shift:

- asymptote

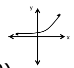
Steps for sketching $y = ab^{x-p} + q$

1. Determine the asymptote ('q')
1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

EXAMPLE 1

Sketch $f(x) = 2^{x+1} + 1$

- Asymptote

  $y = 1$
- Shape: a > 0 $\therefore$ 
- x-intercept (y = 0)

  $0 = 2^{x+1} + 1$

  $-1 = 2^{x+1}$

  Not possible to solve for x

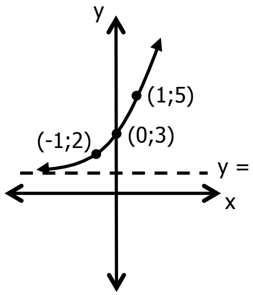
  $\therefore$ No x-intercept
- y-intercept (x = 0)

  $y = 2^{0+1} + 1$

  $y = 3$
- Domain

  $x \in R$
- Range

  $y > 1$



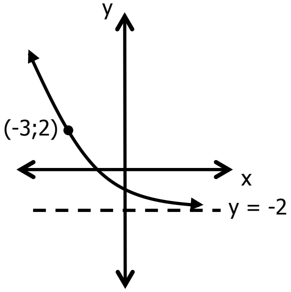
Finding the equation in the form $y = ab^{x-p} + q$

Given the **asymptote** and **another point**

1. Substitute the asymptote into the equation
2. Substitute the other point in
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph given $y = b^{x+1} + q$:



- Asymptote

  $q = -2$

  $y = b^{x+1} - 2$
- Other point

  $(-3; 2)$

  $2 = b^{-3+1} - 2$

  $4 = b^{-2}$

  $4 = \frac{1}{b^2}$

  $b^2 = \frac{1}{4}$

  $b = \pm\frac{1}{2}$ $\qquad b \neq -\frac{1}{2}$

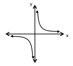
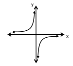
  $b = +\frac{1}{2}$
- Equation

  $y = (\frac{1}{2})^{x+1} - 2$

Hyperbola

$$y = \frac{a}{x-p} + q$$

Shape:

- a > 0 
- a < 0 

Horizontal shift:

- asymptote

Vertical shift:

- asymptote

Steps for sketching $y = \frac{a}{x-p} + q$

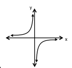
1. Determine the asymptotes (y = 'q' and x = 'p')
1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

EXAMPLE 1

Sketch $f(x) = \frac{-1}{x-2} - 1$

- Asymptotes

  $x = 2$

  $y = -1$
- Shape: a < 0 $\therefore$ 
- x-intercept (y = 0)

  $0 = \frac{-1}{x-2} - 1$

  $1 = \frac{-1}{x-2}$

  $x - 2 = -1$

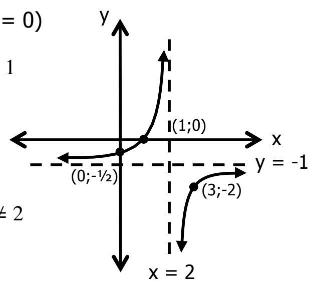
  $x = 1$
- y-intercept (x = 0)

  $y = \frac{-1}{-2} - 1$

  $y = -\frac{1}{2}$
- Domain

  $x \in R; x \neq 2$
- Range

  $y \in R; y \neq -1$



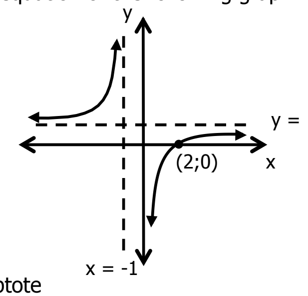
Finding the equation in the form $y = \frac{a}{x-p} + q$

Given the **asymptotes** and **another point**

1. Substitute the asymptotes into the equation
2. Substitute the other point into the equation to find 'a'
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- Asymptote

  $y = 1$ and $x = -1$

  $f(x) = \frac{a}{x-(-1)} + 1$

  $f(x) = \frac{a}{x+1} + 1$
- Other point

  $(2; 0)$

  $0 = \frac{a}{2+1} + 1$

  $-1 = \frac{a}{3}$

  $-3 = a$
- Equation

  $f(x) = \frac{-3}{x+1} + 1$

Lines of Symmetry:

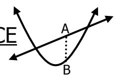
Use point of intersection of asymptotes. (-1;1)

$y = x + c \quad (-1;1)$	$y = -x + c \quad (-1;1)$
$1 = -1 + c$	$1 = 1 + c$
$2 = c$	$0 = c$
$y = x + 2$	$y = -x$

Functions and Graphs

Deductions from Graphs

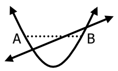
DISTANCE



Steps for determining VERTICAL DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – (bottom graph)
2. Substitute the given x-value to derive your answer

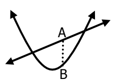
Steps for determining HORIZONTAL DISTANCE



1. Find the applicable x-values
   $AB = x_B - x_A$ (largest – smallest)

Steps for determining MAXIMUM DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – bottom graph
2. Complete the square
   $y = a(x-p)^2 + q$
3. State the maximum distance
   $y = a(x-p)^2 + (\mathbf{q}) \rightarrow$ q is the max distance

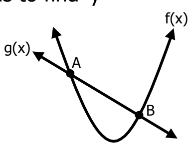


NOTE:
- Distance is always positive
- Distance on a graph is measured in units

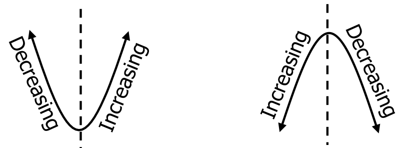
INTERSECTION OF GRAPHS

Steps for determining POINTS OF INTERCEPTION

1. Equate the two functions
   $f(x) = g(x)$
2. Solve for x (look for the applicable x-value: A or B)
3. Substitute the applicable x-value into any of the two equations to find 'y'

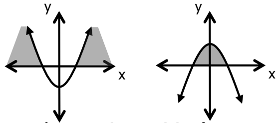


INCREASING/DECREASING



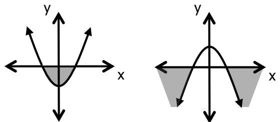
NOTATION

- $f(x) > 0$ ⊕
  (above the line y = 0)



(i.e. where y is positive)

- $f(x) < 0$ ⊖
  (below the line y = 0)



(i.e. where y is negative)

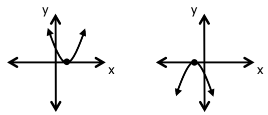
⊕ ⊖
- $f(x) \cdot g(x) \leq 0$ ⊖

⊖ ⊕
(one graph lies above y = 0 and one graph lies below y = 0)

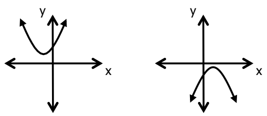
- $f(x) \geq g(x)$
  top bottom
  (i.e. f(x) lies above g(x))
- $f(x) = g(x)$
  (point of intersection)

ROOTS & PARABOLAS

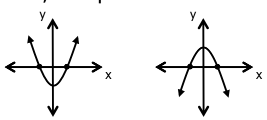
- Equal, real roots



- Non-real/ No real roots

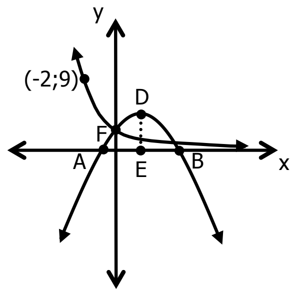


- Real, unequal roots



EXAMPLE 1

$f(x) = ax^2 + bx + c$ and $g(x) = k^x$ are sketched. D is the turning point of f(x) with the axis of symmetry at x=2. AB is 6 units.



Questions:

a. Determine the value of k.

b. Determine the x-values of A and B.

c. Show that $a = \frac{-1}{5}$ and $b = \frac{4}{5}$.

d. Determine the coordinates of D.

e. Determine the maximum distance of DE.

f. Determine the values of p for which:
$\frac{-1}{5}x^2 + \frac{4}{5}x + p < 0$

g. Determine for which values of x:

i. $f(x) \geq 0$

ii. $\frac{f(x)}{g(x)} > 0$

iii. $f(x)$ is increasing

Solutions:

a. $(-2;9)$
$9 = k^{-2}$
$9 = \frac{1}{k^2}$
$k = \pm\frac{1}{3}$
$k = \frac{1}{3}$

b. $E = (2;0)$ and $AB = 6\ units$
$A = (-1;0) \quad x = -1$
$B = (5;0) \quad x = 5$

c. $y = a(x - x_1)(x - x_2)$
$(-1;0)$ and $(5;0)$
$y = a(x+1)(x-5)$
Use $F(0;1)$
$1 = a(+1)(-5)$
$\underline{-\frac{1}{5} = a}$
$y = -\frac{1}{5}(x^2 - 4x + 1)$
$y = -\frac{1}{5}x^2 + \frac{4}{5}x + 1$
$\underline{b = \frac{4}{5}}$

d. $y = -\frac{1}{5}(2)^2 + \frac{4}{5}(2) + 1$
$y = \frac{9}{5} \quad \therefore D = (2; \frac{9}{5})$

e. $\frac{9}{5}$ units (y-value of co-ordinate D is also TP)

f. $-\frac{1}{5}x^2 + \frac{4}{5}x + p < 0$ ⊖
$p < -\frac{9}{5}$

NOTE:
- Interpret question as: How many units must the graph move for the max. value to be < 0

g.

i. $x \in [-1;5]$

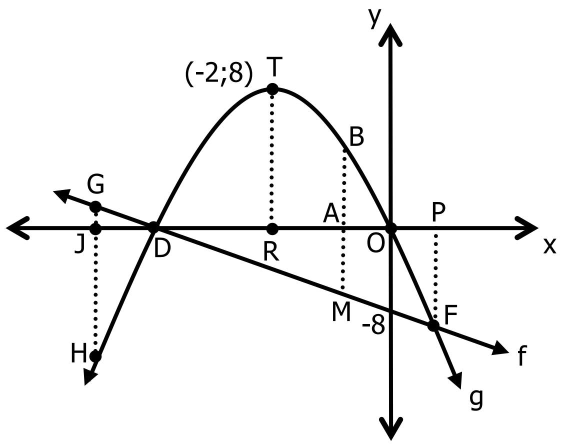
ii. $x \in (-1;5)$

iii. $x \in (-\infty;2)$

Deductions from Graphs

EXAMPLE 2

$f(x) = mx + c$ and $g(x) = a(x-p)^2 + q$ are sketched below. T is the turning point of g(x).



Questions:

Determine:

a. The value of a, p, q, m and c.

b. The length of OD.

c. The length of TR.

d. The equation of TR.

e. BM if OA = 1 unit.

f. OJ if GH = 28 units.

g. The length of FP.

h. The maximum length of BM.

i. The value of k for which $-2x^2 - 8x + k$ has two equal roots.

j. For which value(s) of x will $\frac{f(x)}{g(x)} < 0$?

Solutions:

a. $y = a(x+2)^2 + 8 \quad (0;0)$

$-8 = 4a$

$-2 = a$ and $p = -2$ and $q = 8$

$\therefore g(x) = -2(x+2)^2 + 8$

$D\ (-4;0)$

$\therefore m = \frac{8}{-4} = -2$ and $c = -8$

b. OD = 4 units

c. TR = 8 units

d. TR: $x = -2$

e. $g(x) = -2(x+2)^2 + 8$

$= -2(x^2 + 4x + 4) + 8$

$= -2x - 8x - 8 + 8$

$= -2x^2 - 8x$

BM = $g(x) - f(x)$

BM $= -2x^2 - 8x - (-2x - 8)$

$= -2x^2 - 6x + 8$

BM $= -2(-1)^2 - 6(-1) + 8$

BM = 12 units

f. $28 = -2x - 8 - (-2x^2 - 8x)$

$28 = -2x - 8 + 2x^2 + 8x$

$0 = 2x^2 + 6x - 36$

$0 = 2(x^2 + 3x - 18)$

$0 = 2(x+6)(x-3)$

$x = -6$ or $x = 3$ (NA)

$\therefore$ OJ = 6 units

g. $-2x^2 - 8x = -2x - 8$

$0 = 2x^2 + 6x - 8$

$0 = 2(x-1)(x+4)$

$x = 1$ or $x = -4$ (NA)

$y = -2(1) - 8$

$y = -10$

$\therefore$ FP = 10 units

h. Max length is given by TP of parabola ($L(x)$) given by $L(x) = g(x) - f(x)$. Find the TP by completing the square.

$\therefore$ Max BM = $g(x) - f(x)$

$= -2x^2 - 8x - (-2x - 8)$

$= -2x^2 - 6x + 8$

$= -2(x^2 + 3x - 4)$

$= -2[(x + \frac{3}{2})^2 - 4 - \frac{9}{4}]$

$= -2[(x + \frac{3}{2})^2 - \frac{25}{4}]$

$= -2(x + \frac{3}{2})^2 + \frac{25}{2}$

$\therefore$ Max of BM = $\frac{25}{2}$ units

i. k = -8

j. $x \in (-\infty; 0);\ x \neq -4$

FUNCTIONS AND INVERSES

Relation: set of ordered pairs

Function: relation where each of the values in the domain (x-values) is associated with only ONE value in the range (y-value)

How to determine whether a graph is a function:

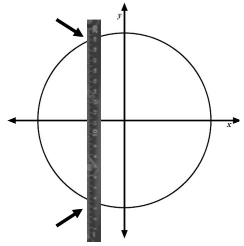
Use the Vertical Ruler Test. If the ruler crosses the graph:

- Once, it **IS** a function
- More than once, it **IS NOT** a function

EXAMPLE

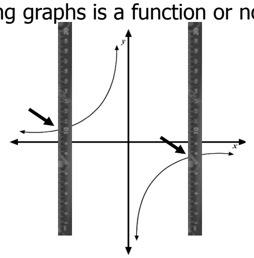
Determine whether each of the following graphs is a function or not

1.



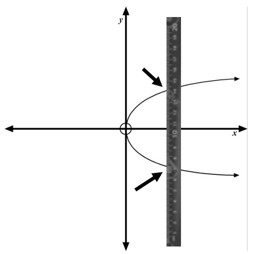
Is **not** a function

2.



Is a function

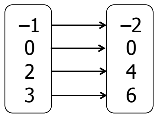
3

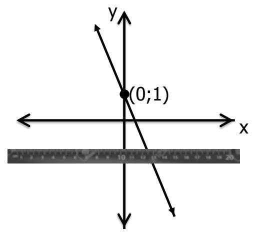


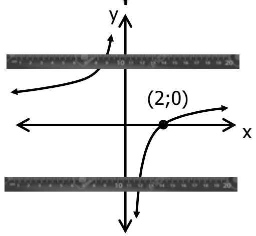
Is **not** a function

ONE-TO-ONE FUNCTION

A function where every element of the domain has a different element in the range. If you do a horizontal line test, the graph will only be cut by your line ONCE.

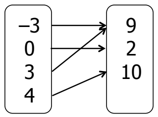


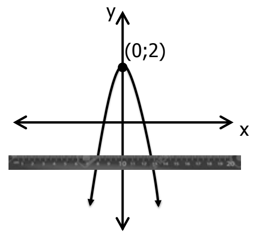


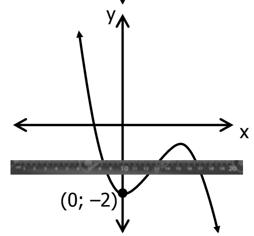


MANY-TO-ONE FUNCTION

A function where two or more elements of the domain may be associated with the same element of the range. If you do a horizontal line test, the graph will be cut by your line MORE THAN ONCE.

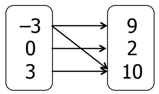


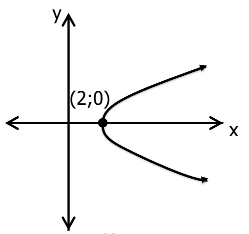


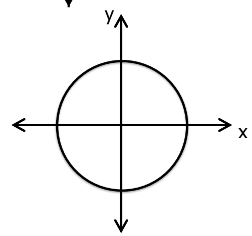


NON-FUNCTION

One element of the domain associated with two or more elements of the range.







INVERSE OF A FUNCTION

The symmetry if a graph is a mirror image of the curve around a specific line. The inverse of a function is the symmetry of a graph about the line $y = x$.

How to determine the inverse of a function:

1. Write in standard form
2. Switch x and y
3. Make y the subject of the formula

NOTE:
For Many-to-One functions (Parabolas): The domain needs to be restricted in order for the inverse to be a function

EXAMPLE

Find the inverse equation of each of the following functions. Write them in the form $f^{-1}(x) = \ldots$

1. $f(x) = 4x - 5$

$x = 4y - 5$

$x + 5 = 4y$

$y = \dfrac{1}{4}x + \dfrac{5}{4}$

$\therefore\ f^{-1}(x) = \dfrac{1}{4}x + \dfrac{5}{4}$

2. $f(x) = (x - 8)^2 + 1$

$x = (y - 8)^2 + 1$

$x - 1 = (y - 8)^2$

$\pm\sqrt{x - 1} = y - 8$

$8 \pm \sqrt{x - 1} = y$

$\therefore\ f^{-1}(x) = 8 \pm \sqrt{x - 1}$

SKETCHING INVERSE FUNCTIONS

Method 1: Table Method

1. Using the original graph, create a table
2. Sketch the graph of the inverse by switching the values of x and y for each point

EXAMPLE

Original function:

x	-1	0	1
y	-2	0	-2

Inverse:

x	-2	0	-2
y	-1	0	-1

Method 2: Using the original function

1. Find important properties on the original function (i.e. x- and y-intercepts, turning points, asymptotes, domain and range)
2. Sketch the graph of the inverse by switching the values of x and y for each important point

EXAMPLE

Original function:

x-intercept - $0 = 2x - 4$
$x = 2$

y-intercept - $y = 2(0) - 4$
$y = -4$

Inverse:

x-intercept - $x = -4 \quad (-4; 0)$

y-intercept - $y = 2 \quad (0; 2)$

Method 3: Using the inverse equation

1. Determine the equation of the inverse
2. Calculate the important properties (i.e. x- and y-intercepts, turning points, asymptotes, intervals that are ascending/descending, domain and range)
3. Sketch the graph of the inverse

EXAMPLE

Original function: $y = -2x^2$

$x = -2y^2$

$\dfrac{-x}{2} = y^2$

Inverse: $y = \pm\sqrt{\dfrac{-x}{2}}$

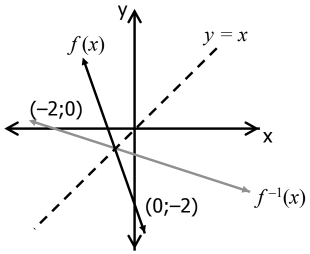
Functions and Inverses

INVERSE OF A LINEAR FUNCTION

Straight line: $f(x) = ax + q$

Inverse: $f^{-1}(x) = \dfrac{x - q}{a}$

Given $f(x) = -3x - 2$, the inverse is $f^{-1}(x) = -\dfrac{1}{3}x - \dfrac{2}{3}$



General properties of the inverse:

Domain: $x \in R$

Range: $y \in R$

Shape: ascending if $a > 0$

descending if $a < 0$

Average Gradient: $\overline{m} = \dfrac{1}{a}$

NOTE:

The inverse of $y = k$ (where k is a constant) will be perpendicular to the x-axis

The inverse of $x = k$ (where k is a constant) will be perpendicular to the y-axis

EXAMPLE

Given $g(x) = x - 3$, sketch the graph of $g(x)$ and its inverse on the same set of axes.

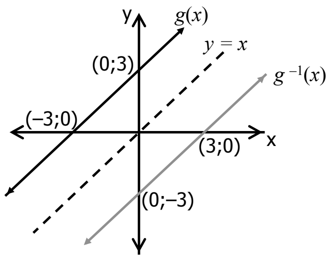
$g(x) = x + 3$

For $g^{-1}(x)$, switch x and y

$x = y - 3$

$y = x + 3$

$\therefore\ g^{-1}(x) = x + 3$



Determine the following relating to the inverse:

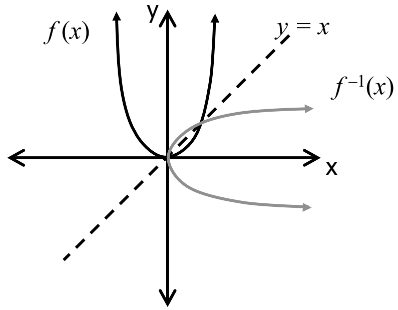
1. Domain: $x \in R$
2. Range: $y \in R$
3. x-intercept: $x = 3$ or $(3;0)$
4. y-intercept: $y = -3$ or $(0;-3)$
5. Average gradient: $m = \dfrac{0-(-3)}{3-0} = 1$

INVERSE OF A PARABOLA

Parabola: $f(x) = ax^2$

Inverse: $f^{-1}(x) = \pm\sqrt{\dfrac{x}{a}}$

Given $f(x) = x^2$, the inverse is $f^{-1}(x) = \pm\sqrt{x}$



General properties of the inverse:

Domain: $x \geq 0$ if $a > 0$

$x \leq 0$ if $a < 0$

Range: $y \in R$

Shape: if $a > 0$

if $a < 0$

Interval ascending (thus average gradient is positive):

if $a > 0; x \in [0;\infty)$

if $a < 0; x \in (-\infty;0]$

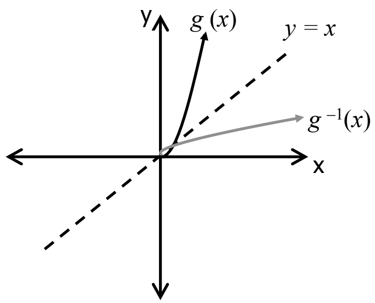
Interval descending (thus average gradient is negative):

if $a > 0; x \in [0;\infty)$

if $a < 0; x \in (-\infty;0]$

The inverse of a parabola is **not** a function, because there are two elements of the range for every element of the domain, **BUT** if we restrict the domain of the original function, we get an inverse that **is** a function and will look like this:

Given $g(x) = 3x^2$ with domain $x \geq 0$, the inverse is $g^{-1}(x) = +\sqrt{\dfrac{x}{3}}$. Both the restricted parabola and its inverse appear on the graph below.



Therefore, the domain of a parabolic function should be limited, either as $x \geq 0$ or $x \leq 0$, in order to create an inverse that is a function.

EXAMPLE

Given $h(x) = -x^2$ with domain $x \leq 0$, sketch the graph of $h(x)$ and its inverse on the same set of axes. Indicate the line of symmetry.

$h(x) = -x^2$

For $h^{-1}(x)$, switch x and y

$x = -y^2$

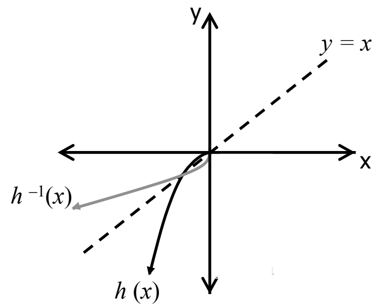
$y^2 = -x$

$y = \sqrt{-x}$

$\therefore\ h^{-1}(x) = \pm\sqrt{-x}$

BUT domain for $h(x)$ given as $x \leq 0$

thus, $h^{-1}(x) = \sqrt{-x}$



Determine the following relating to the inverse:

1. Domain: $x \leq 0$
2. Range: $y \leq 0$
3. x-intercept: $x = 0$ or $(0;0)$
4. y-intercept: $y = 0$ or $(0;0)$
5. Interval ascending: $x \in (-\infty;0]$

FUNCTIONS AND INVERSES

WORKING WITH LOGARITHMS

A logarithm (or 'log') is a mathematical notation that has been defined to allow us to make an exponent the subject of a formula. A logarithm function is the inverse of an exponential function, therefore if $y = a^x$ $(a > 0,\ a \neq 1)$ then the inverse is $x = a^y$. If written in standard function form then $x = a^y$ is $y = \log_a x$.

$\boxed{x = a^y}$ then $\boxed{y = \log_a x}$

Index/(log) : y
Number : x
Base : a

Log/(Index) : y
Number : x
Base : a

EXAMPLES

1. Write the following in log form:

a. $y = \left(\frac{1}{5}\right)^x$

$x = \log_{\frac{1}{5}} y$

b. $3^1 = 3$

$1 = \log_3 3$

c. $x = 4^y$

$y = \log_4 x$

2. Write the following in exponential form:

a. $y = \log_{\frac{1}{2}} x$

$x = \left(\frac{1}{2}\right)^y$

b. $\log_6 1 = 0$

$1 = 6^0$

c. $x = \log_y 5$

$5 = y^x$

3. Solve the following equations:

a. $\log_4 \frac{m}{3} = 3$

$\frac{m}{3} = 4^3$

$m = 64 \times 3 = 192$

b. $\log_3(5x - 3) = 3$

$5x - 3 = 3^3$

$x = \frac{27 + 3}{5} = 6$

c. $\log\left(\frac{x}{4}\right) = -3$

$\frac{x}{4} = 10^{-3}$

$x = \frac{4}{1000}$

4. Determine the value of the following:

a. $\log_2 256$

$x = \log_2 256$

$256 = 2^x$

$x = 8$

b. $\log_{\frac{1}{3}}\left(\frac{1}{729}\right)$

$x = \log_{\frac{1}{3}}\left(\frac{1}{729}\right)$

$\left(\frac{1}{3}\right)^x = \frac{1}{729}$

$x = 6$

c. $\log\sqrt{10}$

$x = \log\sqrt{10}$

$10^x = \sqrt{10}$

$x = \frac{1}{2}$

LOG LAWS

1. Sum to product law:

$\log_a x + \log_a y = \log_a xy$

2. Difference to quotient law:

$\log_a x - \log_a y = \log_a \frac{x}{y}$

3. Power law:

$\log_a x^m = m\log_a x$

4. Change of base law:

$\log_a x = \frac{\log_b x}{\log_b a}$

NOTE:
$\log_a a = 1$
$\log_a 1 = 0$
$\log a\, x = \log_{10} a$

EXAMPLES

Simplify the following:

1. $\log_3 \frac{81}{729}$

$= \log_3 81 - \log_3 729$

$= 4\log_3 3 - 6\log_3 3$

$= 4(1) - 6(1)$

$= -2$

2. $\log_3 \frac{1}{9} + \log_5 1 + \log_4 8$

$= -2\log_3 3 + \log_5 1 + \frac{\log_2 8}{\log_2 4}$

$= -2(1) + 0 + \frac{3\log_2 2}{2\log_2 2}$

$= -2 + \frac{3}{2}$

$= -\frac{1}{2}$

3. $(\log_x 16 - \log_x 4) \div \log_x 8$

$= \log_x \frac{16}{4} \div \log_x 8$

$= \log_x 4 \div \log_x 8$

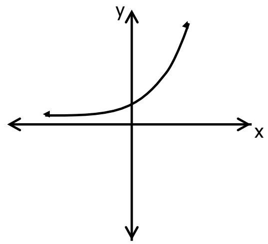
$= \frac{2\log_x 2}{3\log_x 2}$

$= \frac{2}{3}$

EXPONENTIAL AND LOG FUNCTIONS

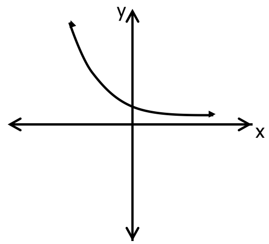
For exponential functions:

if $a > 1$



increasing function

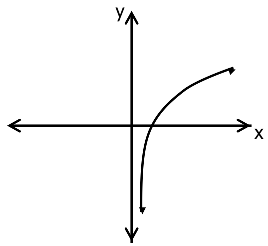
if $0 < a < 1$



decreasing function

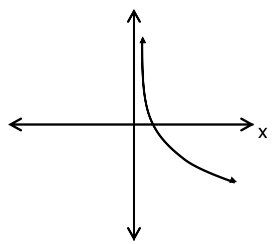
For logarithmic functions:

if $a > 1$



increasing function

if $0 < a < 1$

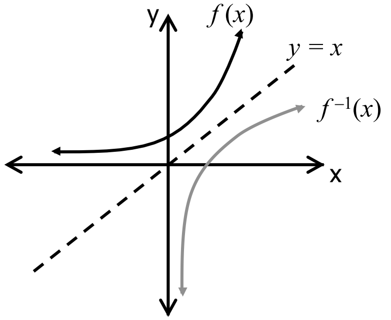


decreasing function

INVERSE OF AN EXPONENTIAL FUNCTION

Exponential: $f(x) = a^x;\ a > 0$ and $a \neq 1$
Inverse: $f^{-1}(x) = \log_a x;\ a > 0$ and $a \neq 1$

Given $f(x) = 3^x$, the inverse is $f^{-1}(x) = \log_3 x$



General properties of the inverse:
Domain: $x > 0$
Range: $y \in R$
Shape: if $a > 0$
if $a < 0$

Asymptote: $x = 0$

If there are no vertical or horizontal shifts (i.e. no p- or q-values) the log graph will cut the x-axis at 1.

EXAMPLE

Given $h(x) = \left(\frac{1}{3}\right)^x$ sketch the graph of $h(x)$ and its inverse on the same set of axes. Indicate the line of symmetry.

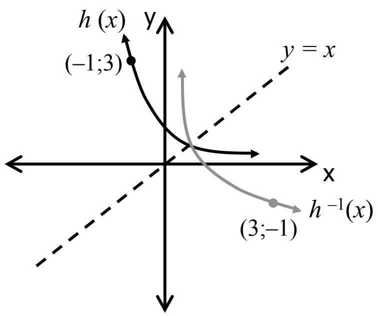
$h(x) = \left(\frac{1}{3}\right)^x$

For $h^{-1}(x)$, switch x and y

$x = \left(\frac{1}{3}\right)^y$

$y = \log_{\frac{1}{3}} x$

$\therefore\ h^{-1}(x) = \log_{\frac{1}{3}} x$



MIXED EXAMPLE 1

Questions:

1. Given $f(x) = \log_{\frac{1}{2}} x$

a. If $f(x) = -3$, determine x.

b. Draw the graph of $f(x)$.

c. For which value of x is $f(x) < -3$?

d. Determine $f^{-1}(x)$ and then draw the graph on the same set of axes as f.

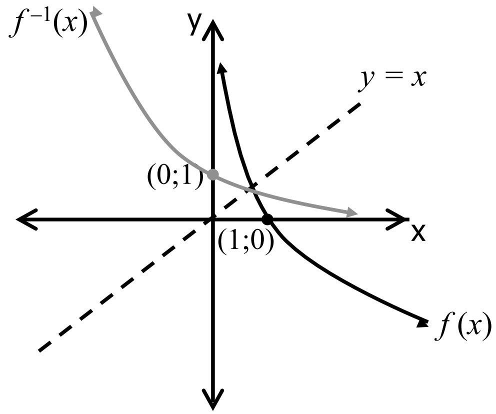
Solutions:

a. $\log_{\frac{1}{2}} x = -3$

$x = \left(\frac{1}{2}\right)^{-3}$

$x = 8$

b.



c. $x \in (8; \infty)$ OR $x > 8; x \in R$

d. $f^{-1}(x) = \left(\frac{1}{2}\right)^x$
$= 2^{-x}$

Questions:

2. The growth of a virus cell is given by $g(t) = 1{,}95^t$, where t is in minutes

a. Determine the number of virus cells after 8,5 minutes

b. After how many minutes will there be 2164 virus cells in the body?

Solutions:

a. $g(8{,}5) - 1{,}95^{8{,}5}$

Number of virus cells ≈ 292

b. $1{,}95^t = 2164$

$\log 1{,}95^t = \log 2164$

$t \log 1{,}95 = \log 2164$

$t = \dfrac{\log 2164}{\log 1{,}95}$

$t \approx 11{,}5$ minutes

Alternative solution (b):

$1{,}95^t = 2164$

$t = \log_{1{,}95} 2164$

$t \approx 11{,}5$ minutes

Functions and Inverses

MIXED EXAMPLE 2

Questions:

Given $g^{-1}(x) = -\sqrt{x}$

a. Write down the domain and range of $g^{-1}(x) = -\sqrt{x}$

b. Determine $g(x)$.

c. Draw sketch graphs of g and g^{-1} on the same set of axes

Solutions:

a. Domain: $x \in [0;\infty)$
Range: $y \in (-\infty;0]$

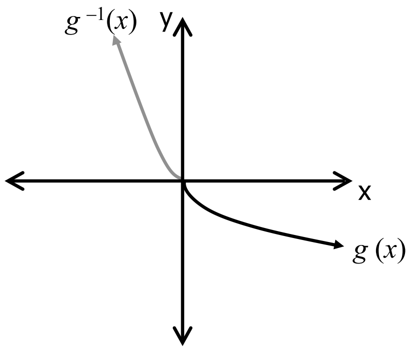
b. $x = -\sqrt{y}$

$(x)^2 = (-\sqrt{y})^2$

$x^2 = y$

$\therefore\ g(x) = x^2;\ x \in (-\infty;0]$

c.



MIXED EXAMPLE 3

Questions:

Given $f(x) = 3^x$

a. Draw a sketch graph of f and give the domain and range.

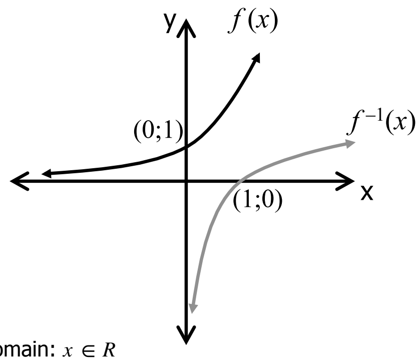
b. Determine $f^{-1}(x) = \dots$.

c. Sketch the graph of f^{-1} on the same set of axes and give the domain and range

d. Give the equation of the line symmetrical to f^{-1} about the x-axis

Solutions:

a.



Domain: $x \in R$
Range: $y \in (0;\infty)$

b. $y = \log_3 x$

c. Domain: $x \in (0;\infty)$
Range: $y \in R$

d. $y = -\log_3 x$

MIXED EXAMPLE 4

Questions:

a. If $h(x) = 2x^2$, determine the equation of the inverse of $h(x)$

b. Is the inverse a function? If it isn't a function, how can we restrict the original function so that it **is** a function?

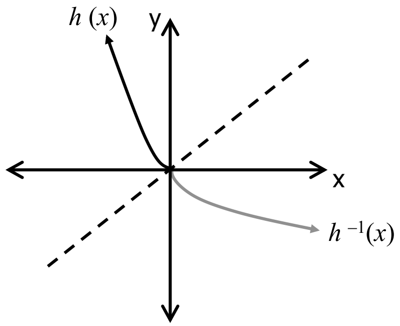
c. Sketch the graph $h(x)$ and it's inverse as determined in (b).

Solutions:

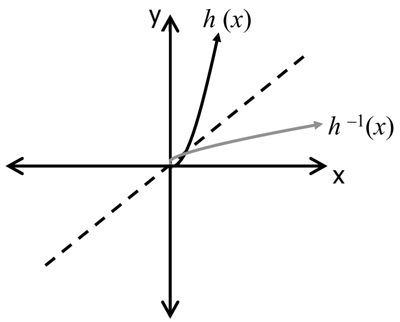
a. $x = y^2$
$y = \pm\sqrt{x}$

b. The inverse is not a function as one x has two different y values. Thus, we must restrict the domain of $h(x)$. There are two ways: i) $x \leq 0$ or ii) $x \geq 0$

c. i)



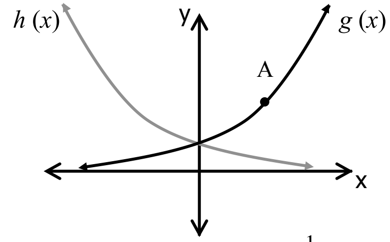
ii)



MIXED EXAMPLE 5

Questions:

The sketch represents the graph of $g(x) = a^x$ and $h(x)$, the reflection of $g(x)$ in the y-axis.



a. Calculate the value of a if $A(2;2\frac{1}{4})$ is a point on $g(x)$.

b. Write down the equation of $h(x)$.

c. Write down the equation of $g^{-1}(x)$ in the form $y = \dots$

d. Give the domain and range of $g(x)$, $h(x)$ and $g^{-1}(x)$

Solutions:

a. $2\frac{1}{4} = a^2$

$\sqrt{\frac{9}{4}} = \sqrt{a^2}$

$\therefore a = \frac{3}{2}$

b. $h(x) = \left(\frac{3}{2}\right)^{-x}$

$h(x) = \left(\frac{2}{3}\right)^{x}$

c. $g^{-1}(x) = \log_{\frac{3}{2}} x$

d. $g:\ x \in R$ and $y \in (0;\infty)$

$h:\ x \in R$ and $y \in (0;\infty)$

$g^{-1}:\ x \in (0;\infty)$ and $y \in R$

Finance – Simple and Compound Interest

REMINDERS:

1. **Inflation**: The rate at which prices increase over time
2. **Consumer Price Index (CPI)**: the average prices of a basket of goods
3. **Exchange Rates**: The value of one currency for the purpose of conversion to another
4. **Population Growth**: Change of population size over time
5. **Hire Purchase**: Short term loan, deposit payable. Calculated using simple interest.
6. **Reducing balance loan**: Interest is paid on the reducing balance, the lower the balance, the less you have to pay.
7. **Nominal interest rate**: Quoted period and compounded period is different eg 15% per annum compounded monthly.
8. **Effective interest rate**: Quoted period and compound period is equal eg 0,75% per month compounded monthly.

COMPOUND PERIODS

Annually:	1 per year
Semi-annually:	2 per year
Quarterly:	4 per year
Monthly:	12 per year
Daily:	365 per year* *(excl leap years)

SIMPLE INTEREST

$$A = P(1 + in)$$

OR

$$A = P\left(1 + \frac{r}{100}n\right)$$

A = accumulated amount
P = original amount
n = number of periods
r = interest rate as a %
i = interest rate $\frac{r}{100}$

COMPOUND INTEREST

$$A = P(1 + i)^n$$

OR

$$A = P\left(1 + \frac{r}{100}\right)^n$$

EXAMPLE

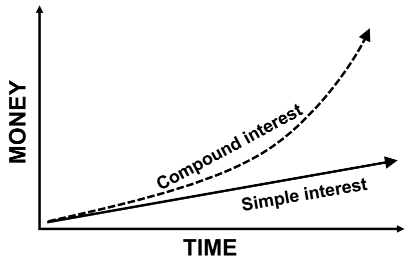
Determine the difference in the accumulated amounts when investing your savings of R 15 000 for 4 years at two different banks, both offer 6,5% however one offers simple interest and the other compound.

SIMPLE INTEREST

$$\begin{aligned} A &= P(1+in) \\ A &= 15\ 000(1+(0{,}065)(4)) \\ A &= R18\ 900 \end{aligned}$$

COMPOUND INTEREST

$$\begin{aligned} A &= P(1+i)^n \\ A &= 15\ 000(1+(0{,}065))^4 \\ A &= R19\ 296{,}99 \end{aligned}$$



HIRE PURCHASE

You buy a washing machine of R4 000 by signing a 2 year hire purchase agreement, you pay an R800 deposit. Calculate the
a) total amount you will repay if the interest rate is 12%
b) your monthly installment.

a) Deposit : R800

$$\begin{aligned} P &= 4\ 000 - 800 \\ &= 3\ 200 \\ A &= P(1+in) \\ &= 3\ 200(1+(0{,}12)(2)) \\ &= R3\ 968 \end{aligned}$$

b) $A = R3\ 968$

2 years = 24 equal payments

$$\frac{3\ 968}{24} = R165{,}33$$

POPULATION GROWTH

$$\boxed{P_{future} = P_{present}(1+i)^n}$$

P_{future} = future population size
$P_{present}$ = present population size
i = average population (%)
n= number of years

The population of lions is 2 567 in 2015.
If the growth rate is 1,34%, calculate the number of lions in 2020.

$2020 - 2015 = 5$

$$\begin{aligned} P_{future} &= P_{present}(1+i)^n \\ &= 2567(1+0{,}0134)^5 \\ &= 2743 \end{aligned}$$

(note that the number of lions will be an integer)

EXAMPLE

Calculate the future value of your investment after three years at an interest rate of 15% per annum compounded:

a) Annually

$$\begin{aligned} A &= P(1+i)^n \\ &= 15\ 000(1+(0{,}15))^3 \\ &= R22\ 813{,}13 \end{aligned}$$

b) Semi-annually

$$\begin{aligned} A &= P(1+i)^n \\ &= 15\ 000\left(1+\left(\frac{0{,}15}{2}\right)\right)^6 \\ &= R23\ 149{,}52 \end{aligned}$$

c) Quarterly

$$\begin{aligned} A &= P(1+i)^n \\ &= 15\ 000\left(1+\left(\frac{0{,}15}{4}\right)\right)^{12} \\ &= R23\ 331{,}81 \end{aligned}$$

d) Monthly

$$\begin{aligned} A &= P(1+i)^n \\ &= 15\ 000\left(1+\left(\frac{0{,}15}{12}\right)\right)^{36} \\ &= R23\ 495{,}16 \end{aligned}$$

Notice: As compounding periods increase during the year, so the accumulated amount increases.

EXAMPLE

If R13 865 is received after 6 years of being invested and the interest rate was 16% compounded annually, what was the original amount invested?

$$\begin{aligned} A &= P(1+i)^n \\ 13\ 865 &= P(1+0{,}16)^6 \\ 13\ 865 &= 2{,}44P \\ \frac{13\ 865}{2{,}44} &= P \\ P &= 5\ 690{,}78 \end{aligned}$$

∴R 5 690,78 was the principal amount invested.

OR use the following formulae:

$A = P(1+i)^n$	To find A
$P = A(1+i)^{-n}$	To find P

FINANCE

NOMINAL VS EFFECTIVE INTEREST RATES (COMPOUND INTEREST)

Annual effective rate is equivalent to the nominal rate per annum compounded monthly, because it produces the same accumulated amount.

$$\boxed{1 + i_{\text{eff}} = \left(1 + \frac{i_{\text{Nom}}}{n}\right)^n}$$

i_{eff} = effective rate (annual)
i_{Nom} = nominal rate
n = number of compoundings per year

EXAMPLE
Convert a nominal rate of 18% per annum compounded monthly to an annual effective rate.

$$\begin{aligned} 1 + i_{\text{eff}} &= \left(1 + \frac{0,18}{12}\right)^{12} \\ i_{\text{eff}} &= \left(1 + \frac{0,18}{12}\right)^{12} - 1 \\ i_{\text{eff}} &= 0,196 \\ \therefore i_{\text{eff}} &= 19,6\,\% \end{aligned}$$

EXAMPLE
You invest R25 000 at 14% per annum compounded monthly for a period of 12 months. Use the annual effective rate to show that the same accumulated amount will be obtained as when using the nominal rate.

$$\begin{aligned} 1 + i_{\text{eff}} &= \left(1 + \frac{0,14}{12}\right)^{12} \\ i_{\text{eff}} &= \left(1 + \frac{0,14}{12}\right)^{12} - 1 \\ i_{\text{eff}} &= 0,1493 \\ \therefore i_{\text{eff}} &= 14,93\,\% \end{aligned}$$

The exponent **(12)** is calculated by noting there will be 12 compounding periods: once a month for 12 months.

$$\begin{aligned} A &= P(1+i)^n \\ &= 25\ 000\left(1 + \left(\frac{0,14}{12}\right)\right)^{12} \\ &= R\,28\ 733,55 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 25\ 000(1 + 0,1493)^1 \\ &= R\,28\ 733,55 \end{aligned}$$

CHANGING INTEREST RATES

If the interest rate changes after a set period of time:
1. Determine the accumulated amount after the first period
2. Set the accumulated amount as the initial amount for the second period
3. Determine the accumulated amount after the second period.

EXAMPLE

R100 000 is invested for 6 years at an interest rate of 16% per annum compounded quarterly. Thereafter the accumulated amount is reinvested for 5 years at an interest rate of 14% compounded semi-annually. Calculate the value of the investment at the end of this period.

$$\begin{aligned} A &= P(1+i)^n \\ A &= 100\ 000\left(1 + \frac{0,16}{4}\right)^{24} \\ A &= R\,256\ 330,42 \end{aligned}$$

24 periods : **4** period (quarterly) p.a. over 6 years

$$\begin{aligned} A &= P(1+i)^n \\ A &= 256\ 330,42\left(1 + \frac{0,14}{2}\right)^{10} \\ A &= R\,504\ 239,91 \end{aligned}$$

10 periods : **2** period (semi-annual) p.a. over 6 years

EXAMPLE

R30 000 was left to you in a savings account. The interest rate for the first 4 years is 12% per annum compounded semi-annually. Thereafter the rates change to 18% per annum compounded monthly and you leave the money for another 3 years. What is the future value of the investment after the savings period.

$$\begin{aligned} A &= P(1+i)^n \\ A &= 30\ 000\left(1 + \frac{0,12}{2}\right)^{8} \\ A &= R\,47\ 815,44 \end{aligned}$$

8 periods : **2** period (semi-annual) p.a. over 4 years

$$\begin{aligned} A &= P(1+i)^n \\ A &= 47\ 815,44\left(1 + \frac{0,18}{12}\right)^{36} \\ A &= R\,81\ 723,25 \end{aligned}$$

36 periods : **12** period (monthly) p.a. over 3 years

Alternatively:
$$\begin{aligned} A &= P(1+i)^n \times (1+i)^n \\ &= 30\ 000\left(1 + \frac{0,12}{2}\right)^{8} \times \left(1 + \frac{0,18}{12}\right)^{36} \\ &= R\,81\ 723,26 \end{aligned}$$

DEPRECIATION (DECAY)

Depreciation is the loss or decrease of value at a specified rate over time.

Depreciation: Loss of value over time
Book value: Value of equipment at a given time after depreciation
Scrap value: Book value of equipment at the end of its useful life

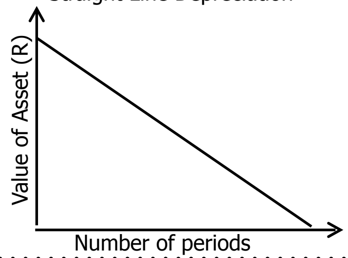
A = Book or scrap value
P = Present value
i = Depreciation rate
n = time period

LINEAR DEPRECIATION

Also known as simple decay or straight line depreciation

$$\mathbf{A = P(1 - in)}$$

Straight Line Depreciation

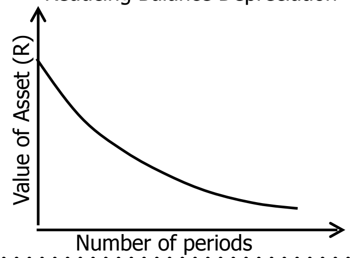


COMPOUND DEPRECIATION

Also known as depreciation on a reducing balance

$$\mathbf{A = P(1 - i)^n}$$

Reducing Balance Depreciation



EXAMPLE
My new car , to the value of R 200 000 depreciates at a rate of 9% per annum. What would the value of my car be after 6 years? Compare a linear depreciation to a reducing balance depreciation.

LINEAR DEPRECIATION
$$\begin{aligned} A &= P(1 - in) \\ &= 200\ 000(1 - (0,09)(6)) \\ &= R\,92\ 000 \end{aligned}$$

REDUCING BALANCE DEPRECIATION
$$\begin{aligned} A &= P(1 - i)^n \\ &= 200\ 000(1 - 0,09)^6 \\ &= R\,113\ 573,85 \end{aligned}$$

EXAMPLE
The value of a piece of equipment depreciates from R15 000 to R5 000 in four years. What is the rate of depreciation calculated on the:

a) Straight line method
$$\begin{aligned} A &= P(1 - in) \\ 5\ 000 &= 15\ 000((1 - (x)4) \\ 5\ 000 &= 15\ 000 - 60\ 000x \\ -10\ 000 &= -60\ 000x \\ x &= 0,1667 \end{aligned}$$

Depreciation rate = 16,67%

b) Reducing balance depreciation
$$\begin{aligned} A &= P(1 - i)^n \\ 5\ 000 &= 15\ 000(1 - i)^4 \\ \frac{1}{3} &= (1 - i)^4 \\ \sqrt[4]{\frac{1}{3}} - 1 &= -x \\ -0,2401\ldots &= -i \\ i &= 0,2401\ldots \times 100 \\ r &= 24\,\% \end{aligned}$$

CALCULATING TIME PERIOD OF A LOAN OR INVESTMENT

When calculating the time of your investment (n) in a compound formula you need to make use of Logs:

$y = b^x$ (Exponential form)

$x = log_b y$ (Logarithmic form)

EXAMPLE

Your car, costing R 250 000, needs to be traded in when its value has depreciated to R 90 000. How long will you drive the car if the reducing balance depreciation rate is 7,5% per annum.

$$\begin{aligned} A &= P(1-i)^n \\ 90\ 000 &= 250\ 000(1-0{,}075)^n \\ 0{,}36 &= 0{,}925^n \\ n &= \log_{0{,}925} 0{,}36 \\ n &= 13{,}1 \end{aligned}$$

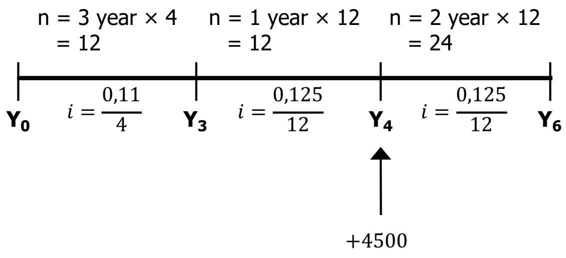
You can drive the car for 13 years before it depreciates to R90 000.

ADDITIONAL PAYMENTS OR WITHDRAWALS

Timelines assist in visualising and keeping track of different rates and payments. Set up each section with information about the number of terms, compound periods and interest rates.

EXAMPLE

You deposit R5 000 into a savings account and 4 years later you add R4 500. The interest for the first 3 years was 11% compounded quarterly, then it changed to 12,5% compounded monthly. Calculate the savings at the end of the 6th year.



$$\begin{aligned} Y_3 &= P(1+i)^n \\ &= 5000\left(1+\frac{0{,}11}{4}\right)^{12} \\ &= R6923{,}92 \end{aligned}$$

$$\begin{aligned} Y_4 &= P(1+i)^n + 4500 \\ &= 6923{,}92\left(1+\frac{0{,}125}{12}\right)^{12} + 4500 \\ &= R12\ 340{,}76 \end{aligned}$$

$$\begin{aligned} Y_6 &= P(1+i)^n \\ &= 12\ 340{,}76\left(1+\frac{0{,}125}{12}\right)^{24} \\ &= R15\ 825{,}37 \end{aligned}$$

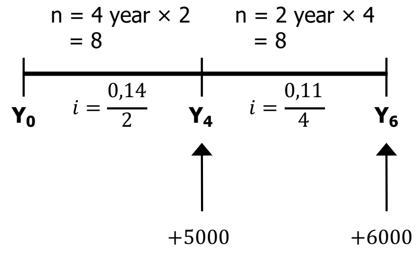
If withdrawals are made from the savings account, use subtraction instead

OR

$$\begin{aligned} Y_6 &= 5000\left[\left(1+\frac{0{,}11}{4}\right)^{12}\right]\left[\left(1+\frac{0{,}125}{12}\right)^{12} + 4500\right]\left[\left(1+\frac{0{,}125}{12}\right)^{24}\right] \\ &= R15\ 825{,}37 \end{aligned}$$

EXAMPLE

You take out a loan to buy a new iPad. You make an additional payment of R 5 000 four years after taking out the loan. Two years later you repay the final amount of R 6 000. During the first four years of the loan the interest rate is 14% per annum compounded semi annually. For the last two years the rate changed to 11% per annum compounded quarterly. How much did you initially borrow?



Working backwards, we need the value of the the loan at Y_4:

$$\begin{aligned} P &= A(1+i)^{-n} \\ P &= 6000\left(1+\frac{0{,}11}{4}\right)^{-8} \\ P &= R4\ 829{,}44 \end{aligned}$$

$A = P(1+i)^n$	To find A
$P = A(1+i)^{-n}$	To find P

That means that the loan amount at the end of Y4, before the payment of R5000 was made, was **R9 829,44.**

Work backing backwards, we determine the initial amount borrowed / loan amount at Y_0.

$$\begin{aligned} P &= A(1+i)^{-n} \\ P &= 9829{,}44\left(1+\frac{0{,}14}{2}\right)^{-8} \\ P &= R5\ 720{,}82 \end{aligned}$$

OR combine the steps into a single calculation

$$\begin{aligned} P &= 6000\left[\left(1+\frac{0{,}11}{4}\right)^{-8} + 5000\right]\left[\left(1+\frac{0{,}14}{2}\right)^{-8}\right] \\ &= R5\ 720{,}82 \end{aligned}$$

FUTURE VALUE ANNUITIES

Equal investment payments made at regular intervals, subject to a rate of interest over a period of time. Eg annuity fund, a retirement fund, a savings account.

$$F = \frac{x[(1+i)^n - 1]}{i}$$

F = total accumulated at the end of the time period
x = monthly installment
i = interest rate per annum
n = number of installments/payments

EXAMPLE

R2000 is deposited into a bank account. One month later a further R2 000 is deposited, and another R2 000 a month after that. The interest rate is 7% per annum compounded monthly. How much will be saved after 2 months?

Compound interest formula: $A = P(1+i)^n$

At the end of each month, a) interest is determined on the accumulated amount, and b) an additional deposit is made:

$M_0 : 2000$

$M_1 : 2000\left(1+\frac{0,07}{12}\right)^1 + 2000$

$M_2 : 2000\left(1+\frac{0,07}{12}\right)^2 + 2000\left(1+\frac{0,07}{12}\right)^1 + 2000 \ = R6\ 035,07$

The money saved at the end is the Future value of the investment

The values van be expressed as a series:

$M_2 : 2000\left(1+\frac{0,07}{12}\right)^2 + 2000\left(1+\frac{0,07}{12}\right)^1 + 2000\left(1+\frac{0,07}{12}\right)^0$

This is a geometric series where $a = 2000$, $r = \left(1+\frac{0,07}{12}\right)$, $n = 3$ (3 payments).

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$F_3 = \frac{2000[(1+\frac{0,07}{12})^3 - 1]}{(1+\frac{0,07}{12}) - 1}$$

$$F_3 = 6035,07$$

Future value annuity formula:

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$= \frac{2000[(1+\frac{0,07}{12})^3 - 1]}{\frac{0,07}{12}}$$

$$= R6\ 035,07$$

Formula only holds when payment commences one period from the present and ends after n periods.

PRESENT VALUE ANNUITIES

Reducing balance loan, a sum of money borrowed and paid back with interest in regular payments at equal intervals.

$$P = \frac{x[1 - (1+i)^{-n}]}{i}$$

P = present value
x = monthly installment
i = interest rate p.a.
n = number of time periods to repay the loan

EXAMPLE

You borrow money for a year at an interest rate of 18% per annum compounded quarterly and monthly repayments will be R 1 367,20. The payments will start one month after receiving the loan. What was the initial amount borrowed?

Compound interest formula: $P = A(1+i)^{-n}$

The payment at the end of each month can be used to determine the present value of that payment at M_0:

$M_1 : P = 1367,20\left(1+\frac{0,18}{4}\right)^{-1}$

$M_2 : P = 1367,20\left(1+\frac{0,18}{4}\right)^{-2}$

The money borrowed in the beginning is the present value of the annuity.

Present value of all payments (loan) at M_0:

$P = 1367,20\left(1+\frac{0,18}{4}\right)^{-1} + 1367,20\left(1+\frac{0,18}{4}\right)^{-2} + \cdots 1367,20\left(1+\frac{0,18}{4}\right)^{-12} = R12\ 466,92$

The values van be expressed as a series:

M_{12}: $1367,20\left(1+\frac{0,18}{4}\right)^{-1} + 1367,20\left(1+\frac{0,18}{4}\right)^{-2} + \cdots 1367,20\left(1+\frac{0,18}{4}\right)^{-12}$

This is a geometric series where $a = 1367,20\left(1+\frac{0,18}{4}\right)^{-1}$, $r = \left(1+\frac{0,18}{4}\right)^{-1}$, $n = 12$ (12 payments)

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$P = \frac{1367,20(1+\frac{0,18}{4})^{-1}[(1+\frac{0,18}{4})^{-12} - 1]}{(1+\frac{0,18}{4})^{-1} - 1}$$

$$P = 12\ 466,92$$

Present value annuity formula:

$$P = \frac{x[1 - (1+i)^{-n}]}{i}$$

$$= \frac{1367,20[1 - (1+\frac{0,18}{4})^{-12}]}{\frac{0,18}{4}}$$

$$= R12\ 466,92$$

Formula only holds when there is a period break between the loan and first payment. Eg first monthly payment starts 1 month after the loan is granted.

CALCULATE MONTHLY INSTALLMENTS

Monthly installments are calculated using the present value annuity formula, and solving for x, the repayment amount.

NB: If repayments commence one month after the initiation of the loan;
a) Calculate the growth of the loan during the first month to determine the new present value
b) Subtract one month from the total repayment terms.

CALCULATE OUTSTANDING BALANCE

Outstanding balance is calculated using the present value annuity formula. The present value after the n^{th} installments is the outstanding balance, also known as the settlement amount.

EXAMPLE

In order to buy a car John takes a loan of R 25 000. The bank charges an annual interest rate of 11% compounded monthly. The installments start a month after he has received the money from the bank.
Calculate
a) his monthly installments if he has to pay back the loan over a period of 5 years.
b) the outstanding balance of his loan after two years immediately after the 24th installment.

a) Repayment is deferred by one month, causing the capital amount to grow;

$$P = 25\ 000\left(1+\frac{0{,}11}{12}\right)^{1}$$

Total number of terms: $(5 \times 12) - 1 = 59$

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$25\ 000\left(1+\frac{0{,}11}{12}\right)^{1} = \frac{x\left[1-\left(1+\frac{0{,}11}{12}\right)^{-59}\right]}{\frac{0{,}11}{12}}$$

$$25\ 299{,}17 \times \frac{0{,}11}{12} = 0{,}4163x$$

$$x = R555{,}53$$

b)

$$P_{24} = \frac{x[1-(1+i)^{-n}]}{i}$$

$$= \frac{555{,}53\left[1-\left(1+\frac{0{,}11}{12}\right)^{-24}\right]}{\frac{0{,}11}{12}}$$

$$= R11\ 919{,}45$$

ANALYSES OF INVESTMENT AND LOAN OPTIONS

When analysing investment and loan options, consider both the payment amounts, as well as the total payment made at completion.

Investments with higher future values are preferable.
Loans with smaller present values are preferable.

EXAMPLE

You have to take a home loan of R8 000 000 and there are 2 options to consider.
A- a 20 year loan at 17% interest per annum compounded monthly.
B- a 30 year loan at 17% interest compounded monthly.

<u>Monthly repayments</u>:

A. $P = \frac{x[1-(1+i)^{-n}]}{i}$

$$8\ 000\ 000 = \frac{x\left[1-\left(1+\frac{0{,}17}{12}\right)^{-240}\right]}{\frac{0{,}17}{12}}$$

$$8\ 000\ 000 \times \frac{0{,}17}{12} = x\left[1-\left(1+\frac{0{,}17}{12}\right)^{-240}\right]$$

$$11\ 333{,}33 = 0{,}9658x$$

$$x = R11\ 734{,}40$$

B. $P = \frac{x[1-(1+i)^{-n}]}{i}$

$$8\ 000\ 000 = \frac{x\left[1-\left(1+\frac{0{,}17}{12}\right)^{-360}\right]}{\frac{0{,}17}{12}}$$

$$8\ 000\ 000 \times \frac{0{,}17}{12} = x\left[1-\left(1+\frac{0{,}17}{12}\right)^{-360}\right]$$

$$11\ 333{,}33 = 0{,}99368x$$

$$x = R11\ 405{,}40$$

Total repayments:
$A : R11\ 734{,}40 \times 240 = R2\ 816\ 256$
$B : R11\ 405{,}40x360 = R4\ 105\ 944$

Option B has a lower monthly repayment, but a total amount of almost double that of OPTION A, therefore A is the better option.

DIFFERENTIAL CALCULUS

AVERAGE GRADIENT:

Gradient between two points on a curve:

$$\text{Ave } m = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE

Determine the average gradient of $f(x) = x^2 + 2$ between $x = 2$ and $x = 4$.

$$\begin{aligned}\text{Ave } m &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{f(4) - f(2)}{4 - 2}\\ &= \frac{[(4)^2 + 2] - [(2)^2 + 2]}{4 - 2}\\ &= 6\end{aligned}$$

GRADIENT AT A POINT

The gradient of $f(x)$ at the point where $x = a$ is given by:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

EXAMPLE

Determine the gradient of $f(x) = 2x^2 - 1$ at the point where $x = 1$.

$$\begin{aligned}&\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\\ &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}\\ &= \lim_{h \to 0} \frac{[2(1+h)^2 - 1] - [2(1)^2 - 1]}{h}\\ &= \lim_{h \to 0} \frac{4h + 2h^2}{h}\\ &= \lim_{h \to 0} \frac{h(4 + 2h)}{h}\\ &= 4 + 2(0)\\ &= 4\end{aligned}$$

Note: DO NOT substitute zero until h is no longer in the denominator $\left(\frac{A}{0}\text{ is undefined}\right)$

FINDING THE DERIVATIVE USING THE DEFINITION (FIRST PRINCIPLES)

The derivative of a curve is the gradient of the curve.

$f'(x)$ is the derivative of $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

EXAMPLE

Find the derivative of $f(x)$ using first principles.

1. $f(x) = x^2 + 4$

$$\begin{aligned}\therefore f(x+h) &= (x+h)^2 + 4\\ &= x^2 + 2xh + h^2 + 4\end{aligned}$$

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 + 4] - [x^2 + 4]}{h}\\ &= \lim_{h \to 0} \frac{2xh + h^2}{h}\\ &= \lim_{h \to 0} \frac{h(2x + h)}{h}\\ &= 2x + 0\\ &= 2x\end{aligned}$$

2. $f(x) = \frac{1}{x}$

$$\therefore f(x+h) = \frac{1}{x+h}$$

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}\\ &= \lim_{h \to 0} \left[\frac{x - (x+h)}{(x+h)x}\right] \div h\\ &= \lim_{h \to 0} \frac{-h}{x(x+h)} \times \frac{1}{h}\\ &= \frac{-1}{x(x+0)}\\ &= \frac{-1}{x^2}\end{aligned}$$

DIFFERENT NOTATIONS USED FOR DERIVATIVES

The derivative of $y = f(x)$ can be written as:

- $f'(x)$
- $\frac{d\,f(x)}{dx}$
- $\frac{d}{dx}[f(x)]$
- y'
- $\frac{dy}{dx}$
- $D_x[f(x)]$
- $\frac{dy}{dt}$ means the derivative of the function of y with respect to the variable t
- $D_k[f(k)]$ means the derivative of $f(k)$ with respect to the variable k

DIFFERENTIATION RULES

NB: Use these rules unless you are specifically asked to use "FIRST PRINCIPLES"

1. The derivative of a constant is ZERO
   a. $\frac{dy}{dx}(-4) = 0$
   b. If $f(x) = 3$ then $f'(x) = 0$

2. The derivative of x^n is $n\,x^{n-1}$ (when n is a constant)
   a. If $y = x^2$ then $\frac{dy}{dx} = 2x^{2-1} = 2x$
   b. $D_x[x^{-3}] = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

3. The derivative of a constant k multiplied by $f(x)$ is $k\,.\,f'(x)$
   a. $\frac{d}{dx}(-2x^3) = -2(3x^{3-1}) = -6x^2$
   b. $D_x\left[\frac{3}{x^6}\right] = D_x[3x^{-6}] = -18x^{-7} = \frac{-18}{x^7}$
   c. $y = \frac{1}{2x^2} = \frac{x^{-2}}{2}$ then $y' = \frac{1}{2}(-2x^{-2-1}) = -x^{-3} = -\frac{1}{x^3}$

Note: First change all monomial denominators to negative exponents

4. The derivative of a sum, is the sum if the derivatives
   a. $\frac{d}{dx}\left(\frac{1}{2}x^3 + \frac{1}{x} - \pi\right) = \frac{d}{dx}\left(\frac{1}{2}x^3 + x^{-1} - \pi\right) = \frac{3}{2}x^2 + \frac{1}{x^2}$
   b. If $s = (t+1)(t-4) = t^2 - 3t - 4$
      then $\frac{d}{dx} = 2t - 3$
   c. If $f(x) = \frac{x^2 - 5x - 6}{x - 6} = \frac{\cancel{(x-6)}(x+1)}{\cancel{(x-6)}} = x + 1$
      then $f'(x) = 1$
   d. If $y = \frac{x^3 - 3}{x^2} = \frac{x^3}{x^2} - \frac{3}{x^2} = x - 3x^{-2}$
      then $\frac{dy}{dx} = 1 + 6x^{-3} = 1 + \frac{6}{x^3}$

DIFFERENTIAL CALCULUS

TANGENTS TO A CURVE AT A POINT

The gradient of a tangent to $y = f(x)$ at the point where $x = c$ is given by $m_{tan} = f'(c)$.

Steps:

1. Find the point of contact (POC)
2. Find $f'(x)$ at the POC (i.e. $f'(c)$)
3. $m_{tan} = f'(c)$
4. Find $y = mx + c$ and sub in m_{tan} and the POC

EXAMPLE 1

Find the equation of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$

- POC: $x = 2$
  $y = 2(2)^2 + 1$
  $= 9$
  $(2;9)$
- $f'(x)$
  Let $y = f(x)$
  $\therefore f'(x) = 4x$
  At $x = 2$:
  $f'(2) = 4(2) = 8$
- $\therefore m_{tan} = 8$
- $y = mx + c$
  $y = 8x + c$ (Sub in $(2;9)$)
  $9 = 8(2) + c$
  $c = -7$
  $\therefore y = 8x - 7$ is the tangent to $y = 2x^2 + 1$ at $x = 2$

EXAMPLE 2

Find the equation of the tangent(s) to $f(x) = x^3 - 1$ that is perpendicular to $y = -\frac{1}{3}x + 8$

1. $\perp$ lines: $m_1 \times m_2 = -1$
   $-\frac{1}{3} \times m_\perp = -1$
   $\therefore m_\perp = 3$

BUT $m_{tan} = f'(x)$
$3 = 3x^2$
$x^2 = 1$
$\therefore x = \pm 1$

So there are 2 tangents to $f(x)$ that have a gradient of 3

Tangent 1:	Tangent 2:
$y = (1)^3 - 1 = 0$	$y = (-1)^3 - 1 = -2$
POC $(1;0)$	POC $(-1;-2)$
$y = 3x + c$	$y = 3x + c$
$0 = 3(1) + c$	$-2 = 3(-1) + c$
$c = -3$	$c = 1$
$\therefore y = 3x - 3$	$\therefore y = 3x + 1$

EXAMPLE 3

Find the normal to the tangent of the curve $f(x) = 2x^2 + 4$ at the point where $x = 0$

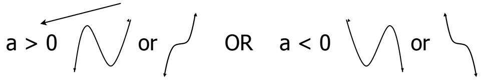
NOTE: THE NORMAL IS THE LINE $\perp$ TO THE TANGENT AT THE POC

- POC: $x = 0$
  $y = 2(0)^2 + 4$
  $= 4$
  $(0;4)$
- $f'(x)$
  $f'(x) = 4x$
  At $x = 0$:
  $f'(0) = 4(0) = 0$
- $m_{tan} = 0$
  thus, tangent is a horizontal line $y = 4$
- tan $\perp$ normal
  $\therefore$ the normal will be a vertical line through the POC
  $\therefore x = 0$

CUBIC CURVES

Standard form:

$y = ax^3 + bx^2 + cx + d \longrightarrow$ y-intercept



SKETCHING CUBIC GRAPHS

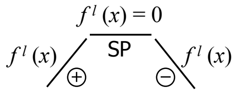
Steps for sketching a cubic curve:

1. Determine the stationary points (SPs) and their natures
2. Determine the x- and y-intercepts
3. Sketch the graph

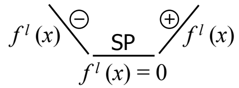
THE FIRST DERIVATIVE TEST

Stationary points (SP) are found when $f'(x) = 0$

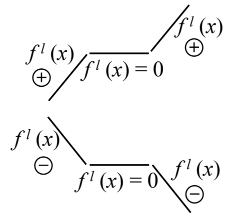
- $f(c)$ is a local max if :
  $f'(x) > 0$ to the left of c
  $f'(x) < 0$ to the right of c



- $f(c)$ is a local min if :
  $f'(x) < 0$ to the left of c
  $f'(x) > 0$ to the right of c



- If $f(c)$ has $f'(x)$ with the same sign on either side of c. Then $f(c)$ is neither a local to the right of c



EXAMPLE 1

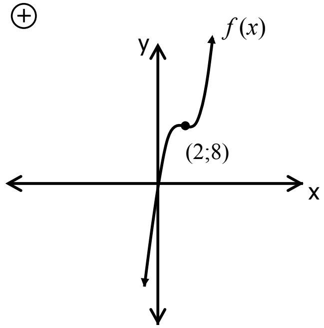
Sketch $f(x) = (x - 2)^3 + 8$

- Standard form:
  $f(x) = (x - 2)(x^2 - 4x + 4) + 8$
  $= x^3 - 6x^2 + 12x$
- Stationary points (SPs at $f'(x) = 0$):
  $3x^2 - 12x + 12 = 0$
  $x^2 - 4x + 4 = 0$
  $(x - 2)^2 = 0$
  $\therefore x = 2$
  $f(2) = 8$
  $\therefore (2;8)$

Nature of SPs → first derivative test

x		SP 2	
$f'(x)$	⊕	0	⊕

- y-intercept (x = 0)
  $f(0) = 0$
- x-intercept (y = 0)
  $f(x) = 0$
  $(x - 2)^3 + 8 = 0$
  $(x - 2)^3 = -8$
  $x - 2 = \sqrt[3]{-8}$
  $x - 2 = -2$
  $\therefore x = 0$



EXAMPLE 2

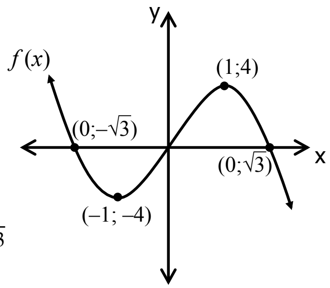
Sketch $f(x) = 6x - 2x^3$

- Stationary points (SPs at $f'(x) = 0$):
  $6 - 6x^2 = 0$
  $x^2 = 1$
  $x = \pm 1$
  $\therefore (1;4)$ or $(-1;-4)$

Nature of SPs → first derivative test

x		SP (-1)		SP (1)	
$f'(x)$	⊕	0	⊕	0	⊖

- y-intercept (x = 0)
  $f(0) = 0$
- x-intercept (y = 0)
  $f(x) = 0$
  $6x - 2x^3 = 0$
  $2x(3 - x^2) = 0$
  $x = 0$ or $x^2 = 3$
  $x = \pm\sqrt{3}$

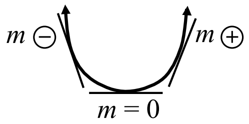


DIFFERENTIAL CALCULUS

CONCAVITY

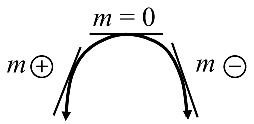
Concavity explains how the gradients of the tangents to a curve change from left to right

- A section of a curve is concave up if the gradient of the tangent increases from left to right



$\ominus \rightarrow 0 \rightarrow \oplus$
increasing gradients
$\therefore$ concave up

- A section of a curve is concave down if the gradient of the tangent decreases from left to right



$\oplus \rightarrow 0 \rightarrow \ominus$
decreasing gradients
$\therefore$ concave down

TEST FOR CONCAVITY

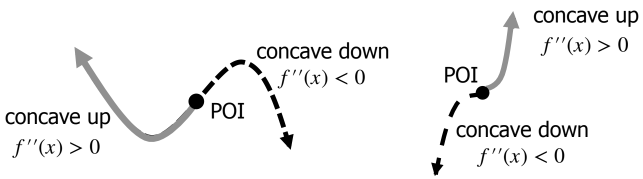
Remove the highest common factor from the coefficients and common variables.

$f''(x) \rightarrow$ second derivative

If: * $f''(x) > 0$ concave up
* $f''(x) < 0$ concave down
* $f''(x) = 0$ no conclusion about concavity

POINT OF INFLECTION (POI)

Occurs where concavity changes (i.e. where $f''(x) = 0$)



Second derivative test:
Another way to find the nature of the stationary points:

If: * $f''(x) > 0$ at the stationary point, is LOCAL MIN
* $f''(x) < 0$ at the stationary point, is LOCAL MAX
* $f''(x) = 0$ at the stationary point, is POINT OF INFLECTION

FACTORISING MORE COMPLEX CUBIC POLYNOMIALS

The factor theorem states that if $f(c) = 0$ then $(x - c)$ is a factor

EXAMPLE 1

Factorise $x^3 - 5x^2 - 2x + 24$ by using inspection

$f(-2) = 0 \quad \therefore \ (x + 2)$ is a factor

$x^3 - 5x^2 - 2x + 24$

$= \overbrace{(x + 2)(x^2 - 7x + 12)}^{x^3 \qquad 24}$

$\underbrace{\qquad}_{2x^2}$, $\underbrace{\qquad}_{-7x^2}$

$= (x + 2)(x - 4)(x - 3)$

EXAMPLE 2

Factorise $x^3 - 7x^2 + 14x - 8$ by using synthetic division

$f(1) = 0 \quad \therefore \ (x - 1)$ is a factor

$$\begin{array}{c|l} 1 & 1 - 7 + 14 - 8 \\ & \downarrow + 1 - 6 + 8 \\ \hline & 1 - 6 + 8 \quad \cdot \end{array}$$
$\leftarrow$ coefficients of the polynomial

$= (x - 1)(x^2 - 4x + 5)$
$= (x - 1)(x - 2)(x - 4)$

MORE SKETCHING (USING THE SECOND DERIVATIVE)

EXAMPLE 1

Sketch $f(x) = x^3 - 5x^2 + 8x - 4$

- Stationary points (SPs at $f'(x) = 0$):

$3x^2 - 10x + 8 = 0$
$(3x - 4)(x - 2) = 0$
$x = \frac{4}{3}$ or $x = 2$
$(\frac{4}{3}; \frac{4}{27})$ $\quad (2; 0)$

Nature of SPs $\rightarrow$ second derivative test

$f''(x) = 6x - 10$
$f''\left(\frac{4}{3}\right) = -2$ and $-2 < 0$ $\quad$ $f''(2) = 2$ and $2 > 0$
$\therefore$ local max $\quad$ $\therefore$ local min

- y-intercept (x = 0)

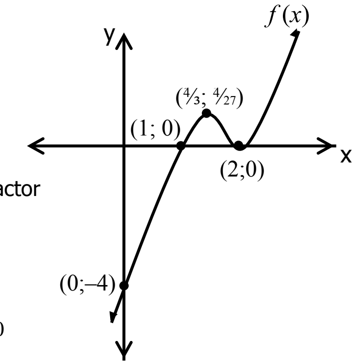
$f(0) = 4$

- x-intercept (y = 0)

$f(x) = 0$
$x^3 - 5x^2 + 8x - 4 = 0$
$f(1) = 0 \quad \therefore \ (x - 1)$ is a factor

$$\begin{array}{c|l} 1 & 1 - 5 + 8 - 4 \\ & \downarrow + 1 - 4 + 4 \\ \hline & 1 - 4 + 4 \quad \cdot \end{array}$$

$(x - 1)(x^2 - 4x + 4) = 0$
$(x - 1)(x - 2)^2 = 0$
$x = 1$ or $x = 2$
($(x = 2)$ is double x-cut, $\therefore$ also a SP)



EXAMPLE 2

Sketch $f(x) = x^3 - x^2 - 5x - 3$

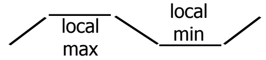
- Stationary points (SPs at $f'(x) = 0$):

$3x^2 - 2x - 5 = 0$
$(3x - 5)(x + 1) = 0$
$x = \frac{5}{3}$ or $x = -1$
$(\frac{5}{3}; \frac{-256}{27})$ $\quad (-1; 0)$

Nature of SPs $\rightarrow$ 2nd derivative test **OR** 1st derivative test

$f''(x) = 6x - 2$
$f''\left(\frac{5}{3}\right) = 8$ and $8 > 0$
$\therefore$ local min
$f''(-1) = -8$ and $-8 < 0$
$\therefore$ local max

x		SP (-1)		SP (5/3)	
$f'(x)$	$\oplus$	0	$\ominus$	0	$\oplus$



- y-intercept (x = 0)

$f(0) = 4$

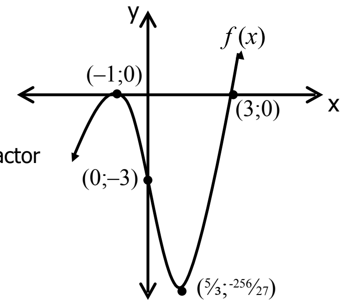
- x-intercept (y = 0)

$f(x) = -3$
$x^3 - 5x^2 + 8x - 4 = 0$
$f(-1) = 0 \quad \therefore \ (x + 1)$ is a factor
$x^3 - x^2 - 5x - 3 = 0$

$= \overbrace{(x + 1)(x^2 - 2x - 3)}^{x^3 \qquad -3}$

$\underbrace{\qquad}_{x^2}$, $\underbrace{\qquad}_{-2x^2}$

$= (x + 1)(x + 1)(x - 3) = 0$
$x = -1$ or $x = 3$



DIFFERENTIAL CALCULUS

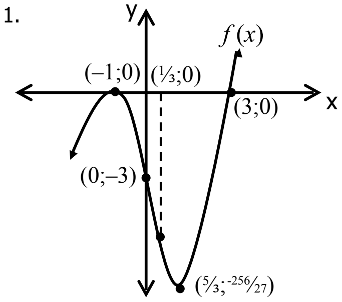
EXAMPLE 2: APPLICATION QUESTIONS

Questions:

1. Find the x-value of the POI and add it onto your sketch of $f(x)$.
2. State the values of x for which:
   a. $f(x) \geq 0$ (above/on the x-axis)
   b. $f'(x) > 0$ (gradient positive and increasing)
   c. $f''(x) < 0$ (concave down)
   d. $f(x) \cdot f'(x) > 0$ (positive product)
   e. $f''(x) \cdot f'(x) \leq 0$ (negative product)

Solutions:

1.



2.

a. $x = -1$ or $x \in [3; \infty)$

b. $x \in (-\infty; -1) \cup \left(\frac{5}{3}; \infty\right)$

c. $x \in \left(-\infty; \frac{1}{3}\right)$

d.

	$f(x)$	$f'(x)$
below x-axis	⊖	⊖ ↘
above x-axis	⊕	⊕ ↗

$x \in \left(-1; \frac{5}{3}\right) \cup (3; \infty)$

e.

	$f''(x)$	$f'(x)$
conc down	⊖ ∩	⊕ ↗
conc up	⊕ ∪	⊖ ↗

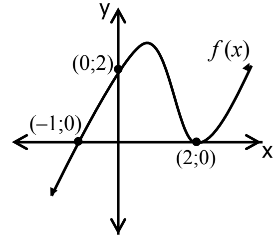
$x \in (-\infty; -1] \cup \left[\frac{1}{3}; \frac{5}{3}\right]$

FINDING THE EQUATION OF A CUBIC CURVE

x-intercepts given

$y = a(x - x_1)(x - x_2)(x - x_3)$

$[x_1;\ x_2;\ x_3:\ x$-intercepts$]$



$y = a(x - x_1)(x - x_2)(x - x_3)$

$y = a(x+1)(x-2)(x-2)$

$(0; 2)$

$2 = a(0+1)(0-2)^2$

$4a = 2$

$a = \frac{1}{2}$

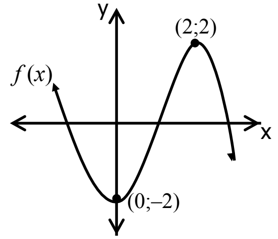
$\therefore\ y = \frac{1}{2}(x+1)(x-2)^2$

$= \frac{1}{2}(x+1)(x^2 - 4x + 4)$

$= \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2$

Stationary Points given

Remember: $f'(x) = 0$



If $f(x) = -x^3 + bx^2 + cx + d$

then $f'(x) = -3x^2 + 2bx + c$

at $x = 0$ and $x = 2$

$2 = a(0+1)(0-2)^2$

$-3(0)^2 + 2b(0) + c = 0$

$c = 0$

$-3(2)^2 + 2b(2) = 0$

$b = 6$

$\therefore\ y = -x^3 + 6x^2 + d$ (Sub in $(0; -2)$ or $(2; 2)$

$-2 = -(0)^3 + 6(0)^2 + d$

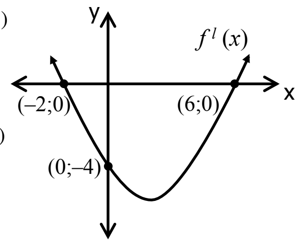
$\therefore\ d = -2$

$\therefore\ y = -x^3 + 6x^2 - 2$

Derivative graph given

This is a sketch of $f'(x)$, the derivative of $f(x) = ax^3 + bx^2 + cx + 8$

$f'(x)$ graph is a parabola with x-intercepts given



$y = p(x - x_1)(x - x_2)$

$y = p(x+2)(x-6)$

Sub in $(0; -4)$

$-4 = p(0+2)(0-6)$

$p = \frac{1}{3}$

$\therefore\ y = \frac{1}{3}(x+2)(x-6)$

$y = \frac{1}{3}x^2 - \frac{4}{3}x - 4$

$\therefore f'(x) = \frac{1}{3}x^2 - \frac{4}{3}x - 4$

But $f'(x) = 3ax^2 + 2bx + c$

$\therefore\ 3a = \frac{1}{3}$ $\quad 2b = -\frac{4}{3}$ $\quad c = -4$

$a = \frac{1}{9}$ $\quad b = -\frac{2}{3}$

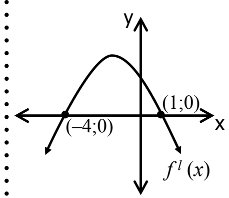
$f(x) = \frac{1}{9}x^3 - \frac{2}{3}x^2 - 4x + 8$

ANALYSING DERIVATIVE GRAPHS

$f'(x)$ graph →	$f(x)$ graph
Negative (i.e. $f'(x) < 0$)	Slope is decreasing
Positive (i.e. $f'(x) > 0$)	Slope if increasing
Zero/x-intercepts (i.e. $f'(x) = 0$)	Stationary points (SPs)
Turning point (i.e. $f''(x) = 0$)	Point of inflection (POI)

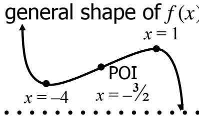
EXAMPLE

The graph of $f'(x)$ is given



For which value of x does $f(x)$ have the following characteristics?

- Decreasing (i.e. $f'(x) < 0$)
  $x \in (-\infty; -4) \cup (1; \infty)$
- Increasing (i.e. $f'(x) > 0$)
  $x \in (-4; 1)$
- SPs (i.e. $f'(x) = 0$)
  $x = -4$ or $x = 1$
- POI (i.e. $f'(x)$ – TP)
  $x = \frac{-4+1}{2} = \frac{-3}{2}$
- Concave up (grad incr L→R)
  $x \in \left(-\infty; -\frac{3}{2}\right)$
- Concave down (grad decr L→R)
  $x \in \left(-\frac{3}{2}; \infty\right)$
- Hence, the general shape of $f(x)$ would be:

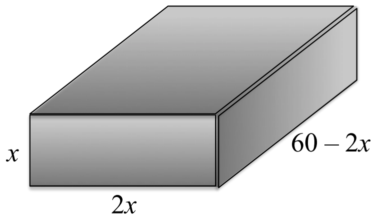


APPLICATIONS OF DIFFERENTIAL CALCULUS

MINIMA AND MAXIMA

Minima/ maxima are found when the derivative is zero (i.e. $f'(x) = 0$)

EXAMPLE



A rectangular box has the dimensions $x \times 2x \times (60 - 2x)$ cm.

Questions:

1. Determine an expression for the volume of the box in terms of x.
2. Determine the dimensions of the box that would give the maximum volume.
3. What is the maximum volume?

Solutions:

1. $\text{v} = \ell\,\text{bh} = x(2x)(60 - 2x)$

   $= 120x^2 - 4x^3 \text{ cm}^3$

2. $\text{v}' = 0$ at maximum volume

   $240x - 12x^2 = 0$

   $12x(20 - x) = 0$

   $x \neq 0$ or $x = 20$

   Dimensions: $x \times 2x \times (60 - 2x)$ cm

   $= 20 \times 40 \times 20$ cm

3. Method 1:

   $\text{v} = 120x^2 - 4x^3$

   $= 120(20)^2 - 4(20)^3$

   $= 16\ 000 \text{ cm}^3$

   Method 2:

   $\text{v} = \ell\,\text{bh}$

   $= (20)(40)(20)$

   $= 16\ 000 \text{ cm}^3$

RATE OF CHANGE

If y is a function of x, then the instantaneous rate of change is $\frac{dy}{dx}$.

EXAMPLE

The volume of a tank of water at a given time (t in minutes) is given by $\text{v} = 10 + 8t - 2t^2 \text{ m}^3$

Questions:

1. What is the rate at which the volume is changing after 1 minute?
2. After how long will the water's volume be a maximum?
3. When will the tank be empty?

Solutions:

1. $\text{v} = 10 + 8t - 2t^2 \text{ m}^3$

   Rate of change: $\frac{d\text{v}}{dt} = 8 - 4t$

   At $t = 1$: $8 - 4(1)$

   $= 4\text{m}^3$/minute

2. Max when $\frac{d\text{v}}{dt} = 0$

   $\therefore\ 8 - 4t = 0$

   $t = 2$ minutes

3. $\text{v} = 0$

   $10 + 8t - 2t^2 = 0$

   $t^2 - 4t - 5 = 0$

   $(t - 5)(t + 1) = 0$

   $t = 5$ or $t \neq -1$

   $\therefore\ t = 5$ minutes

CALCULUS OF MOTION

Displacement	s	m	
Velocity	v	m/s	$\text{v} = \frac{ds}{dt}$
Acceleration	a	m/s^2	$\text{a} = \frac{dv}{dt}$ or $\text{a} = \frac{d^2s}{dt^2}$

EXAMPLE

The displacement of a projectile is given by $s = 5t^2 - 20t$ with s in metres and t in seconds.

Questions:

1. What is the time when the displacement is a maximum?
2. What is the velocity of the projectile after 5 seconds?
3. What is the acceleration of the projectile?

Solutions:

1. $\frac{ds}{dt} = \text{v} = 0$

   $10t - 20 = 0$

   $\therefore\ t = 2$ seconds

2. $\text{v} = 10t - 20$

   $= 10(5) - 20$

   30 m/s

3. $\text{v} = 10t - 20$

   $\text{a} = \frac{d\text{v}}{dt} = 10$ m/s^2

PROBABILITY

Theoretical Probability of an event happening:

$$P(E) = \frac{\text{number of possible times event can occur}}{\text{number of possible outcomes}}$$

$$= \frac{\text{n}(E)}{\text{n}(S)}$$

E = event S = sample space

Relative frequency or Experimental probability:

$$P(E) = \frac{\text{number of times the event occured}}{\text{number of trials done}}$$

Theoretical Probability of an event happening:

$S = \{\text{sample set}\}$

$A = \{\text{event A}\}$

$B = \{\text{event B}\}$

$A \cup B = \{\text{A union B}\}$ = in sets A or B

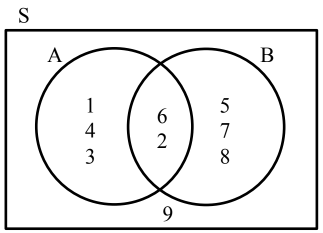
$A \cap B = \{\text{A intersection B}\}$ = in sets A and B

Addition Rule:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A and B are **inclusive** events as they have elements in common.



$A \cap B = \{2; 6\}$

$A \cup B = \{1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8\}$

$n(A) = 5$

$n(B) = 5$

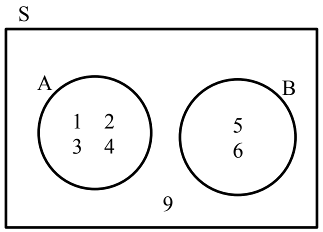
$n(A \cap B) = 2$

$n(A \cup B) = 8$

Mutually Exclusive:

$$A \cap B = \{\}$$

A and B are **mutually exclusive** events as they have no elements in common.



NOTE:

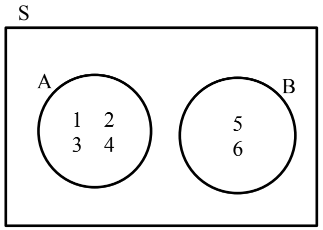
$n(A \cup B) = n(A) + n(B)$

$n(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

Exhaustive Events:

Events are **exhaustive** when they cover all elements in the sample set.



Complimentary Events:

Events A and B are **complimentary** events if they are mutually exclusive **and** exhaustive.

$$n(\text{not in } A) = n(A') = 1 - n(A)$$

$$P(A') = 1 - P(A) = P(B)$$

Independent Events:

Independent events are two events that do not affect each other's outcomes. E.g. choosing two coloured marbles from a bag, with replacement, thus, the first choice doesn't affect the outcome of the second choice.

Thus, the multiplication rule holds

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Dependent Events:

Dependent events are when the first event (A) affects the outcome of the second event (B). E.g. choosing two coins from a wallet without replacing the first coin. The first choice affects the second choice as the coin in hand in no longer available for the second choice.

PROBABILITY

TREE DIAGRAMS

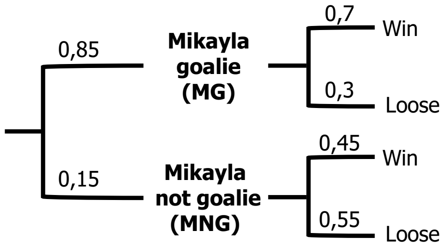
EXAMPLE 1

Questions:

The girls' hockey team has a match on Saturday. There is an 85% chance Makayla will play goalie. If she does there is a 70% chance the team will win but if she doesn't play there is a 45% chance they will win the game.
Draw a tree diagram to help answer the following questions:

1. What is the probability of Makayla playing goalie and the team winning?
2. What is the probability of the team losing the match on Saturday?

Solutions:



1. $\text{P(MG and win)} = 0{,}85 \times 0{,}7$
$= 0{,}6$

2. $\text{P(MG and loose)} + \text{P(MNG and loose)}$
$= 0{,}85 \times 0{,}3 + 0{,}15 \times 0{,}55$
$= 0{,}34$

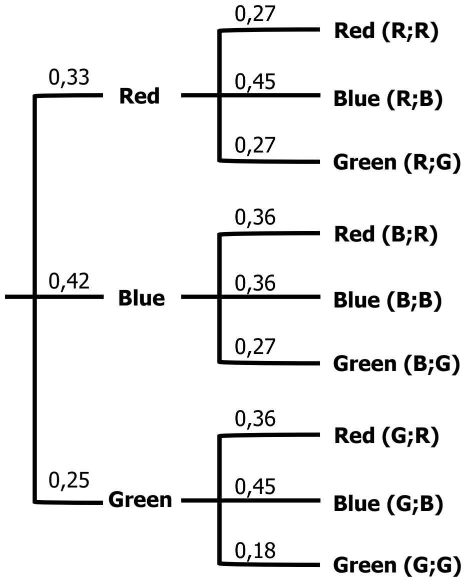
EXAMPLE 2

Questions:

A bag containing 4 red metal balls, 5 blue metal balls and 3 green metal balls. A ball is chosen at random and not replaced and then another ball is chosen at random again. Draw a tree diagram to list all possible events
Use the tree diagram to determine the probabilities of the following:

1. a blue metal ball in the first drawn.
2. a red metal ball then a green metal ball is drawn.
3. two metal balls of the same colour are drawn.

Solutions:



1. $\text{P(B and any)} = 0{,}42 \times 0{,}36 + 0{,}42 \times 0{,}36 + 0{,}42 \times 0{,}27$
$= 0{,}42$

2. $\text{P(R then G)} = 0{,}33 \times 0{,}27$
$= 0{,}09$

3. $\text{P(RR or BB or GG)} = 0{,}33 \times 0{,}27 + 0{,}42 \times 0{,}36 + 0{,}25 \times 0{,}18$
$= 0{,}29$

CONTINGENCY TABLE (OR TWO-WAY TABLE)

EXAMPLE 1

Questions:

In a group of 26 learners, 10 wear glasses, 8 are left handed and 6 of the students are both left-handed and wear glasses.

1. Complete the missing information in the table below.

	Left-handed	Right-handed	TOTAL
Glasses	a	b	c
No glasses	d	e	f
TOTAL	g	h	i

2. Calculate that if you pick any one of the 26 learners at random that they will be:
   a. Right handed
   b. Right handed and not wear glasses

Solutions:

1. $a = 6;\ b = 4;\ c = 10;\ d = 2;\ e = 14;\ f = 16;\ g = 8;\ h = 18;\ i = 26$
2. $\text{P(RH)} = \frac{18}{26} = 0{,}69$
3. $\text{P(RH and NG)} = \frac{14}{26} = 0{,}54$

EXAMPLE 2

Questions:

A survey was conducted asking learners which hand they use to write with and what colour ink they prefer. The results are summarised below.

		Hand used to write with		
		Left (L)	**Right (R)**	**Total**
Ink colour	**Blue (Bu)**	a	b	24
	Black (Ba)	c	d	56
	Total	50	30	80

The survey concluded that "the hand used to write with" and "Ink colour" are independent. Calculate the values of a, b and c.

Solutions:

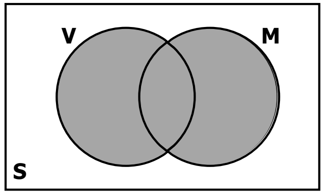
$$\text{P(L and BU)} = \text{P(L)} \times \text{P(Bu)}$$
$$\frac{a}{80} = \frac{50}{80} \times \frac{24}{80}$$
$$a = 15$$

$\therefore b = 24 - 15 = 9$

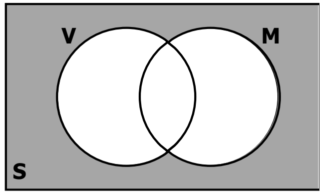
$c = 50 - 15 = 35$

PROBABILITY – VENN DIAGRAM

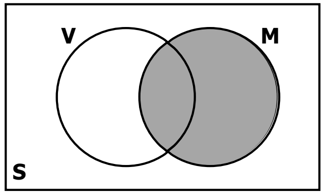
$V \cup M$



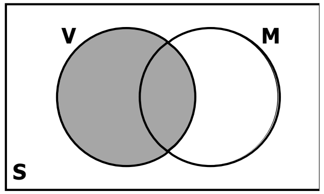
$(V \cup M)'$ or $V' \cap M'$



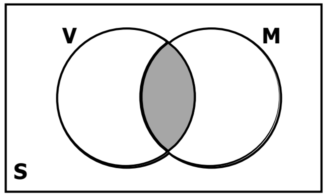
M



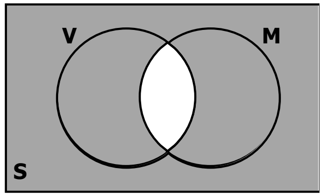
V



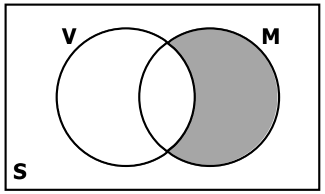
$V \cap M$



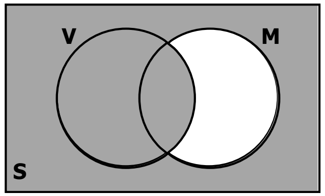
$(V \cap M)'$ or $V' \cup M'$



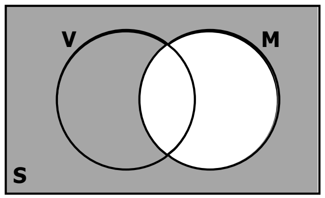
$V' \cup M$



$V \cup M'$



M'

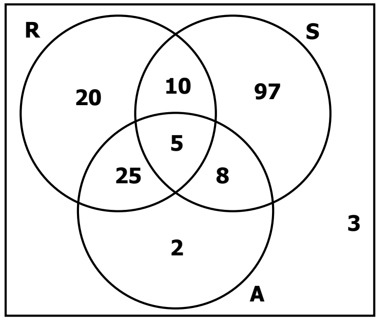


EXAMPLE 1

Questions:

Calculate from the Venn diagram for a grade 6 group in which the number of equally likely ways the events (Reading(R); Sports(S) and Art(A)) can occur has been filled in:

Gr 6



1. P(A∩R∩S)
2. P(R and A and not S)
3. P(A or R)
4. P(S or R and not A)

Solutions:

1. $P(A \cap R \cap S) = \frac{5}{170} = \frac{1}{34}$
2. $P(R \text{ and } A \text{ and not } S) = \frac{25}{170} = \frac{5}{34}$
3. $P(A \text{ or } R) = \frac{70}{170} = \frac{7}{17}$
4. $P(S \text{ or } R \text{ and not } A) = \frac{127}{170}$

EXAMPLE 2

Questions:

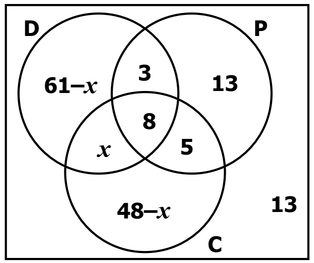
120 Gr 12 girls at Girls High where asked about their participation in the school's culture activities:

- 61 girls did drama (D)
- 29 girls did public speaking (P)
- 48 girls did choir (C)
- 8 girls did all three
- 11 girls did drama and public speaking
- 13 girls did public speaking and choir
- 13 girls did no culture activities

1. Draw a Venn diagram to represent this information.
2. Determine the number of Girls who participate in drama and choir.
3. Determine the probability that a grade 12 pupil chose at random will:
   a. only do choir.
   b. not do public speaking.
   c. participate in at least two of these activities.

Solutions:

1. **Gr 12**



2. $(61 - x) + 3 + 13 + x + 8 + 5 + (48 - x) + 13 = 120$

$-x = 120 - 151$

$\therefore x = 31$

$\therefore 61 - x = 30$ and $48 - x = 17$

3.

a. $P(C \text{ only}) = \frac{17}{120} = 0{,}14$

b. $P(P') = \frac{30 + 31 + 17 + 13}{120} = \frac{91}{120} = 0{,}76$

c. $P(\text{at least } 2) = \frac{3 + 30 + 8 + 5}{120} = \frac{46}{120} = 0{,}38$

Probability – Fundamental Counting Principles

Given different choices c, d and e

$$n(s) = c \times d \times e$$

EXAMPLE:

How many different outfits could you put together with 4 shirts, 6 skits and 2 pairs of shoes?

$n(s) = 4 \times 6 \times 2$

$= 48$ outfits

Arrangements **with repetition:**

$$n(s) = k^x$$

Where;
k = number of choices
x = number of times you can choose

EXAMPLE:

How many ways van the letters in 'ERIN' be arranged with repetition?

$n(s) = 4^4$

$= 256$

EXAMPLE:

How many three letter codes can be made from the letters d, g, h, m, r, and t, if the letters can be repeated?

$n(s) = 6^3$

$= 216$

Arrangements **without repetition:**

$n(s) = p!$ (factorial notation)

$= p \times (p-1) \times (p-2) \times (p-3) \times \ldots$

EXAMPLE:

How many ways can the letters in Erin be arranged without repetition?

$n(s) = 4!$

$= 4 \times 3 \times 2 \times 1$

$= 24$

Order of arrangement is important (**Permutation**):

$$n(s) = \frac{n!}{(n-r)!} \quad or \quad n(s) = nPr$$

where;
r = number of specific choices

EXAMPLE:

There are 7 players in a netball team who hope to be shooter or goal attack. How many different options are there?

$n(s) = \frac{7!}{(7-2)!}$

$= 42$

or

$n(s) = 7P2$

$= 42$

Use $[nPr]$ key on calculator:

[7][nPr][2] [=]

Identical items (repetition) in an arrangement:

$$n(s) = \frac{n!}{m! \times p!}$$

where;
m and p: number of times different items are repeated

EXAMPLE:

How many times can the letters in the name 'VANESSA' be arranged?

There are 2 A's and 2 S's:

$n(s) = \frac{7!}{2! \times 2!}$

$= 1260$

Arrangements and **Set Positions:**

$n(s)$ = number of positions $\times$ number of arrangements in each position

EXAMPLE:

How many ways can 5 Maths books, 2 Afrikaans books and 3 English books be arranged if they are grouped in their subjects?

Number of positions = 3
Number of arrangements for Maths books = 5!
Number of arrangements for Afrikaans books = 2!
Number of arrangements for English books = 3!

$n(s) = 3 \times (5! \times 2! \times 3!)$

$= 4\ 320$

EXAMPLE:

Questions:

A four-digit code can be made from four numbers 1 to 9 or 4 vowels.

1. How many possible codes can be made with repetition?
2. How many codes can be formed if the vowels cannot be repeated?
3. What is the probability of a code been created with no numbers been repeated?

Solutions:

1. $n(S) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9^4 + 5^4$
   $= 7\ 186$
2. $n(E) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9^4 + 5P4$
   $= 6\ 681$
3. $n(E_2) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9P4 + 5^4$
   $= 3\ 649$

   $\therefore P(E_2) = \frac{n(E_2)}{n(S)}$

   $= \frac{3649}{7186}$

   $= 0{,}51$

EXAMPLE:

Questions:

A password consists of 8 characters. The first two characters must be any consonant and may not be repeated. The third letter is a vowel. The next four characters form a four-digit number which must not start with 0 but digits can repeat. The last character is a vowel which must be different from the first vowel.
For example: **HG E 2558 A**

1. How many different passwords are possible?
2. What is the Probability the code will have an even number between the letters and end with an A?

Solutions:

1. 26 letters - 5 vowels = 21 consonants
   0;1;2;...9 is 10 numbers

   $n(S) = 21 \times 20 \times 9 \times 10^3 \times 4$
   $= 75\ 600\ 000$
2. $n(E) = 21 \times 20 \times 4 \times 9 \times 10^2 \times 5 \times 1$
   $= 7\ 560\ 000$

   $\therefore P(E) = \frac{n(S)}{n(E)}$

   $= \frac{7\ 560\ 000}{75\ 600\ 000}$

   $= 0{,}1$

PROBABILITY

EXAMPLE:

Questions:
Consider the word Matric-Vacation.
1. How many different ways can the letters be arranged? Treat all letters as different.
2. What is the probability that the M and V will always be next to each other? Treat all letters as different.
3. What is the probability that the 'word' will never start with cc? Treat all letters as different.
4. If the letters are not different, how many different ways can the letters be arranged?

Solutions:
1. $n(s) = 14!$
   $= 87\ 178\ 291\ 200$

2. MV or VM give 2!; MV can be treated as one letter group thus 13 letters
   $n(E_1) = 2! \times 13!$
   $= 12\ 454\ 041\ 600$
   $$\therefore P(E_1) = \frac{12\ 454\ 041\ 600}{87\ 178\ 291\ 200} = \frac{1}{7} = 0{,}143$$

3. $P(cc) = 2!$
   cc has to be a group and first, therefore 12 options remain following cc
   $$\text{P(not starting with cc)} = 1 - \text{P(starting with cc)} = 1 - \frac{2! \times 12!}{14!} = \frac{90}{91} = 0{,}989$$

4. Note that there are 3 A's, 2 T's, I's and C's
   $$n(S) = \frac{14!}{3! \times 2! \times 2! \times 2!} = 1\ 816\ 214\ 400$$

EXAMPLE:

Questions:
1. Events A and B are mutually exclusive. It is further given that:
   • 3P(B) = P(A), and
   • P(A or B) = 0,64;

   Calculate P(A).

2. A and B are independent events such that $P(A \cap B) = 0{,}27$ and $P(B) = 0{,}36$.

   Find P(A).

Solutions:
1. For mutually exclusive events;$\text{P(A)} + \text{P(B)} = \text{P(A or B)}$
   $$3P(B) = P(A)$$
   $$\therefore P(B) = \frac{P(A)}{3}$$
   $$P(A) + \frac{P(A)}{3} = 0{,}64$$
   $$3P(A) + P(A) = 1{,}92$$
   $$\therefore P(A) = 0{,}48$$

2. For independent events;
   $\text{(A)} \times \text{P(B)} = \text{P(A} \cap \text{B)}$
   $$P(A) \times P(B) = P(A \cap B)$$
   $$P(A) \times 0{,}36 = 0{,}27$$
   $$P(A) = 0{,}75$$

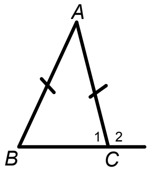
EXAMPLE:

Questions:
James has four R10, six R20, two R100 and three R200 notes in his wallet.
1. In how many different ways can the notes be arranged?
2. What is the probability that all the R200 notes are next to each other?

Solutions:
1. $$n(S) = \frac{15!}{4! \times 6! \times 2! \times 3!} = 6\ 306\ 300$$

2. $$n(\text{R200 next to each other}) = \frac{13!}{4! \times 6! \times 2!} = 180\ 180$$
   $$P(\text{R200 next to each other}) = \frac{180\ 180}{6\ 306\ 300} = \frac{1}{35} = 0{,}03$$

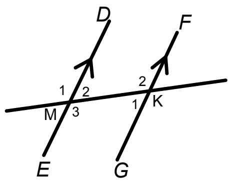
CIRCLE GEOMETRY



$\hat{B} = \hat{C}_1$ ($\angle$'s opp. = sides)

$\hat{A} + \hat{B} + \hat{C}_1 = 180°$ (sum $\angle$'s of Δ)

$\hat{C}_2 = \hat{A} + \hat{B}$ (ext. $\angle$'s of Δ)



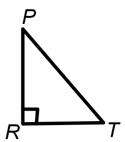
$\hat{K}_2 = \hat{M}_1$ (corres. $\angle$'s DE//GF)

$\hat{K}_2 = \hat{M}_3$ (alt. $\angle$'s DE//GF)

$\hat{K}_2 + \hat{M}_2 = 180°$ (co-int. $\angle$'s DE//GF)

$\hat{M}_1 = \hat{M}_3$ (vert. opp. $\angle$'s)

$\hat{K}_2 + \hat{K}_1 = 180°$ ($\angle$'s on a str. line)

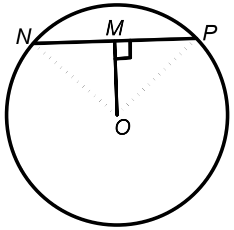


$PT^2 = PR^2 + RT^2$ (Pythag. Th.)

Theorem 1:

(line from centre $\perp$ chord)

A line drawn from the centre of a circle perpendicular to a chord bisects the chord.



GIVEN: Circle centre O with chord $NP \perp MO$.

RTP: $NM = MP$

PROOF:

Join ON and OP

In ΔMON and ΔMOP

$N\hat{M}O = P\hat{M}O$ (OM⊥PN, given)

$ON = OP$ (radii)

$OM = OM$ (common)

$\therefore \Delta MON = \Delta MOP$ (RHS)

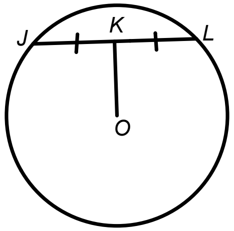
$NM = MP$

Converse of Theorem 1:

(line from centre mid-pt. chord)

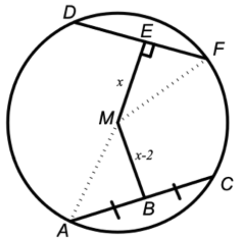
The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

If $JK = KL$, then $OK \perp JL$



EXAMPLE

Given circle centre M with a diameter of 20 cm and chord DF of 12 cm.



Determine the length of of chord AC.

Join MF

$DE = EF = 6$ cm (line from centre $\perp$ chord)

$MF = 10$ cm (radius)

$x^2 = 10^2 - 6^2$ (Pythag. Th.)

$x^2 = 64$

$x = 8$ cm

$\therefore MB = 8 - 3 = 5$ cm (given)

Join MA

$MA \perp AC$ (line from centre mid-pt. chord0

$MA = 10$ cm (radius)

$AB^2 = 10^2 - 5^2$ (Pythag. Th.)

$AB^2 = 75$

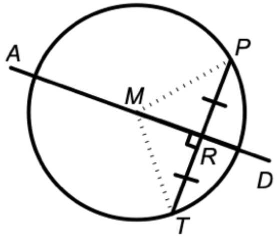
$AB = 8{,}66$ cm

$\therefore AC = 17{,}32$ cm

Converse two of Theorem 1:

(perp bisector of chord)

The perpendicular bisector of a chord passes through the centre of the circle.



GIVEN: $RT = RP$ and $MR \perp TP$

RTP: MR goes through the centre of the circle.

PROOF:

Choose any point, say M, on AD.

Join MT and MP

In ΔMRP and ΔMRT

$PR = RT$ (given)

$MR = MR$ (common)

$M\hat{R}P = M\hat{R}T = 90°$ ($\angle$'s on a str. line)

$\Delta MRT \equiv \Delta MRP$ (SAS)

$\therefore MT = MP$

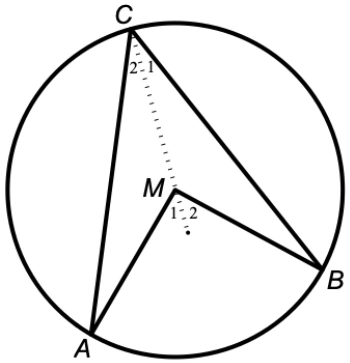
$\therefore$All points on AD are equidistant from P and T and the centre is equidistant from P and T.

$\therefore$The centre lies on AD.

CIRCLE GEOMETRY

Theorem 2:
(∠ at centre = 2 x ∠ at circum.)

The angle subtended by an arc at the centre of the circle is twice the angle the arc subtends at any point on the circumference of the circle.



GIVEN: Circle centre M with arc AB subtending $A\hat{M}B$ at the centre and $A\hat{C}B$ at the circumference.

RTP: $A\hat{M}B = 2 \times A\hat{C}B$

PROOF:

$AM = BM = CM$ (radii)

$\hat{A} = \hat{C}_2$ (∠ 's opp. = sides)

$\hat{B} = \hat{C}_1$ (∠ 's opp. = sides)

$\hat{M}_1 = \hat{A} + \hat{C}_2$ (ext. ∠ of Δ)

$\therefore \hat{M}_1 = 2\hat{C}_2$

$\hat{M}_2 = \hat{B} + \hat{C}_1$ (ext. ∠ of Δ)

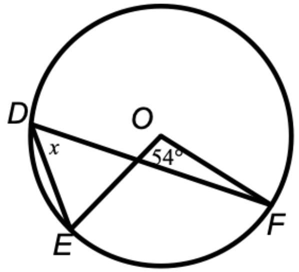
$\therefore \hat{M}_2 = 2\hat{C}_1$

$\therefore \hat{M}_1 + \hat{M}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{M}B = 2 \times A\hat{C}B$

EXAMPLE 1

Determine the value of x:

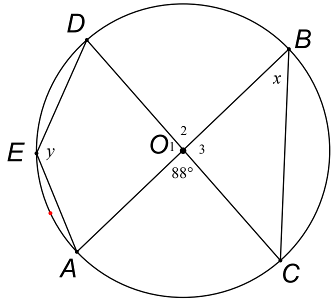


$x = 54° \div 2$ (∠ at centre = 2 x ∠ at circum.)

$\therefore x = 27°$

EXAMPLE 2

Determine the value(s) of x and y:



$x = 44°$ (∠ at centre = 2 x ∠ at circum.)

$OB = OC$ (radii)

$\hat{C} = 44°$ (∠ 's opp. = sides)

$\hat{O}_3 = 92°$ (sum ∠ 's of Δ)

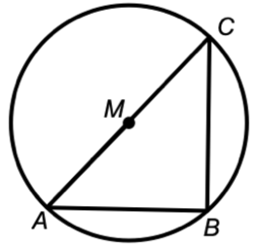
$\hat{O}_2 = 88°$ (vert. opp. ∠ 's)

$$y = \frac{88° + 92° + 88°}{2}$$

$y = 137{,}5°$ (∠ at centre = 2 x ∠ at circum.)

Theorem 3:
(∠ in semi-circle)

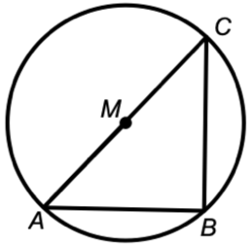
The angle subtended by the diameter at the circumference of a circle is a right angle.



If AMC is the diameter then $\hat{B} = 90°$.

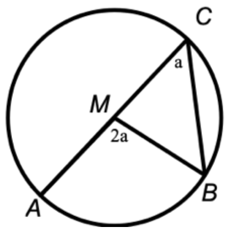
Converse Theorem 3:
(chord subtends 90°)

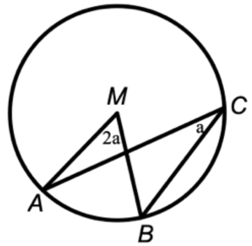
If a chord subtends an angle of 90° at the circumference of a circle, then that chord is a diameter of the circle.

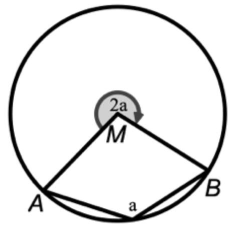


If $\hat{B} = 90°$ then AMC is the diameter.

ALTERNATIVE DIAGRAMS:

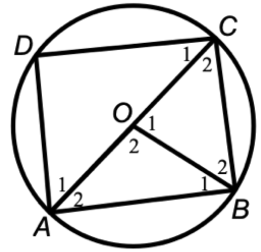






EXAMPLE

In circle O with diameter AC, $DC = AD$ and $\hat{B}_2 = 56°$. Determine the size of $D\hat{A}B$



$CO = OB$ (radii)

$\hat{C}_2 = \hat{B}_2 = 56°$ (∠ 's opp. = sides)

$\hat{O}_1 = 68°$ (sum ∠ 's of Δ)

$\hat{A}_2 = 34°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{D} = 90°$ (∠ in semi-circle)

$\hat{A}_1 = \hat{C}_1$ (∠'s opp. = sides, DC = AD)

$\hat{A}_1 = 45°$ (sum ∠ 's of Δ)

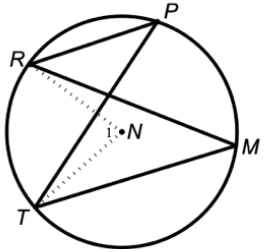
$\therefore D\hat{A}B = 34° + 45° = 79°$

EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

Theorem 4:
(∠ in same seg.)

Angles subtended by a chord (or arc) at the circumference of a circle on the same side of the chord are equal.



GIVEN: Circle centre N with arc RT subtending $R\hat{P}T$ and $R\hat{M}T$ in the same segment.

RTP: $R\hat{P}T = R\hat{M}T$

PROOF:
Join NR and NT to form $\hat{N}_1$.

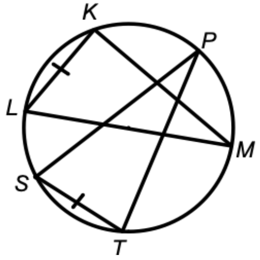
$\hat{M} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\hat{P} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\therefore R\hat{M}T = R\hat{P}T$

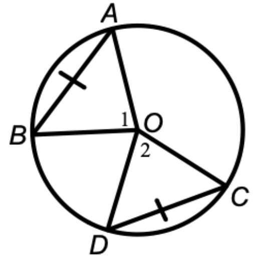
COROLLARIES:

a) Equal chords (or arcs) subtend equal angles at the circumference.



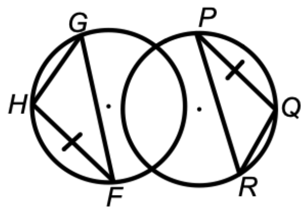
$KL = ST$ then $\hat{P} = \hat{M}$ (= chords, = ∠'s)

b) Equal chords subtend equal angles at centre of the circle.



If $AB = CD$ then $\hat{O}_1 = \hat{O}_2$ (= chords, = ∠'s)

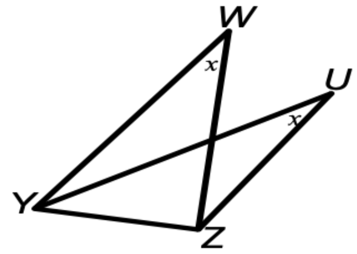
c) Equal chords in equal circles subtend equal angles at their circumference.



If $HF = PQ$ then $\hat{G} = \hat{R}$ (= chords, = ∠'s)

Converse Theorem 4:
(line subt. = ∠'s)

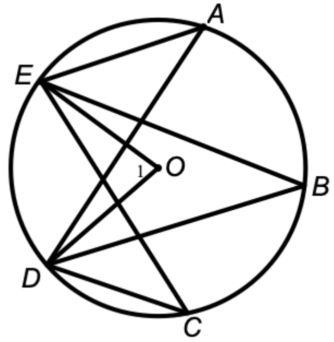
If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (that is, they lie on the circumference of a circle.)



If $\hat{W} = \hat{U}$, then $WUZY$ is a cyclic quadrilateral.

EXAMPLE 1

Given circle centre O with $\hat{C} = 36°$



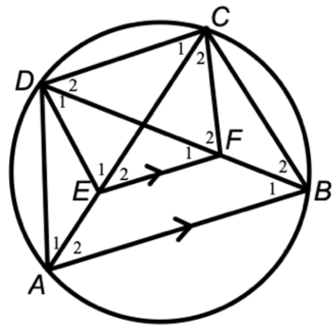
Calculate the values of angles:
$\hat{O}_1$, $\hat{A}$ and $\hat{B}$.

$\hat{O}_1 = 2 \times 36° = 72°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{A} = \hat{B} = \hat{C} = 36°$ (∠'s same seg.)

EXAMPLE 2

Given circle $ABCD$ with $AB || EF$.



Questions:
a) Prove $CDEF$ is a cylindrical quad.
b) If $\hat{D}_2 = 38°$, calculate $\hat{E}_2$

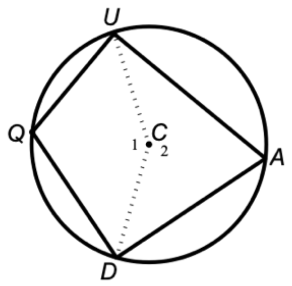
Solutions:
a) $\hat{B}_1 = \hat{C}_1$ (∠'s same seg.)
$\hat{B}_1 = \hat{F}_1$ (corres. ∠'s, AB||EF)
$\therefore \hat{C}_1 = \hat{F}_1$
$\therefore CDEF$ cyc. quad (line subt = ∠'s)

b) $\hat{D}_2 = \hat{E}_2 = 38°$ (∠'s same seg quad CDEF)

Theorem 5:

(opp. ∠'s cyc. quad)

The opposite angles of a cyclic quadrilateral are supplementary.



GIVEN: Circle centre C with quad $QUAD$.

RTP: $\hat{Q} + \hat{A} = 180°$

PROOF:

Join UC and DC

$\hat{C}_1 = 2\hat{A}$ (∠ at centre = 2 x ∠ at circum.)

$\hat{C}_2 = 2\hat{Q}$ (∠ at centre = 2 x ∠ at circum.)

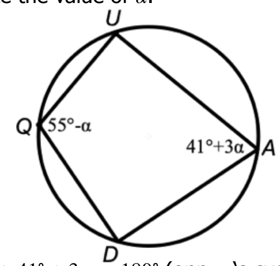
$\hat{C}_1 + \hat{C}_2 = 360°$ (∠'s around a pt.)

$\therefore 2\hat{A} + 2\hat{Q} = 360°$

$\therefore \hat{A} + \hat{Q} = 180°$

EXAMPLE 1

Calculate the value of α.



$55° - \alpha + 41° + 3\alpha = 180°$ (opp. ∠'s cyc. quad)

$2\alpha = 180° - 96°$

$2\alpha = 84°$

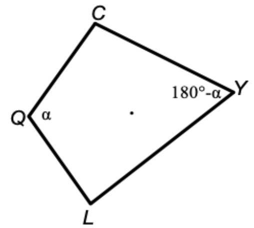
$\therefore \alpha = 42°$

Converse Theorem 5:

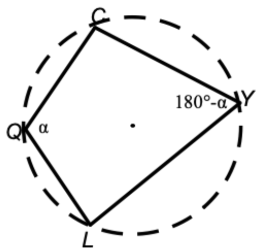
(opp. ∠'s quad supp)

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

If $\hat{Q} + \hat{Y} = 180°$
or $\hat{C} + \hat{L} = 180°$

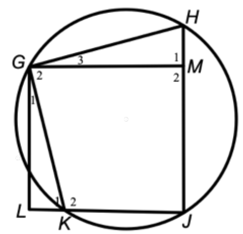


Then $QCYL$ is cyclic



EXAMPLE 2

Given circle $GHJK$ with $GM \perp HJ$ and $GL \perp LJ$. $\hat{G}_3 = 24°$



a) Is quadrilateral $GLJM$ a cyclic quad?

b) Is quadrilateral $GLJH$ a cyclic quad?

a) $\hat{M}_2 = 90°$ (Given $GM \perp HJ$)

$\hat{L} = 90°$ (Given $GL \perp LJ$)

$\therefore GLJM$ cyc quad (opp ∠'s quad suppl)

b) $\hat{H} = 180° - 24° - 90°$ (sum ∠'s of Δ)

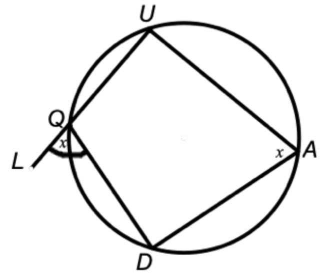
$\hat{H} = 66°$

$GLJH$ not cyclic (opp ∠'s = 156° not 180°)

Theorem 6:

(ext. ∠ cyc quad)

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

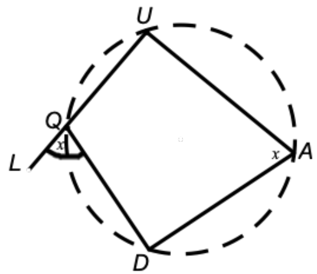


$L\hat{Q}D = \hat{A}$ (ext. ∠ cyc quad)

Converse Theorem 6:

(ext. ∠ = int. opp. ∠)

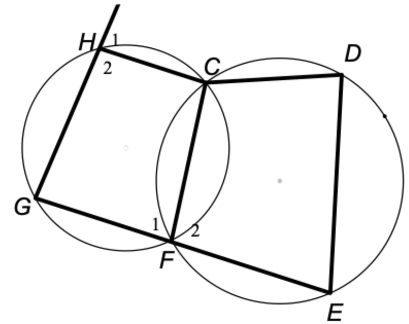
If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.



If $L\hat{Q}D = \hat{A}$ then $QUAD$ is cyclic

EXAMPLE 1

GFE is a double chord and $\hat{H}_1 = 75°$



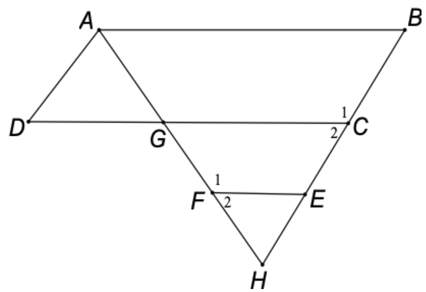
Determine the value of $\hat{D}$.

$\hat{H}_1 = \hat{F}_1 = 75°$ (ext. ∠ cyc quad)

$\hat{F}_1 = \hat{D} = 75°$ (ext. ∠ cyc quad)

EXAMPLE 2

$ABCD$ is a parallelogram and $B\hat{A}D = \hat{F}_1$.
Prove that $CEFG$ is a cyclic quad.



$B\hat{A}D = \hat{C}_1$ (opp. ∠'s parm)

$B\hat{A}D = \hat{F}_1$ (given)

$\therefore \hat{C}_1 = \hat{F}_1$

$\therefore CEFG$ is a cyc quad (ext. ∠ = int. opp. ∠)

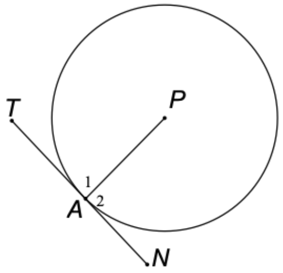
EUCLIDEAN GEOMETRY

Grade 11 Recap

CIRCLE GEOMETRY

Theorem 7:
(tan ⊥ radius)

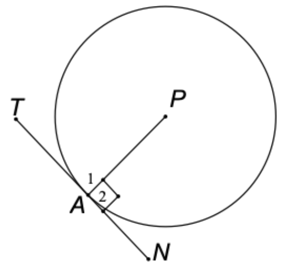
A tangent to a circle is perpendicular to the radius at its point of contact.



If TAN is a tangent to circle P, then $PA \perp TAN$

Converse Theorem 7:
(line seg ⊥ radius)

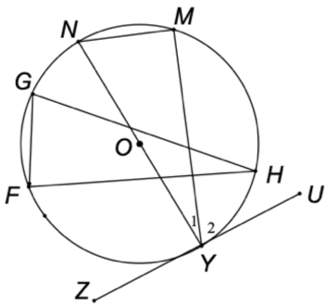
A line drawn perpendicular to the radius at the point where the radius meets the circumference is a tangent to the circle.



If $PA \perp TAN$, then TAN is a tangent to circle P.

EXAMPLE 1

Given circle centre O with tangent ZYU and $MN = FG$. If $\hat{H} = 18°$ determine the size of $\hat{Y}_2$.



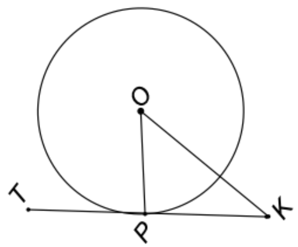
$\hat{Y}_1 = \hat{H} = 18°$ (equal chords, = ∠'s)

$\hat{Y}_1 + \hat{Y}_2 = 90°$ (tan ⊥ radius)

$\therefore \hat{Y}_2 = 90° - 18° = 72°$

EXAMPLE 2

Prove that TPK is a tangent to circle centre O and radius of 8 cm, if $OK = 17$ cm and $PK = 15$ cm.



$OK^2 = 17^2 = 289$

$OP^2 + PK^2 = 8^2 + 15^2$
$= 289$

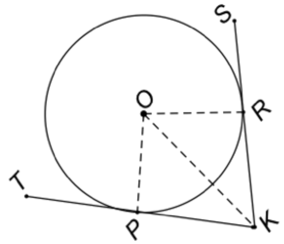
$\therefore OK^2 = OP^2 + PK^2$

$\therefore OP \perp TPK$ (conv. Pythag. Th.)

$\therefore TPK$ is a tan to circle O (line seg ⊥ radius)

Theorem 8:
(tan from same pt.)

Two tangents drawn to a circle from the same point outside the circle are equal in length.



GIVEN: Tangents TPK and SRK to circle centre O.

RTP: $PK = RK$

PROOF:

Construct radii OR and OP and join OK.

In ΔOPK and ΔORK

$OP = OR$ (radii)

$OK = OK$ (common)

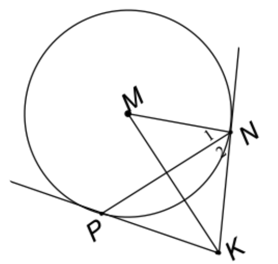
$O\hat{P}K = O\hat{R}K = 90°$ (tan ⊥ radius)

$\therefore \Delta OPK \equiv \Delta ORK$ (RHS)

$\therefore PK = RK$

EXAMPLE

PK and KN are tangents to circle centre M. If $\hat{N}_1 = 24°$, determine the size of $P\hat{K}N$.



$M\hat{N}K = 90°$ (tan ⊥ radius)

$\therefore \hat{N}_2 = 66°$

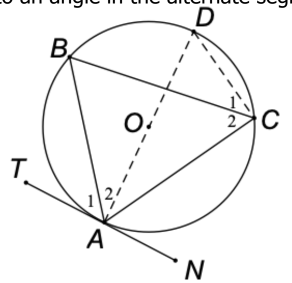
$PK = NK$ (tan from same pt.)

$\hat{N}_2 = N\hat{P}K = 66°$ (∠'s opp. = sides)

$\therefore P\hat{K}N = 48°$ (sum ∠'s of Δ)

Theorem 9:
(tan-chord th.)

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.



GIVEN: Tangent TAN to circle O, and chord AC subtending $\hat{B}$.

RTP: $\hat{A}_1 = \hat{C}_2$

PROOF:

Draw in diameter AOD an join DC.

$\hat{A}_1 + \hat{A}_2 = 90°$ (tan ⊥ radius)

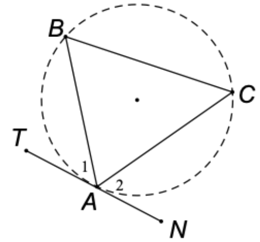
$\hat{C}_1 + \hat{C}_2 = 90°$ (∠ in semi-circle)

$\hat{A}_2 = \hat{C}_1$ (∠'s in same seg)

$\therefore \hat{A}_1 = \hat{C}_2$

Converse Theorem 9:
(∠ betw. line and chord)

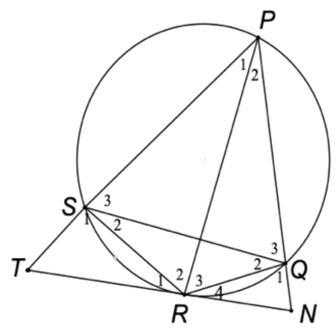
If a line is drawn through the end point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.



If $\hat{A} = \hat{C}$ or $\hat{A}_2 = \hat{B}$, TAN a tangent

EXAMPLE 1

TRN is a tangent at R and $SR = RQ$. If $\hat{R}_1 = x$, find five angles equal to x.



$\hat{R}_1 = \hat{P}_1 = x$ (tan-chord th.)

$\hat{Q}_2 = x$ (tan-chord or ∠'s in same seg)

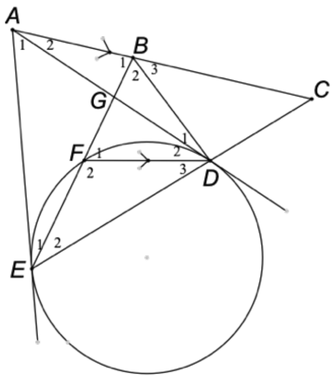
$\hat{Q}_2 = \hat{S}_2 = x$ (∠'s opp. = sides)

$\hat{S}_2 = \hat{P}_2 = x$ (∠'s same seg)

$\hat{P}_2 = \hat{R}_4 = x$ (tan-chord th.)

EXAMPLE 2

In the figure, AD and AE are tangents to the circle DEF. The straight line drawn through A, parallel to FD meets ED produced at C and EF produced at B. The tangent AD cuts EB at G.



a) Prove that $ABDE$ is a cyclic quadrilateral given $\hat{E}_2 = x$.

b) If it is further given that $EF = DF$, prove that ABC is a tangent to the circle passing through the points B, F and D.

a) $\hat{E}_2 = \hat{D}_2 = x$ (tan-chord th.)

$\hat{D}_2 = \hat{A}_2 = x$ (alt ∠'s AB||FD)

$\therefore ABDE$ a cyc quad (line seg subt. = ∠'s)

b) $\hat{E}_2 = \hat{D}_3 = x$ (∠'s opp. = sides)

$\hat{F}_1 = \hat{E}_2 + \hat{D}_3 = 2x$ (ext. ∠ of Δ)

$AE = AD$ (tan from same pt.)

$\hat{E}_1 + \hat{E}_2 = \hat{D}_2 + \hat{D}_3 = 2x$ (∠'s opp. = sides)

$\therefore \hat{B}_3 = 2x$ (ext. ∠ cyc quad)

$\hat{B}_3 = \hat{F}_1$

$\therefore ABC$ tan to circle (∠ betw. line and chord)

ALTERNATIVE

$\hat{F}_1 = \hat{B}_1$ (alt ∠'s AB||FD)

$\hat{B}_1 = \hat{D}_2 + \hat{D}_3$ (∠'s same seg)

$\hat{D}_1 = \hat{E}_1$ (∠'s same seg)

$\hat{E}_1 = \hat{D}_3$ (tan-chord th.)

$\therefore \hat{B}_1 = \hat{D}_2 + \hat{D}_1$

$\therefore ABC$ tan to circle (∠ betw. line and chord)

Hints when answering Geometry Questions

- Read the given information and mark on to the diagram if not already done.
- Never assume anything. If not given or marked on diagram is not true unless proved.
- As you prove angles equal or calculate angles mark them on to the diagram and write down statement and reason there and then.
- Make sure that by the end of the question you have used all the given information.
- If asked to prove something, it is true.
  For EXAMPLE if asked to prove ABCD a cyclic quad, then it is; but if you can't then you can use it as one in the next part of the question.

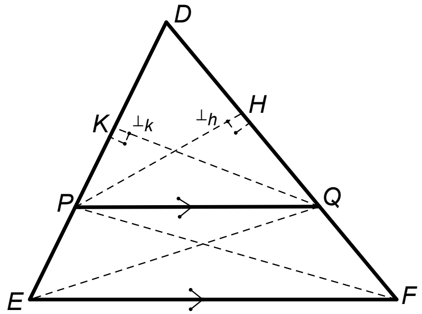
Note for Matric:
All Grade 11 Theorems and their required proofs are also examinable in Grade 12

EUCLIDEAN GEOMETRY

Theorem 1:

(Proportion Theorem)

A line drawn parallel to one side of a triangle cuts the other two sides in proportion. (Proportion Th. or side 1 ∥ side 2 or line ∥ one side $\triangle$)



GIVEN: $\triangle DEF$, with P on DE and Q on DF and $PQ \parallel EF$

RTP: $\frac{DP}{PE} = \frac{DQ}{QF}$

PROOF:

Construct PF and EQ and draw in altitudes (perpendicular heights)

QK (⊥ k) and PH (⊥ h)

Area of triangle: $= \frac{1}{2} \times \text{base} \times \perp \text{h}$

Area $\triangle DPQ = \frac{1}{2} \times DQ \times \perp \text{h}$

Area $\triangle FPQ = \frac{1}{2} \times FQ \times \perp \text{h}$

$\therefore \quad \frac{\text{Area} \triangle DPQ}{\text{Area} \triangle FPQ} = \frac{\frac{1}{2} \times DQ \times \perp \text{h}}{\frac{1}{2} \times FQ \times \perp \text{h}} = \frac{DQ}{FQ}$

Similarly: $\frac{\text{Area} \triangle DPQ}{\text{Area} \triangle EQP} = \frac{\frac{1}{2} \times DP \times \perp \text{k}}{\frac{1}{2} \times PE \times \perp \text{k}} = \frac{DP}{PE}$

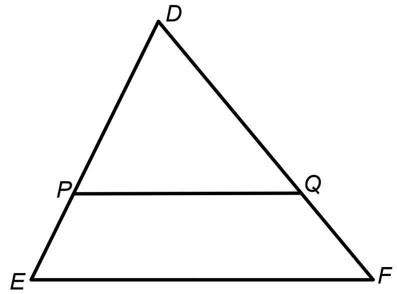
But $\text{Area} \triangle FPQ = \text{Area} \triangle EQP$

(Same base PQ and between same ∥ lines)

$\therefore \quad \frac{DQ}{FQ} = \frac{DP}{PE}$

Converse of Theorem 1:

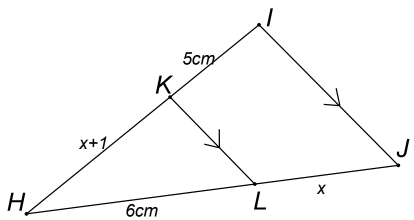
If a line divides two sides of a triangle in proportion, then the line is parallel to the third side of the triangle.
(line divides 2 sides $\triangle$ in prop. or conv. Prop. Th.)



Given $\frac{DP}{PE} = \frac{DQ}{QF}$, therefore $PQ \parallel EF$.

EXAMPLE 1

Calculate x



$KL \parallel IJ$ (given)

$\therefore \frac{HK}{KI} = \frac{HL}{LJ}$ (proportion Th., $KL \parallel IJ$ or line ∥ one side Δ)

$\therefore \quad \frac{x+1}{5} = \frac{6}{x}$

$x(x+1) = 30$

$x^2 + x - 30 = 0$

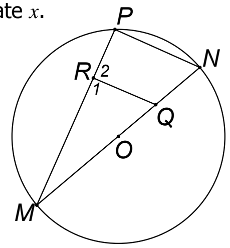
$(x+6)(x-5) = 0$

$\therefore \ x \neq -6$ or $x = 5$

EXAMPLE 2

In circle O, $RQ \perp PM$, $OQ = QN$ and $x = \frac{MR}{MP}$.

Calculate x.



$\hat{P} = 90°$ (∠ in semi-circle)

$\therefore \ \hat{P} = \hat{R}_1 = 90°$ (given $RQ \perp PM$)

$\therefore \ RQ \parallel PN$ (corresp. ∠'s equal)

$\therefore \ \frac{MR}{MP} = \frac{MQ}{MN}$ (line ∥ one side $\triangle$)

But $MO = ON = r$ (radii)

$OQ = QN = \frac{1}{2}r$ (given)

$\therefore \ \frac{MQ}{MN} = \frac{r + \frac{1}{2}r}{r + r}$

$= \frac{\frac{3}{2}r}{2r}$

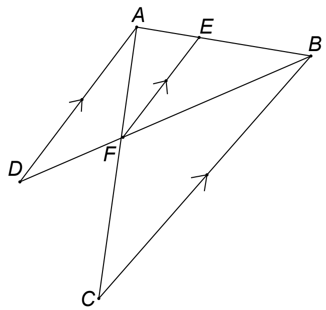
$= \frac{3}{2} \times \frac{1}{2}$

$= \frac{3}{4}$

$\therefore \ x = \frac{3}{4}$

EXAMPLE 3

In the diagram below $AD \parallel EF \parallel BC$. Prove that $AF \cdot BD = AC \cdot DF$.



In $\triangle ABD$: $\frac{AB}{AE} = \frac{BD}{DF}$ (Proportion Th. $AD \parallel EF$)

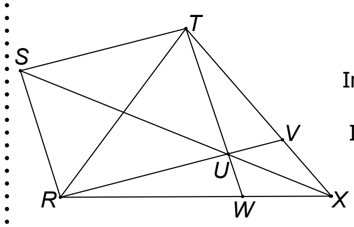
In $\triangle ABC$: $\frac{AB}{AE} = \frac{AC}{AF}$ (Proportion Th. $ST \parallel RV$)

$\therefore \ \frac{BD}{DF} = \frac{AC}{AF}$ (both $= \frac{AB}{AE}$)

$\therefore BD \cdot AF = AC \cdot DF$

EXAMPLE 4

$STUR$ is a parallelogram, with SUX, TUW and RUV straight lines. Prove $RT \parallel VW$.



$ST \parallel RU$ and $SR \parallel TU$ (opp. sides parm)

In $\triangle SRX$: $\frac{SU}{UX} = \frac{RW}{WX}$ (Proportion Th. $SR \parallel TW$)

In $\triangle STX$: $\frac{SU}{UX} = \frac{TV}{VX}$ (Proportion Th. $ST \parallel RV$)

$\therefore \ \frac{RW}{WX} = \frac{TV}{VX}$ (both $= \frac{SU}{UX}$)

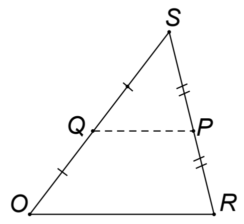
$\therefore RT \parallel VW$ (line divides 2 sides $\triangle$ in prop.)

EUCLIDEAN GEOMETRY

Mid-Point Theorem:

(mid-pt. Th.)

The line segment joining the mid-points of two sides of a triangle, is parallel to the third side and half the length of the third side.

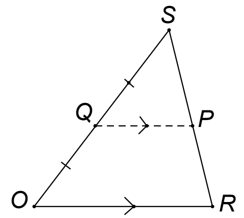


Therefore if $SQ = QO$ and $SP = PR$ then $PQ \parallel OR$ and $QP = \frac{1}{2}OR$ (mid-pt. Th.)

Converse:

(conv. mid-pt. Th.)

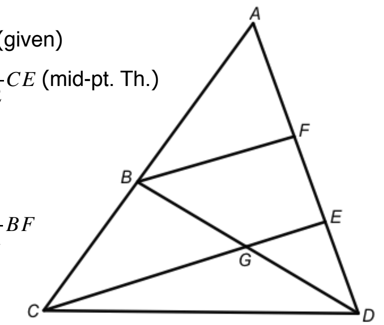
The line passing through the mid-point of one side of a triangle and parallel to another side, bisects the third side. The line is also equal to half the length of the side it is parallel to.



Therefore if $SQ = QO$ and $PQ \parallel OR$ then $SP = PR$ and $QP = \frac{1}{2}OR$ (conv. mid-pt. Th.)

EXAMPLE

In $\triangle ACE$, $AB = BC$, $GE = 15$ cm and $AF = FE = ED$.
Determine the length of CE.



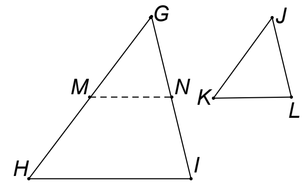
In $\triangle ACE$
$AB = BC$ and $AF = FE$ (given)
$\therefore\ BF \parallel CE$ and $BF = \frac{1}{2}CE$ (mid-pt. Th.)

In $\triangle DFB$
$FE = ED$ (given)
$BF \parallel GE$ (proven)
$\therefore\ BG = GD$ and $GE = \frac{1}{2}BF$
(conv. mid-pt. Th.)
$\therefore BF = 2GE$
$\therefore BF = 2(15) = 30$ cm
$CE = 2BF$ (proven)
$\therefore\ CE = 2(30) = 60$ cm

Theorem 2: Similarity Theorem

(AAA)

If the corresponding angles in two triangles are equal, then the corresponding sides are in proportion (AAA)



GIVEN: $\triangle GHI$ and $\triangle JKL$, $\hat{G} = \hat{J}$; $\hat{H} = \hat{K}$ and $\hat{I} = \hat{L}$

RTP: $\frac{GH}{JK} = \frac{GI}{JL} = \frac{HI}{KL}$

PROOF:

Construct M on GH such that $GM = JK$ and N on GI such that $GN = JL$
Join MN
In $\triangle GMN$ and $\triangle JKL$
$\hat{G} = \hat{J}$ (given)
$GM = JK$ (construction)
$GN = JL$ (construction)
$\therefore\ \triangle GMN \equiv \triangle JKL$ (SAS)
$\therefore G\hat{M}N = \hat{K}$ and $G\hat{N}M = \hat{L}$ ($\equiv \triangle$'s)
BUT $\hat{H} = \hat{K}$ and $\hat{I} = \hat{L}$

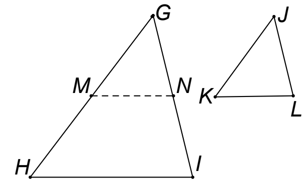
$\therefore \hat{H} = G\hat{M}N$ and $\hat{I} = G\hat{N}M$
$\therefore MN \parallel HI$ (corresp. $\angle$'s equal)
$\therefore \frac{GH}{GM} = \frac{GI}{GN}$ (proportion Th., $MN \parallel HI$)
$\therefore \frac{GH}{JK} = \frac{GI}{JL}$

Similarly, $\therefore \frac{GH}{JK} = \frac{HI}{KL}$ (by constructing P on HI such that $HP = KL$)
$\therefore \frac{GH}{JK} = \frac{GI}{JL} = \frac{HI}{KL}$

Converse Theorem

(sides in prop.)

If the corresponding sides of two triangles are in proportion, then the corresponding angles are in equal (sides in prop.)



GIVEN: $\therefore \frac{GH}{JK} = \frac{GI}{JL} = \frac{HI}{KL}$

RTP: $\hat{G} = \hat{J}$; $\hat{H} = \hat{K}$ and $\hat{I} = \hat{L}$

PROOF:

Construct M on GH such that $GM = JK$ and N on GI such that $GN = JL$
Join MN
$\frac{GH}{JK} = \frac{GI}{JL}$ (given)
$\therefore \frac{GH}{GM} = \frac{GI}{GN}$ (construction $GM = JK$ and $GN = JL$)
$\therefore MN \parallel HI$ (line divides 2 sides $\triangle$ in prop.)
$\therefore \hat{H} = G\hat{M}N$ and $\hat{I} = G\hat{N}M$
(corresp. $\angle$'s $MN \parallel HI$)

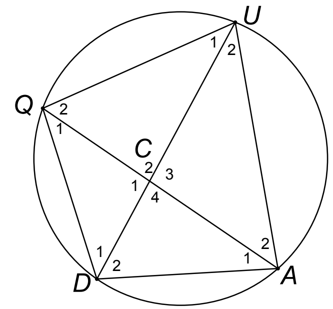
$\therefore\ \triangle GHI \mid\mid\mid \triangle GMN$ (AAA)
$\therefore \frac{HI}{MN} = \frac{GH}{GM}$
$= \frac{GH}{JK}$ (construction)
$= \frac{HI}{KL}$ (given)
$\therefore\ MN = KL$
$\therefore\ \triangle GMN \equiv \triangle JKL$ (SSS)
$\therefore \hat{G} = \hat{J}$; $\hat{M} = \hat{K}$ and $\hat{N} = \hat{L}$
$\therefore \hat{G} = \hat{J}$; $\hat{H} = \hat{K}$ and $\hat{I} = \hat{L}$

EXAMPLE 1

Given $QU = 24$ cm, $QC = 16$ cm, $DA = 6$ cm and $CD = x$.

Questions:

1. Prove $\triangle QUC \mid\mid\mid \triangle ACD$
2. Calculate x



Solutions:

1. In $\triangle QUC$ and $\triangle ACD$
$\hat{C}_2 = \hat{C}_4$ (vert. opp. $\angle$'s)
$\hat{Q}_2 = \hat{D}_2$ ($\angle$'s in same seg.)
$\hat{U}_1 = \hat{A}_1$ (sum $\angle$'s of $\triangle$ or $\angle$'s in same seg.)
$\therefore \triangle CQU \mid\mid\mid \triangle CDA$ (AAA)
$\therefore\ \frac{CQ}{CD} = \frac{QU}{DA} = \frac{CU}{CA}$ ($\mid\mid\mid \triangle$'s)

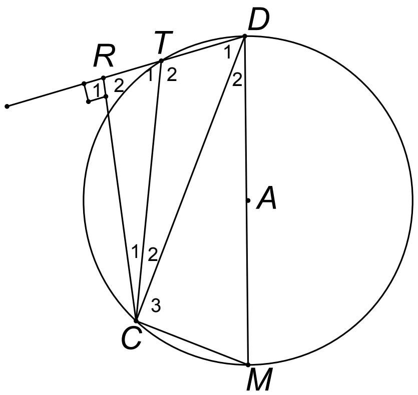
2. $\frac{CQ}{CD} = \frac{QU}{DA}$
$\frac{16}{x} = \frac{24}{6}$
$96 = 24x$
$\therefore x = 4$ cm

EXAMPLE 2

DM is a diameter of circle A. $RTD \perp RC$, $RT = 3$ cm, $RC = 4$ cm, $MC = 6$ cm and $DM = 2x$.

Questions:

1. Prove $\triangle CRT \mid\mid\mid \triangle CDM$
2. Calculate x



Solutions:

1. $\hat{C}_3 = 90°$ ($\angle$ in semi-circle)
   In $\triangle CRT$ and $\triangle CDM$
   $\hat{R}_2 = \hat{C}_3 = 90°$ (given and proven)
   $\hat{T}_1 = \hat{M}$ (ext. $\angle$ of cyc. quad)
   $\hat{C}_1 = \hat{D}_2$ (sum $\angle$'s of $\triangle$)
   $\therefore \triangle RTC \mid\mid\mid \triangle CMD$ (AAA)
   $\therefore \dfrac{RT}{CM} = \dfrac{TC}{MD} = \dfrac{RC}{CD}$ ($\mid\mid\mid \triangle$'s)

2. $TC^2 = 3^2 + 4^2$ (Pythag.)
   $\therefore TC = 5$ cm
   $\dfrac{RT}{CM} = \dfrac{TC}{MD}$
   $\dfrac{3}{6} = \dfrac{5}{2x}$
   $6x = 30$
   $\therefore x = 5$ cm

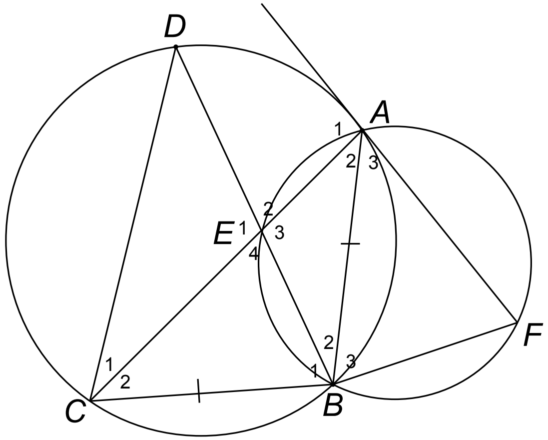
EXAMPLE 3

A, B, C and D are points on the circumference of a circle such that $AB = BC$. AF is a tangent to the circle through $ABCD$ at A.

Questions:

Prove:

1. $\triangle BCD \mid\mid\mid \triangle BEC$
2. $B\hat{C}D = B\hat{F}A$
3. $\triangle BCD \mid\mid\mid \triangle BAF$
4. $BD \cdot AF = BC \cdot CD$



Solutions:

1. In $\triangle BCD$ and $\triangle BEC$
   i. $\hat{B}_1 = \hat{B}_1$ (common)
   ii. $\hat{D} = \hat{A}_2$ ($\angle$'s same seg.)
   but $\hat{A}_2 = \hat{C}_2$ ($\angle$'s opp. = sides, $AB = BC$)
   $\therefore \hat{D} = \hat{C}_2$
   iii. $B\hat{C}D = \hat{E}_4$ (sum $\angle$'s of $\triangle$)
   $\therefore \triangle RTC \mid\mid\mid \triangle CMD$ (AAA)

2. $\hat{F} = \hat{E}_4$ (ext. $\angle$'s of cyc. quad)
   $B\hat{C}D = \hat{E}_4$ (proven)
   $\therefore B\hat{C}D = B\hat{F}A$

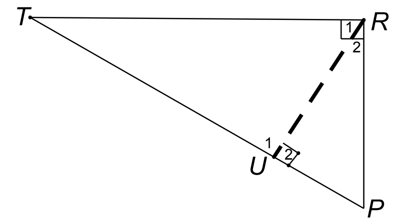
3. In $\triangle BCD$ and $\triangle BFA$
   i. $B\hat{C}D = B\hat{F}A$ (proven)
   ii. $\hat{A}_3 = \hat{C}_2$ (tan-chord th.)
   but $\hat{D} = \hat{C}_2$ (proven)
   $\therefore \hat{D} = \hat{A}_3$
   iii. $\hat{B}_1 = \hat{B}_3$ (sum $\angle$'s of $\triangle$)
   $\therefore \triangle CDB \mid\mid\mid \triangle FAB$ (AAA)

4. $\dfrac{CD}{FA} = \dfrac{DB}{AB} = \dfrac{CB}{FB}$ ($\mid\mid\mid \triangle$'s)
   $\therefore \dfrac{CD}{FA} = \dfrac{DB}{AB}$
   but $CD \cdot AB = BC$ (given)
   $\therefore CD \cdot BC = DB \cdot FA$

Theorem 3: Similar Right Angled Triangles

($\perp$ from rt $\angle$ vert. to hyp.)

The perpendicular drawn from the vertex of the right angle of a right angled triangle to the hypotenuse divides the triangle into two similar right angled triangles, which are similar to the original triangle. ($\perp$ from rt $\angle$ vert. to hyp.)



GIVEN: $\triangle PRT$ with $\hat{R} = 90°$ and $RU \perp TP$.

RTP: $\triangle PRT \mid\mid\mid \triangle PUR \mid\mid\mid \triangle TUR$

PROOF:

$\hat{T} + \hat{R}_1 = 90°$ (sum $\angle$'s of $\triangle$)
$\hat{R}_1 = \hat{R}_2 = 90°$ (given)
$\therefore \hat{T} = \hat{R}_2$
$\therefore \hat{P} = \hat{R}_1$ (sum $\angle$'s of $\triangle$)

In $\triangle PUR$ and $\triangle TUR$
$\hat{U}_1 = \hat{U}_2 = 90°$
$\hat{R}_2 = \hat{T}$ (proven)
$\hat{P} = \hat{R}_1$ (proven)
$\therefore \triangle PUR \mid\mid\mid \triangle RUT$ (AAA)

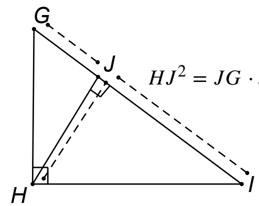
In $\triangle PRT$ and $\triangle PUR$
$P\hat{R}T = \hat{U}_2 = 90°$
$\hat{P} = \hat{P}$ (common)
$\hat{T} = \hat{R}_2$ (proven)
$\therefore \triangle PRT \mid\mid\mid \triangle PUR$ (AAA)
$\therefore \triangle PRT \mid\mid\mid \triangle PUR \mid\mid\mid \triangle RUT$

$\therefore \dfrac{PR}{PU} = \dfrac{PT}{PR} \quad \therefore PR^2 = PT \cdot PU$

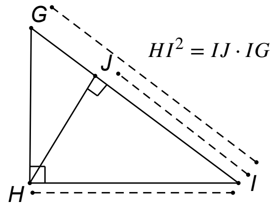
$\therefore \dfrac{UR}{UT} = \dfrac{PU}{UR} \quad \therefore UR^2 = PU \cdot UT$

$\therefore \dfrac{RT}{UT} = \dfrac{PT}{RT} \quad \therefore RT^2 = PT \cdot UT$

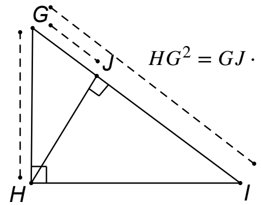
NOTE:



$HJ^2 = JG \cdot JI$



$HI^2 = IJ \cdot IG$

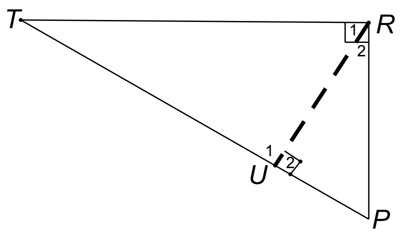


$HG^2 = GJ \cdot GI$

Theorem 4: Theorem of Pythagoras

(Pythag.)

In a right angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas on the other two sides. (Pythag.)



GIVEN: $\triangle PRT$ with $\hat{R} = 90°$.

RTP: $TP^2 = PR^2 + TR^2$

PROOF:

Construct $RU \perp TP$

$\triangle PRT \,|||\, \triangle PUR$ ($\perp$ from rt $\angle$ vert. to hyp.)

$\therefore \dfrac{PR}{PU} = \dfrac{PT}{PR} \quad \therefore PR^2 = PT \cdot PU$

$\triangle PRT \,|||\, \triangle TUR$ ($\perp$ from rt $\angle$ vert. to hyp.)

$\therefore \dfrac{RT}{UT} = \dfrac{PT}{RT} \quad \therefore RT^2 = PT \cdot UT$

$\therefore PR^2 + RT^2 = PT \cdot PU + PT \cdot UT$

$= PT(PU + UT)$

$= PT(PT)$

$\therefore TP^2 = PR^2 + TR^2$

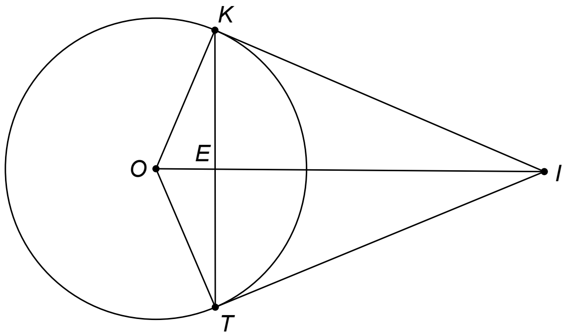
EXAMPLE 1

O is the centre of circle with tangents KI and IT. OEI is a straight line.

Questions:

If EI is 15 cm and IT is 17 cm, calculate:

1. ET
2. OE
3. TO



Solutions:

1. $KI = IT$ (tan from same pt.)

$OK = OT$ (radii)

$\therefore KITO$ is a kite (both pairs adj. sides =)

$\therefore O\hat{E}T = 90°$ (diag. kite $\perp$)

$\therefore ET^2 = 17^2 - 15^2$ (Pythag)

$\therefore ET = 8$ cm

2. $O\hat{T}I = 90°$ (tan$\perp$rad)

$\therefore \triangle TOE \,|||\, \triangle ITE$ ($\perp$ from rt $\angle$ vert. to hyp.)

$\therefore ET^2 = EO \cdot EI$

$8^2 = OE \cdot 15$

$\therefore OE = 4{,}27$ cm

3. $\triangle TOE \,|||\, \triangle IOT$ ($\perp$ from rt $\angle$ vert. to hyp.)

$TO^2 = OE \cdot OI$

$TO^2 = 4{,}27 \cdot (4{,}27 + 15)$

$\therefore TO = 9{,}07$ cm

EXAMPLE 2

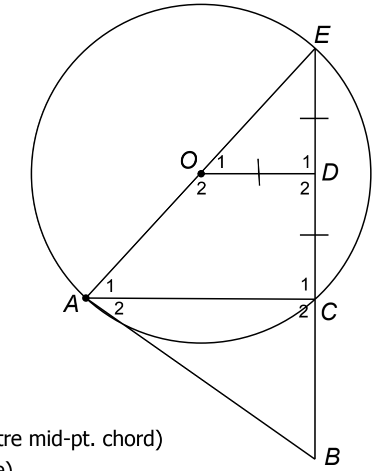
O is the centre of the circle with tangent BA and secant BCE.

$OD = DE = DC$ and AOE is a straight line.

Questions:

Prove:

1. $OD \parallel AC$
2. $\hat{A}_1 = \hat{A}_2$
3. $CB = 2ED$
4. $AE = 2\sqrt{2OD}$



Solutions:

1. $\hat{D}_1 = 90°$ (line from centre mid-pt. chord)

$\hat{C}_1 = 90°$ ($\angle$ in semi-circle)

$\therefore \hat{D}_1 = \hat{C}_1$

$\therefore OD \parallel AC$ (corresp. $\angle$'s =)

2. $\hat{A}_1 = \hat{O}_1$ (corresp. $\angle$'s, $OD \parallel AC$)

$\hat{O}_1 = \hat{E}$ ($\angle$'s opp = sides, $OD = DE$)

$\hat{A}_2 = \hat{E}$ (tan-chord)

$\therefore \hat{A}_1 = \hat{A}_2$

3. In $\triangle ACE$: $DE = DC$ (given)

$OA = OE$ (radii)

$\therefore AC = 2OD$ (mid-pt. Th.)

$\triangle ABE \,|||\, \triangle CAE \,|||\, \triangle CBA$ ($\perp$ from rt $\angle$ vert. to hyp.)

$\therefore AC^2 = CE \cdot CB$

$\therefore (2OD^2) = CE \cdot CB$

$4OD^2 = (2OD) \cdot CB \quad (OD = DE = DC)$

$2OD = CB$

$\therefore 2DE = CB$

4. $AE^2 = AC^2 + CE^2$ (Pythag.)

$\therefore AE^2 = (2OD)^2 + (2OD)^2$

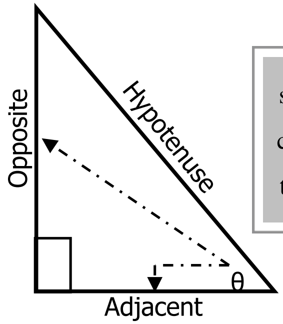
$\therefore AE^2 = 8OD^2$

$\therefore AE^2 = 2\sqrt{2OD}$

TRIGONOMETRY

Grade 11 Recap

BASIC DEFINITIONS



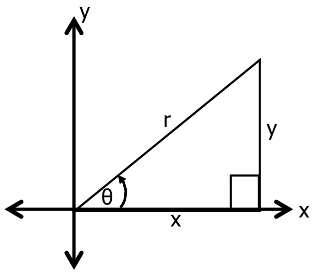
$\sin\theta = \frac{o}{h}$ soh

$\cos\theta = \frac{a}{h}$ cah

$\tan\theta = \frac{o}{a}$ toa

These are our basic trig ratios.

On the Cartesian Plane



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

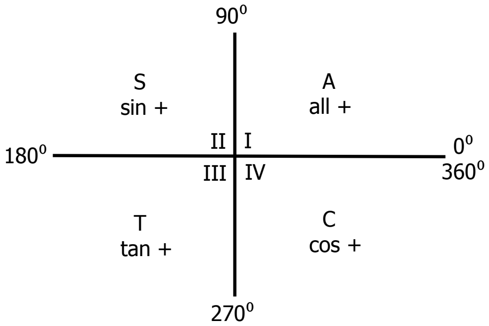
$\tan\theta = \frac{y}{x}$

Remember:

- $x^2 + y^2 = r^2$ (Pythagoras)
- Angles are measured upwards from the positive (+) x-axis (anti-clockwise) up to the hypotenuse (r).

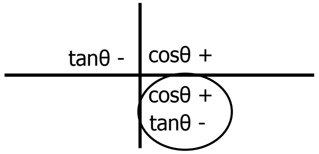
BASIC CAST DIAGRAM

Shows the quadrants where each trig ratio is +



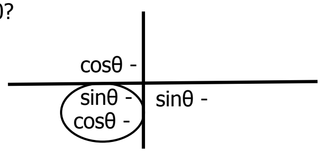
EXAMPLE

1. In which quadrant does θ lie if tanθ < 0 and cosθ > 0?



Quadrant IV

2. In which quadrant does θ lie if sinθ < 0 and cosθ < 0?



Quadrant III

FUNDAMENTAL TRIG IDENTITIES

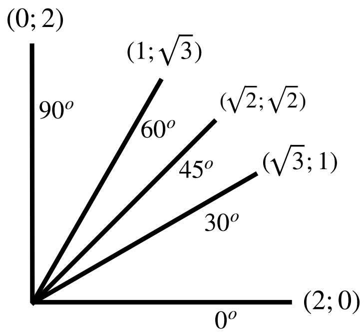
Memorise:

$$\frac{\sin A}{\cos A} = \tan A$$

$\sin^2 B + \cos^2 B = 1$
can be written as
$\sin^2 B = 1 - \cos^2 B$
$\cos^2 B = 1 - \sin^2 B$

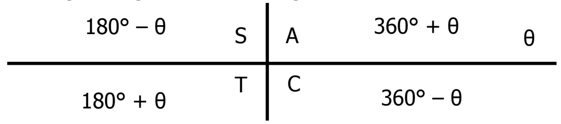
Special Angles

$r = 2 \qquad (x; y)$



REDUCTION FORMULAE

Reducing all angles to **acute** angles.



EXAMPLES

Reduce to an acute angle and simplify if possible (without a calculator):

1. $\sin 125^o$
$= \sin(180^o - 55^o)$
$= \sin 55^o$
(QII so sin is +)

2. $\cos 260^o$
$= \cos(180^o + 80^o)$
$= -\cos 80^o$
(QIII so cos is -)

3. $\tan 660^o$
$= \tan(360^o + 300^o)$
$= \tan 300^o$ (QI so tan is +)
$= \tan(360^o - 60^o)$
$= -\tan 60^o$ (QIV so tan is -)
$= -\frac{\sqrt{3}}{1}$
$= -\sqrt{3}$

Remember: 60^o is a special angle

4. $\frac{\tan(180^o - \beta)\cos(180^o + \beta)\cos^2(360^o - \beta)}{\sin(360 + \beta)} + \sin^2(180^o + \beta)$
$= \frac{(-\tan\beta)(-\cos\beta)(\cos\beta)^2}{\sin\beta} + (-\sin\beta)^2$
$= \tan\beta . \frac{\cos^3\beta}{\sin\beta} + \sin^2\beta$
$= \frac{\sin\beta}{\cos\beta} . \frac{\cos^3\beta}{\sin\beta} + \sin^2\beta$
$= \cos^2\beta + \sin^2\beta$
$= 1$

Remember: Identities

Pythagoras Problems

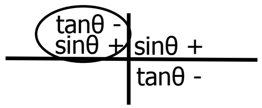
Steps:
1. Isolate the trig ratio
2. Determine the quadrant
3. Draw a sketch and use Pythagoras
4. Answer the question

EXAMPLE

If 3sinθ – 2 = 0 and tanθ < 0, determine $2\cos\theta + \frac{1}{\tan\theta}$ without using a calculator and using a diagram.

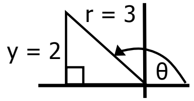
1. $3\sin\theta - 2 = 0$
$\sin\theta = \frac{2}{3} \quad \frac{y}{r}$

2.



$\therefore$ *Quadrant II*

3.



$x^2 + y^2 = r^2$
$x^2 + (2)^2 = (3)^2$
$x^2 = 5$
$x = \pm\sqrt{5}$
$\therefore x = -\sqrt{5}$

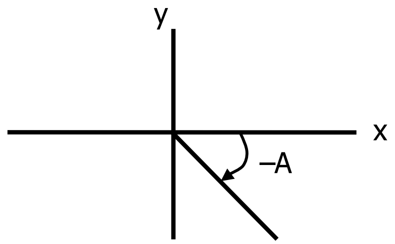
4. $2\cos\theta + \frac{1}{\tan\theta}$
$= 2(\frac{-\sqrt{5}}{3}) + \frac{1}{(\frac{2}{-\sqrt{5}})}$
$= \frac{-2\sqrt{5}}{3} - \frac{\sqrt{5}}{2}$
$= \frac{-4\sqrt{5} - 3\sqrt{5}}{6}$
$= \frac{-7\sqrt{5}}{6}$

Remember:
$\cos\theta = \frac{x}{r}$
and
$\tan\theta = \frac{y}{x}$

TRIGONOMETRY

NEGATIVE ANGLES

Angles measured downwards (clockwise) from the positive x-axis, which can be seen as Quadrant IV.



Method 1: Q IV

sin(–A) = –sinA
cos(–A) = cosA
tan(–A) = –tanA

Method 2: Get rid of negative

Add 360° to the angle to make it positive.

EXAMPLES

Simplify without the use of a calculator: sin(–330°)

NB: Negative Angle

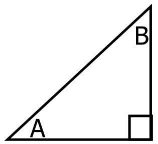
1) QIV

$\sin(-330^o)$
$= -\sin 330^o$
$= -\sin(360^o - 30^o)$
$= -(-\sin 30^o)$
$= \sin 30^o$
$= \dfrac{1}{2}$

2) $+360^o$

$= \sin(-330^o)$
$= \sin(360^o - 330^o)$
$= \sin 30^o$
$= \dfrac{1}{2}$

CO-FUNCTIONS



If A + B = 90° then sinA and cosB are known as co-functions.

sinA = sin(90° – B)
= cosB

EXAMPLES

1. $\sin 30^o$
$= \sin(90^o - 60^o)$
$= \cos 60^o$

2. $\cos 25^o$
$= \cos(90^o - 65^o)$
$= \sin 65^o$

NOTE:
Look at the quadrant first, **THEN** use the reduction/co-function formulae

3. $\sin(90^o - \alpha)$
$= \cos\alpha$

QI, *so* sin +
$90^o \therefore \sin \leftrightarrow \cos$

4. $\cos(90^o + \beta)$
$= -\sin\beta$

QII, *so* cos –
$90^o \therefore \sin \leftrightarrow \cos$

5. $\sin(\theta - 90^o)$
$= -\cos\theta$

QIV, *so* sin –
$90^o \therefore \sin \leftrightarrow \cos$

6. Simplify to a ratio of 10°:

a) $\cos 100^o$
$= \cos(90^o + 10^o)$
$= -\sin 10^o$

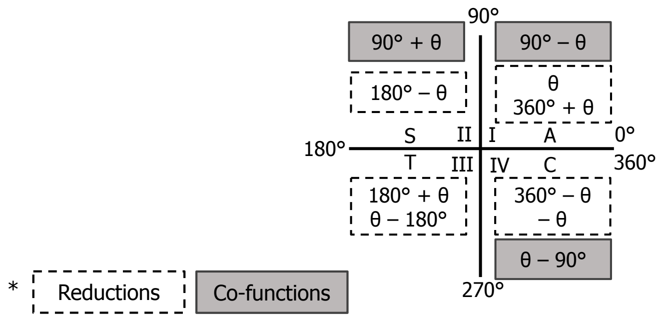
QII, *so* cos –
$90^o \therefore \sin \leftrightarrow \cos$

b) $\tan 170^o$
$= \tan(180^o - 10^o)$
$= -\tan 10^o$

QII, *so* tan –
$180^o \therefore reduction$

FULL CAST DIAGRAM

Memorise the following diagram:



PROVING IDENTITIES

Steps:
1. Separate LHS and RHS
2. Start on the more complex side
3. Prove that the sides are equal.

EXAMPLES

1. $\cos^2 x \,.\, \tan^2 x = \sin^2 x$

LHS $= \cos^2 x \,.\, \mathbf{\tan^2 x}$

$= \cos^2 x \,.\, \dfrac{\sin^2 x}{\cos^2 x}$

$= \sin^2 x =$ RHS

2. $1 - 2\sin x \,.\, \cos x = (\sin x - \cos x)^2$

RHS $= (\sin x - \cos x)^2$

$= \mathbf{\sin^2 x + \cos^2 x} - 2\sin x \,.\, \cos x$

$= 1 - 2\sin x \,.\, \cos x =$ LHS

3. $\tan x + \dfrac{\cos x}{1 + \sin x} = \dfrac{1}{\cos x}$

LHS $= \mathbf{\tan x} + \dfrac{\cos x}{1 + \sin x}$

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x}$

$= \dfrac{\sin x(1 + \sin x) + \cos x(\cos x)}{\cos x(1 + \sin x)}$

$= \dfrac{\sin x + \mathbf{\sin^2 x + \cos^2 x}}{\cos x(1 + \sin x)}$

$= \dfrac{\cancel{\sin x + 1}}{\cos x\cancel{(1 + \sin x)}}$

$= \dfrac{1}{\cos x} =$ RHS

PROBLEM SOLVING:

If cos25° = p, express the following in terms of p (i.e. get all angles to 25°):

1. $\cos(-385^o)$
$= \cos(-25^o)$
$= \cos 25^o$
$= p$

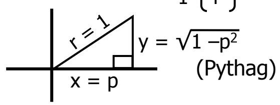
negative angle, so a) + 360°
or b) Q IV
- 385 Q I
$\therefore + \cos$

2. $\sin(65^o)$
$= \sin(90^o - 25^o)$ Q I, sin +
$= \cos(25^o)$
$= p$

3. $\sin(335^o)$
$= \sin(360^o - 25^o)$ Q IV, sin -
$= -\sin 25^o$

REMEMBER: Correct angle is 25° BUT wrong sin ratio. Thus draw sketch.

Given $\cos 25^0 = \dfrac{p}{1}\left(\dfrac{x}{r}\right)$



So, $-\sin 25^o = \dfrac{-y}{r}$

$= \dfrac{-\sqrt{1-p^2}}{1} = -\sqrt{1-p^2}$

4. $\tan(155^o)$
$= \tan(180^o - 25^o)$ Q II, tan -
$= -\tan 25^o$

This can be solved in two ways:

Method 1: Ratio

$= \dfrac{-\sin 25^o}{\cos 25^o}$

$= \dfrac{-\sqrt{1-p^2}}{p}$

Method 2: Sketch

$= \dfrac{-y}{x}$

$= \dfrac{-\sqrt{1-p^2}}{p}$

BASICS

Steps:

- Isolate trig ratios
- Reference angle (don't put negative into calculator)
- Choose quadrants
  - ➡ sin or cos: 2 Quadrants
  - ➡ tan: 1 Quadrant
- General solutions
  - ➡ $\sin\theta$ or $\cos\theta + k360°;\ k \in \mathbb{Z}$
  - ➡ $\tan\theta + k180°;\ k \in \mathbb{Z}$

REMEMBER: Only round off at the end

Common formulae:

$$\theta = \sin^{-1} a + k360° \text{ or}$$
$$\theta = (180° - \sin^{-1} a) + k360°\ (k \in \mathbb{Z})$$
$$\theta = \pm\cos^{-1} a + k360°\ (k \in \mathbb{Z})$$
$$\theta = \tan^{-1} a + k180°\ (k \in \mathbb{Z})$$

EXAMPLES

Solve for θ:

1. $3\sin\theta - 1 = 0$

$\sin\theta = \dfrac{1}{3}$

$\sin +$ in **QI and QII**

Reference$\angle$: 19,47°

QI: $\theta = 19{,}47° + k360°;\ k \in \mathbb{Z}$
QII: $\theta = 180° - 19{,}47° + k360°;\ k \in \mathbb{Z}$
$= 160{,}53° + k360°$

2. $\tan(3\theta + 30^o) + 1 = 0$

$\tan(3\theta + 30^o) = -1$

$\tan -$ in **QII**

Reference$\angle$: 45°

QII: $3\theta + 30° = 180° - 45° + k180°;\ k \in \mathbb{Z}$
$3\theta = 105° + k180°$
$\theta = 35° + k60°$

SQUARES

Hints:

- Do all four quadrants (± means the ratio **must** be both + **and** -)

EXAMPLE

Solve for β:

$4\sin^2\beta - 3 = 0$

$\sin^2\beta = \dfrac{3}{4}$

$\sin\beta = \pm\sqrt{\dfrac{3}{4}}$

Reference$\angle$: 60°

QI: $\beta = 60° + k360°;\ k \in \mathbb{Z}$
QII: $\beta = 180° - 60° + k360°$
$= 120° + k360°$
QIII: $\beta = 180° + 60° + k360°$
$= 240° + k360°$
QIV: $\beta = 360° - 60° + k360°$
$= 300° + k360°$

SINθ AND COSθ

Steps:

- sin and cos with the same angle
- Divide by cos to get tan

EXAMPLE

Solve for α:

$2\sin 2\alpha - \cos 2\alpha = 0$

$2\sin 2\alpha = \cos 2\alpha$

$\dfrac{2\sin 2\alpha}{\cos 2\alpha} = \dfrac{\cos 2\alpha}{\cos 2\alpha}$

$2\tan 2\alpha = 1$

$\tan 2\alpha = \dfrac{1}{2}$

$\tan +$ in **QI**

Reference$\angle$: 26,57°

QI: $2\alpha = 26{,}57° + k180°;\ k \in \mathbb{Z}$
$\alpha = 13{,}28° + k90°$

CO-FUNCTIONS

Hints:

- sin and cos with different angles
- Introduce the co-function with 90° - z
- The angle you change is the reference angle

EXAMPLES

Solve for x:

1. $\cos x = \sin(x - 10^o)$

$\cos x = \cos(90^o - (x - 10^o))$

$\cos x = \cos(100^o - x)$

Reference$\angle$: $100° - x$

QI: $x = 100° - x + k360°;\ k \in \mathbb{Z}$
$2x = 100° + k360°$
$x = 50° + k180°$
QII: $x = 360° - (100° - x) + k360°$
$x - x = 260° + k360°$
$0 = 260° + k360°$
No real solution

2. $\sin(x + 30^o) = \cos 2x$

$\sin(x + 30^o) = \sin(90^o - 2x)$

Reference$\angle$: $90° - 2x$

QI: $x + 30° = 90° - 2x + k360°;\ k \in \mathbb{Z}$
$3x = 60° + k360°$
$x = 20° + k120°$
QII: $x + 30° = 180° - (90° - 2x) + k360°$
$x + 30° = 90° + 2x + k360°$
$-x = 60° + k360°$
$x = -60° - k360°$

NOTE: Specific Solutions

If they ask for $x \in [-360°; 360°]$, choose integer values for k (...-3; -2; -1; 0; 1; 2; 3...) so that x falls in the given intervals.

$x = 30° + k120°$
$x = -330°; -210°; -90°; 30°; 150°; 270°$
$k = -3;\ k = -2;\ k = -1;\ k = 0;\ k = 1;\ k = 2$

OR

$x = -60° + k360°$
$x = -60°;\ 300°$
$k = 0;\ k = 1$

FACTORISING

Steps:

- Solve as you would a quadratic equation

EXAMPLES

Solve for x:

1. $\tan^2 x - 2\tan x + 1 = 0$

$(\tan x - 1)(\tan x - 1) = 0$

$\tan x = 1$

Reference$\angle$: 45°

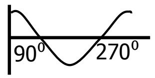
QI: $x = 45° + k180°;\ k \in \mathbb{Z}$

2. $\cos^2 x + \sin x \,.\cos x = 0$

$\cos x(\cos x + \sin x) = 0$

$\cos x = 0$ OR $\cos x = -\sin x$

Use trig graph:



$\dfrac{\cos x}{\cos x} = \dfrac{-\sin x}{\cos x}$

$\tan x = -1$

Reference$\angle$: 45°

$x = 90° + k180°;\ k \in \mathbb{Z}$ QII: $x = 135° + k180°$

3. $2\cos^2 x + 3\sin x = 0$

$2(1 - \sin^2 x) + 3\sin x = 0$

$2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$\sin x = \dfrac{-1}{2}$ OR $\sin x = 2$

Reference$\angle$: 30° No real solution

QIII: $x = 180° + 30° + k360°;\ k \in \mathbb{Z}$
$x = 210° + k360°$
QIV: $x = 360° - 30° + k360°;\ k \in \mathbb{Z}$
$x = 330° + k360°$

Trig Graphs

Grade 11 Recap

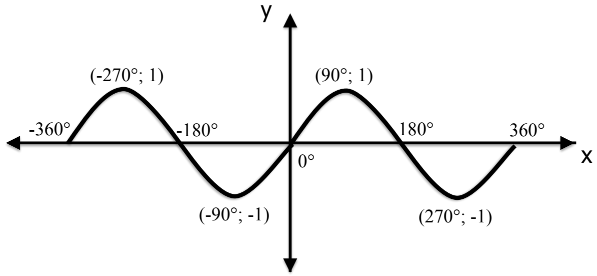
IMPORTANT!

When sketching trig graphs, you need to label the following:

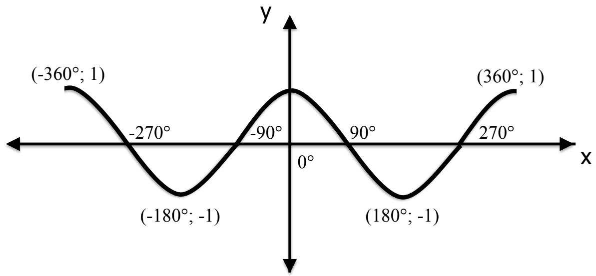
- **both axes**
- **x- and y-intercepts**
- **turning points**
- **endpoints** (if not on the axes)
- **asymptotes** (tan graph only)

BASICS

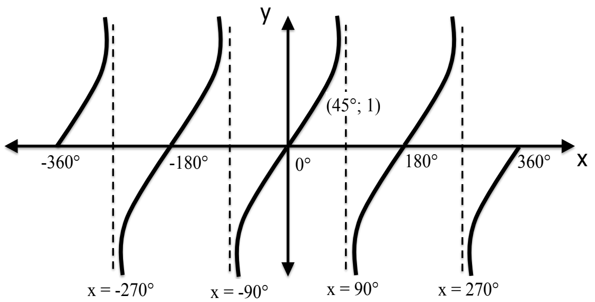
- $y = \sin x$ for $x \in [-360°; 360°]$



- $y = \cos x$ for $x \in [-360°; 360°]$



- $y = \tan x$ for $x \in [-360°; 360°]$



Notes for $\sin x$ and $\cos x$:

❖ Key points (intercepts/turning pts) every 90°

❖ Period (1 complete graph): 360°

❖ Amplitude (halfway between min and max): 1

Notes for $\tan x$:

❖ Key points every 45°

❖ Period (1 complete graph): 180°

❖ No amplitude can be defined

❖ Asymptotes at $x = 90^o + k180^o, k \in \mathbb{Z}$

VERTICAL SHIFT

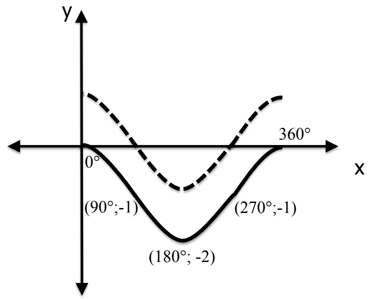
- $y = \sin x + q$ or $y = \cos x + q$ or $y = \tan x + q$

If q > 0 : upwards (e.g: $y = \sin x + 1$)
If q < 0 : downwards (e.g: $y = \cos x - 2$)

EXAMPLE

$y = \cos x - 1 \quad x \in [0°; 360°]$ (solid line)

$y = \cos x$ (dotted line - for comparison)



AMPLITUDE CHANGE

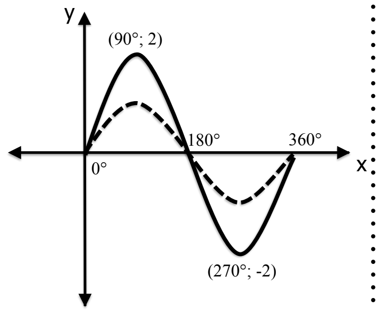
- $y = a\,.\sin x$ or $y = a\,.\cos x$ or $y = a\,.\tan x$

If a > 1 : stretch upwards
0 < a < 1 : compress downward
a < 0 : reflection in x-axis

EXAMPLES

1. $y = 2\sin x$ (solid line)

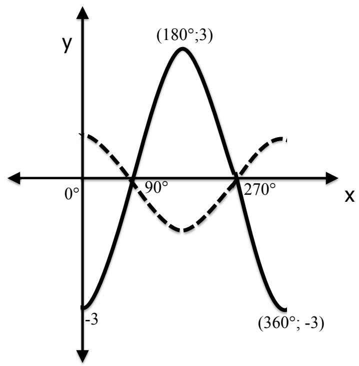
$y = \sin x$ (dotted line - for comparison)



* Amplitude = 2

2. $y = -3\cos x$ (solid line)

$y = \cos x$ (dotted line - for comparison)



* Range: $y \in [-3; 3]$

PERIOD CHANGE

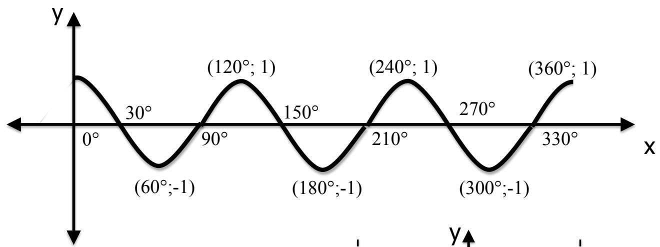
- $y = \sin bx$ or $y = \cos bx$ or $y = \tan bx$

The value of b indicates how many graphs are completed in the 'regular' period of that graph (i.e. $\sin x / \cos x$: 360° and $\tan x$: 180°)

EXAMPLES

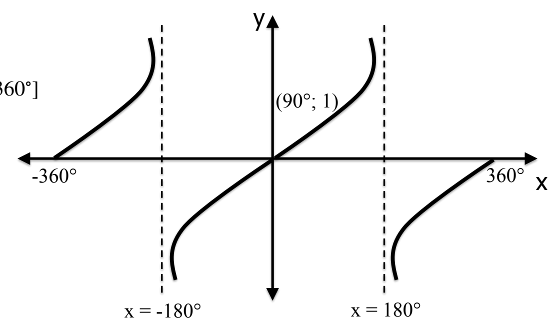
1. $y = \cos 3x \quad x \in [0°; 360°]$

* Normal period: 360°
* New period: 120° (3 graphs in 360°)
* Critical points every 90/3 = **30°**



2. $y = \tan \frac{1}{2}x \quad x \in [0°; 360°]$

* Normal period: 180°
* New period: 360°
(½ graph in 180°)
* Critical points:
every 45/0,5 = **90°**



HORIZONTAL SHIFT

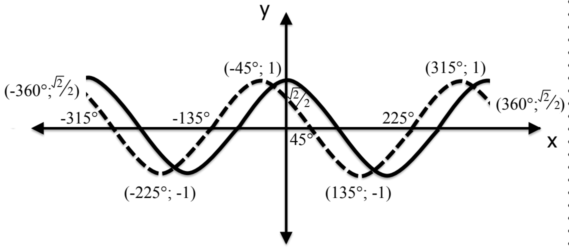
- $y = \sin(x - p)$ or $y = \cos(x - p)$ or $y = \tan(x - p)$

If p > 0 : shift right (e.g: $y = \sin(x - 30°)$)
p < 0 : shift left (e.g: $y = \cos(x + 45)$)

How to plot a horizontal shift:
- Plot the original curve
- Move the critical points left/right
- Label the x-cuts and turning points
- Calculate and label the endpoints and y-cut

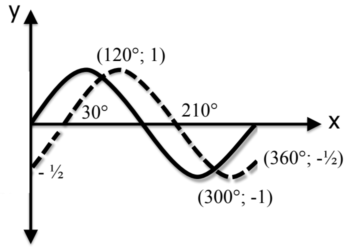
EXAMPLES

1. $y = \cos(x + 45°)$ for $x \in [-360°; 360°]$ (dotted line)
$y = \cos x$ (solid line - for comparison)



Endpoints:
$\cos(-360° + 45°) = \frac{\sqrt{2}}{2}$ and $cos(-360° + 45°) = \frac{\sqrt{2}}{2}$
y-cut:
$\cos(0° + 45°) = \frac{\sqrt{2}}{2}$

2. $y = \sin(x - 30°)$ for $x \in [0°; 360°]$ (dotted line)
$y = \sin x$ (solid line - for comparison)



Endpoints:
$\sin(0° + 45°) = -\frac{1}{2}$ and $\sin(360° - 30°) = -\frac{1}{2}$
y-cut:
The y-cut is one of the endpoints

EXAMPLE

Given $f(x) = \cos(x + 60°)$ and $g(x) = \sin 2x$

Questions:

1. Determine algebraically the points of intersection of $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
2. Sketch $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
3. State the amplitude of $f(x)$
4. Give the period of $g(x)$
5. Use the graphs to determine the values of x for which:
   a. $g(x)$ is increasing and positive
   b. $f(x)$ is increasing and positive
   c. $f(x) \geq g(x)$ - i.e. $f(x)$ is above $g(x)$
   d. $f(x) \cdot g(x) \geq 0$ - i.e. product is + or 0
6. Explain the transformation that takes $y = sin x$ to $y = \sin(2x - 60°)$

Solutions:

1. $\cos(x + 60°) = \sin 2x$

$\cos(x + 60°) = \cos(90° - 2x)$

Reference $\angle$: $90° - 2x$

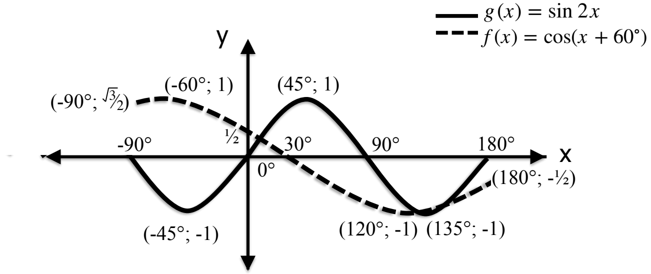
QI: $x + 60° = 90° - 2x + k360°;\ k \in \mathbb{Z}$
$3x = 30° + k360°$
$x = 10° + k120°$

QIV: $x + 60° = 360° - (90° - 2x) + k360°;\ k \in \mathbb{Z}$
$x + 60° = 270° + 2x + k360°$
$-x = 210° + k360°$
$x = -210° + k360°$

but $x \in [-90°; 180°]$

$\therefore\ x = 10°;\ 130°;\ 150°$

2.



For f(x):
Endpoints: $\cos(-90° + 60°) = \frac{\sqrt{3}}{2}$ and $\cos(180° + 60°) = -\frac{1}{2}$
y-cut: $\cos(0° + 60°) = \frac{1}{2}$

3. 1

4. 180°

5. a. $x \in (0°; 45°)$
   b. $x \in [-90°; -60°)$
   c. $x \in [-90°; 10°] \cup (130°; 150°)$
   d. $x \in [0°; 30°] \cup [90°; 180°]$ also at $x = -90°$

6. Rewrite $y = \sin(2x - 60°)$ in the form $y = \sin b(x - p) = \sin(2(x - 30°))$
Transformation: b = 2 ∴ period is halved
p = 30 ∴ shifted 30 to the right°

USING TRIG GRAPHS TO FIND RESTRICTIONS ON IDENTITIES

i.e. answering the question
"for which values of x will this identity be undefined?"

Identities are undefined if:

- the function is undefined
  $\tan x$ has asymptotes at $x = 90° + k180°;\ k \in \mathbb{Z}$
- any denominator is zero

Reminder:
$\frac{A}{0}$ is undefined

EXAMPLES

1. For which values of x will $\cos^2 x \cdot \tan^2 x = \sin^2 x$ be <u>defined</u>?

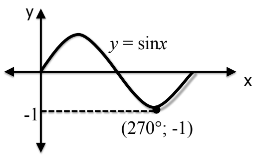
- $\tan x$ is undefined at $x = 90° + k180°;\ k \in \mathbb{Z}$
  $\therefore$ will be defined at $x \in \mathbb{R}$ and $x \neq 90° + k180°;\ k \in \mathbb{Z}$
- no denominators that could be zero

2. For which values of x will $\tan x + \frac{\cos x}{1 + \sin x} = \frac{1}{\cos x}$ be undefined?

- $\tan x$ is undefined at $x = 90° + k180°;\ k \in \mathbb{Z}$
- fractions are undefined if the denominator = 0

$\therefore$ if $1 + \sin x = 0$ or if $\cos x = 0$

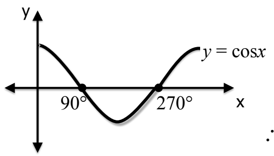
* $1 = \sin x = 0$
$\therefore \sin x = -1$



Use trig graphs for $0; \pm 1$

$\therefore x = 270° + k360°;\ k \in \mathbb{Z}$

* $\cos x = 0$



Use trig graphs for $0; \pm 1$

$\therefore x = 90° + k180°;\ k \in \mathbb{Z}$

$$\left.\begin{array}{l} x = 90° + k180°;\ k \in \mathbb{Z} \\ x = 270° + k360°;\ k \in \mathbb{Z} \\ x = 90° + k180°;\ k \in \mathbb{Z} \end{array}\right\}$$ can be summarised as: $x = 90° + k180°;\ k \in \mathbb{Z}$

TRIGONOMETRY

CAST DIAGRAM

$180° - \theta$

1. $\sin(180° - \theta) = \sin\theta$
2. $\cos(180° - \theta) = -\cos\theta$
3. $\tan(180° - \theta) = -\tan\theta$

$\theta + 360°$

1. $\sin(\theta + 360°) = \sin\theta$
2. $\cos(\theta + 360°) = \cos\theta$
3. $\tan(\theta + 360°) = \tan\theta$

$90° + \theta$

1. $\sin(90° + \theta) = \cos\theta$
2. $\cos(90° + \theta) = -\sin\theta$

$90° - \theta$

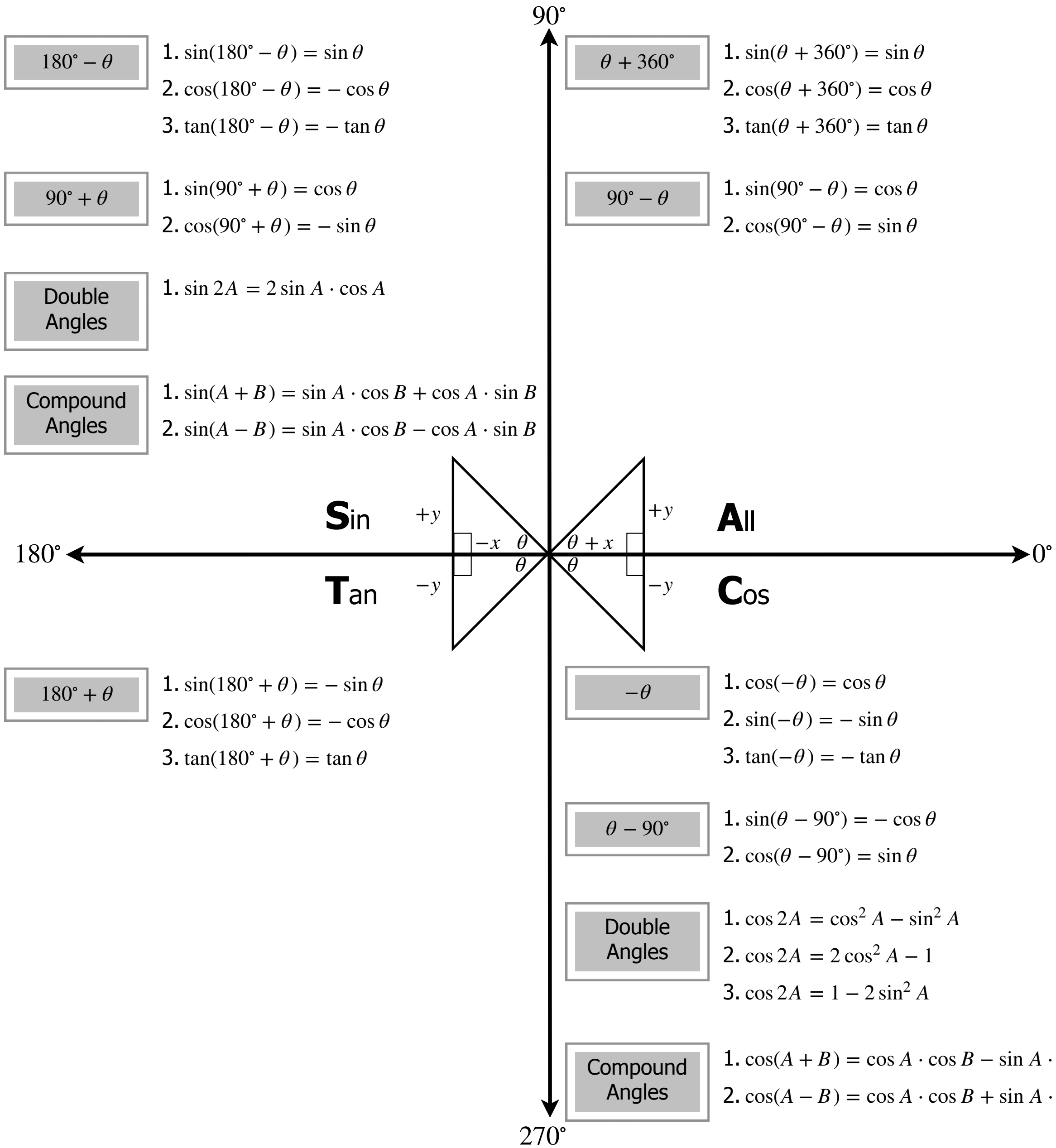
1. $\sin(90° - \theta) = \cos\theta$
2. $\cos(90° - \theta) = \sin\theta$

Double Angles

1. $\sin 2A = 2\sin A \cdot \cos A$

Compound Angles

1. $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
2. $\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$



$180° + \theta$

1. $\sin(180° + \theta) = -\sin\theta$
2. $\cos(180° + \theta) = -\cos\theta$
3. $\tan(180° + \theta) = \tan\theta$

$-\theta$

1. $\cos(-\theta) = \cos\theta$
2. $\sin(-\theta) = -\sin\theta$
3. $\tan(-\theta) = -\tan\theta$

$\theta - 90°$

1. $\sin(\theta - 90°) = -\cos\theta$
2. $\cos(\theta - 90°) = \sin\theta$

Double Angles

1. $\cos 2A = \cos^2 A - \sin^2 A$
2. $\cos 2A = 2\cos^2 A - 1$
3. $\cos 2A = 1 - 2\sin^2 A$

Compound Angles

1. $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
2. $\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

GENERAL SOLUTIONS

- $\sin A = \sin B$ → ref angle
  - $A = B + k.360°$ or $A = 180° - B + k.360°;\ k \in \mathbb{Z}$
- $\cos A = \cos B$
  - $A = \pm B + k.360°;\ k \in \mathbb{Z}$
- $\tan A = \tan B$
  - $A = B + k.360°;\ k \in \mathbb{Z}$
- Other/ Co-function
  - $\sin A = \cos B \quad \therefore \sin A = \sin(90° - B)$ → ref angle
  - $\cos A = \sin B \quad \therefore \cos A = \cos(90° - B)$ → ref angle

EXAMPLE 1

Solve for x if $\cos 2x = -\sin x$

$1 - 2\sin^2 x = -\sin x$

$0 = 2\sin^2 x - \sin x - 1$

$0 = (\sin x - 1)(2\sin x + 1)$

$\sin x = 1$

$x = 0° + k.360°$

or

$x = 180° + k.360°;\ k \in \mathbb{Z}$

or

$\sin x = -\frac{1}{2}$

$x = -30° + k.360°$

or

$x = 210° + k.360°;\ k \in \mathbb{Z}$

EXAMPLE 2

Solve for x if $\cos(90° - x) \cdot \sin x - \cos 2x = 0$

$\sin x \cdot \sin x - (1 - 2\sin^2 x) = 0$

$\sin^2 x - 1 + 2\sin^2 x = 0$

$3\sin^2 x = 1$

$\sin^2 x = \frac{1}{3}$

$\sin x = \frac{\pm\sqrt{3}}{3}$

$x = 35{,}26° + k.360°$

or

$x = 180° - 35{,}26° + k.360°;\ k \in \mathbb{Z}$

$x = 144{,}74° + k.360°$

or

$x = -35{,}26° + k.360°$

or

$x = 180° - (-35{,}26°) + k.360°$

$x = 215{,}26° + k.360°$

TRIGONOMETRY

TRIG IDENTITIES

- $\tan x = \dfrac{\sin x}{\cos x}$
- $\dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
- $\sin 2x = 2\sin x \cdot \cos x$
- $\sin 3x = \sin(2x + x)$

  $= \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$ (to be expanded further)
- $\sin 4x = \sin 2(2x)$

  $= 2\sin 2x \cdot \cos 2x$

  $= 4(\sin x \cdot \cos x)(\cos^2 x - \sin^2 x)$

  $= 4\sin x \cdot \cos^3 x - 4\sin^2 x \cdot \cos x$ (can be expanded further)
- $\cos 2x = \cos^2 x - \sin^2 x$

  $= 1 - 2\sin^2 x$

  $= 2\cos^2 x - 1$

HINTS FOR PROVING IDENTITIES

1. Start on the side with the least number of "terms" and simplify if possible.
2. Go to the other side and simplify until you get the same answer.
3. Look for a conjugate and multiply with the "opposite" sign (to make a difference of squares in the denominator of your fraction)
4. Always try to factorise where possible

EXAMPLE 1

Show\Prove that: $\dfrac{\sin 2x}{\cos 2x + \sin^2 x} = 2\tan x$

RHS $= 2 \cdot \dfrac{\sin x}{\cos x}$

LHS $= \dfrac{2\sin x \cdot \cos x}{(2\cos^2 x - 1) + (1 - \cos^2 x)}$

$= \dfrac{2\sin x \cdot \cos x}{\cos^2 x}$

$= \dfrac{2\sin x}{\cos x}$

$\therefore$ LHS = RHS

EXAMPLE 2

Show\Prove that: $\sin 3x = 3\sin x - 4\sin^3 x$

LHS $= \sin(2x + x)$

$= \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$

$= 2\sin x \cdot \cos x \cdot \cos x + (1 - \sin^2 x) \cdot \sin x$

$= 2\sin x \cdot \cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$\therefore$ LHS = RHS

EXAMPLE 3

Show\Prove that: $\dfrac{1 - \sin x}{1 + \sin x} = \left(\dfrac{1}{\cos x} - \tan x\right)^2$

LHS $= \dfrac{1 - \sin x}{1 + \sin x} \times \dfrac{1 - \sin x}{1 - \sin x}$

$= \dfrac{1 - 2\sin x + \sin^2 x}{1 - \sin^2 x}$

RHS $= \left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right)^2$

$= \dfrac{1 - 2\sin x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1 - 2\sin x + \sin^2 x}{1 - \sin^2 x}$

$\therefore$ LHS = RHS

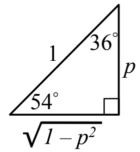
MIXED EXAMPLE 1

If $\sin 54° = p$, express the following in terms of p:

1. $\cos 36°$
2. $\sin 108°$
3. $\sin 84°$

Solutions:

$\sin 54° = \dfrac{p}{1}\ \left(\dfrac{o}{h}\right)$



1. $\cos 36° = p$

2. $\sin 108° = \sin 2(54°)$

$= 2\sin 54° \cdot \cos 54°$

$= 2(p) \cdot \left(\sqrt{1 - p^2}\right)$

3. $\sin 84° = \sin(54° + 30°)$

$= \sin 54° \cdot \cos 30° + \cos 54° \cdot \sin 30°$

$= p \cdot \dfrac{\sqrt{3}}{2} + \left(\sqrt{1 - p^2}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{\sqrt{3}\,p + \sqrt{1 - p^2}}{2}$

MIXED EXAMPLE 2

Find the value of k if: $\cos 75° \cdot \sin 25° - \sin 75° \cdot \sin k = \sin 50°$

$\cos 75° \cdot \sin 25° - \sin 75° \cdot \sin k = \sin 50°$

$\cos 75° \cdot \sin 25° - \sin 75° \cdot \cos(90° - k) = \sin 50°$

$\sin(75° - 25°) = \sin 50°$

$\therefore k = 65°$

MIXED EXAMPLE 3

Express the following in terms of p if $\cos 73° \cdot \cos 31° + \sin 73° \cdot \sin 31° = p$

1. $\cos^2 21° - \sin^2 21° + 7$
2. $\sin 42°$

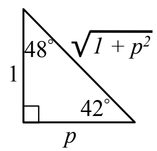
Solutions:

$\cos 73° \cdot \cos 31° + \sin 73° \cdot \sin 31°$

$= \cos(73° - 31°)$

$= \cos 42°$

$\therefore \cos 42° = \dfrac{p}{1}\ \left(\dfrac{a}{h}\right)$



1. $\cos 2(21°) + 7 = \cos 42° + 7$

$= p + 7$

2. $\sin 42° = \dfrac{1}{\sqrt{1 + p^2}}$

What is Analytical Geometry?

Analytical Geometry (Co-ordinate Geometry): Application of straight line functions in conjunction with Euclidean Geometry by using points on a Cartesian Plane.

FLASHBACK

Straight line parallel to the x-axis: m = 0

Straight line parallel to the y-axis: m = undefined

Straight line equation:

$y = mx + c$

Gradient formula:

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Parallel gradients:

$m_1 = m_2$

Perpendicular gradients:

$m_1 \times m_2 = -1$

Distance:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Co-linear:

$m_{AB} = m_{BC}$ OR $d_{AB} + d_{BC} = d_{AC}$

Collinear points A, B and C lie on the same line

Midpoint formula:

$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

Midpoint Theorem: If two midpoints on adjacent sides of a triangle are joined by a straight line, the line will be parallel to and half the distance of the third side of the triangle.

EXAMPLE

Given: $A(-2; 3)$ and $C(p; -5)$ are points on a Cartesian Plane.

1. If $AC = 10$ units determine the value(s) of p.
2. If $C(4; -5)$, determine the equation of the line AC.
3. Determine the co-ordinates of M, the midpoint of AC.
4. If $B\left(-1; \frac{5}{3}\right)$ determine if A, B and C are collinear.
5. Determine the equation of the line perpendicular to AC passing through B.

SOLUTION

1. Draw a sketch diagram. C has two potential x-coordinates for p.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$10 = \sqrt{(p - (-2))^2 + (-5 - 3)^2}$

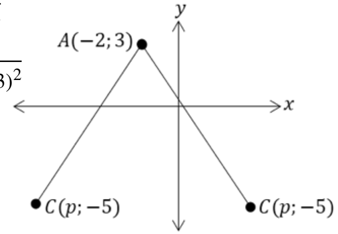
$100 = (p + 2)^2 + 64$

$100 = p^2 + 4p + 4 + 64$

$0 = p^2 + 4p - 32$

$0 = (p + 4)(p + 8)$

$p = 4$ or $p = -8$



2. **Line equation** requires solving m and c.

$m = \frac{\Delta y}{\Delta x}$

$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3 - (-5)}{-2 - 4}$

$= -\frac{4}{3}$

$y = mx + c$

$(3) = -\frac{4}{3}(-2) + c$

$c = \frac{1}{3}$

$\therefore y = -\frac{4}{3}x + \frac{1}{3}$

3. **Midpoint formula**

$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

$= \left(\frac{-2 + 4}{2}; \frac{3 + (-5)}{2}\right)$

$M(1; -1)$

4. **Prove collinearity** by proving that the points share a common gradient.

$m = \frac{\Delta y}{\Delta x}$ $\qquad$ $m = \frac{\Delta y}{\Delta x}$

$m_{AB} = \frac{3 - \frac{5}{3}}{-2 - (-1)}$ $\qquad$ $m_{BC} = \frac{\frac{5}{3} - (-5)}{-1 - 4}$

$m_{AB} = -\frac{4}{3}$ $\qquad$ $m_{BC} = -\frac{4}{3}$

$\therefore A, B$ and C are collinear

5. **Line equation** requires solving m_2 and c w.r.t. B.

$m_{AC} \times m_2 = -1$

$-\frac{4}{3} \times m_2 = -1$

$m_2 = \frac{3}{4}$

$y = mx + c$

$\left(\frac{5}{3}\right) = \frac{3}{4}(-1) + c$

$c = \frac{29}{12}$

$\therefore y = \frac{4}{3}x + \frac{29}{12}$

Converting gradient (m) into angle of inclination (θ)

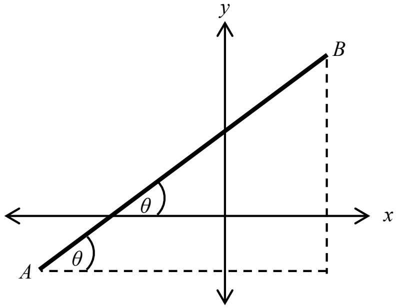
$m_{AB} = \frac{\Delta y}{\Delta x}$

and

$\tan\theta = \frac{o}{a} = \frac{\Delta y}{\Delta x}$

therefore;

$\therefore m_{AB} = \tan\theta$

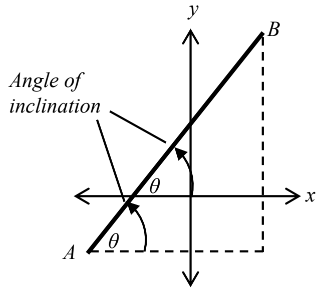


The angle of inclination (θ) is always in relation to a horizontal plane in an anti-clockwise direction.

<u>Positive gradient:</u>

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

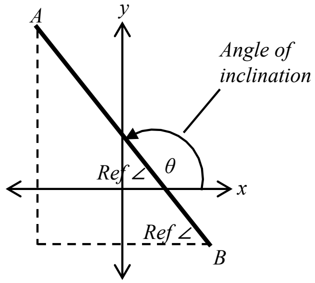


<u>Negative gradient:</u>

$m < 0$
$\tan^{-1}(m) = \text{ref}\angle$

Angle of inclination:
$\theta + \text{ref}\angle = 180°$ ($\angle$'s on str. line)

The angle of inclination must be calculated from the reference angle.



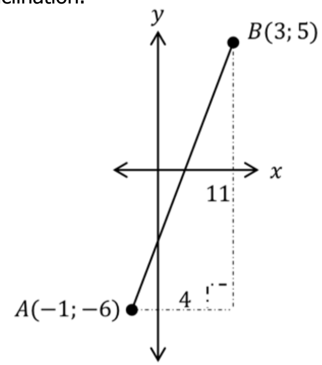
Converting a positive gradient into an angle

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

Given: $A(-1;-6)$ and $B(3;5)$ are two points on a straight line. Determine the angle of inclination.

$$m = \tan\theta$$
$$\frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$
$$\frac{5-(-6)}{3-(-1)} = \tan\theta$$
$$\tan^{-1}\left(\frac{11}{4}\right) = \theta$$
$$\therefore \theta = 70°$$

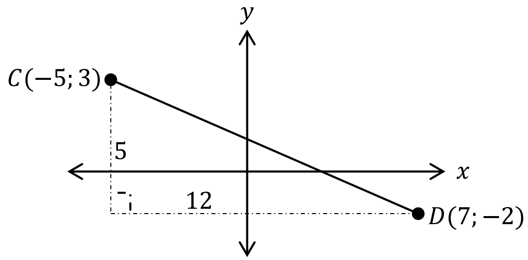


Converting a negative gradient into an angle

$m < 0$
$\tan^{-1}(m) = \text{ref}\angle$

Angle of inclination:
$\theta + \text{ref}\angle = 180°$ ($\angle$'s on str. line)

Given: $C(-5;3)$ and $D(7;-2)$ are two points on a straight line. Determine the angle of inclination.



$$m = \tan\theta$$
$$\frac{5}{12} = \tan\theta$$
$$\tan^{-1}\left(\frac{5}{12}\right) = \theta$$
$$\therefore \text{ref}.\angle = 22{,}6°$$

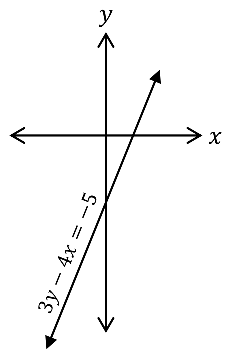
$$\theta + \text{ref}.\angle = 180°$$
$$\theta = 180° - 22{,}6°$$
$$= 157{,}4°$$

EXAMPLE

Given: straight line with the equation $3y - 4x = -5$. Determine the angle of inclination correct to two decimal places.

$3y - 4x = -5$ - make y the subject
$3y = 4x - 5$
$y = \frac{4}{3}x - \frac{5}{3}$ - note that m>0

$m = \tan\theta$ - sub. m and solve θ
$\frac{4}{3} = \tan\theta$
$\tan^{-1}\left(\frac{4}{3}\right) = \theta$
$\therefore \theta = 53{,}13°$ - m>0; ref. $\angle$ = angle of inclination



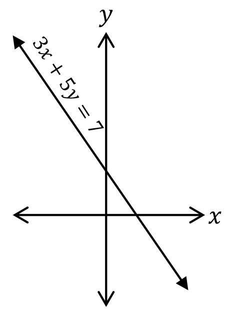
EXAMPLE

Given: straight line with the equation $3x + 5y = 7$. Determine the angle of inclination correct to two decimal places.

$3x + 5y = 7$ - make y the subject
$5y = -3x + 7$
$y = -\frac{3}{5}x - \frac{7}{5}$ - note that m<0

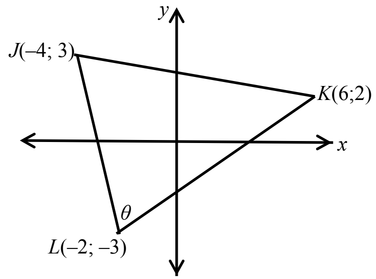
$m = \tan\theta$ - sub. m as a positive value to determine the ref. $\angle$
$\frac{3}{5} = \tan\theta$
$\tan^{-1}\left(\frac{3}{5}\right) = \theta$
$\therefore \text{ref}.\angle = 30{,}96°$

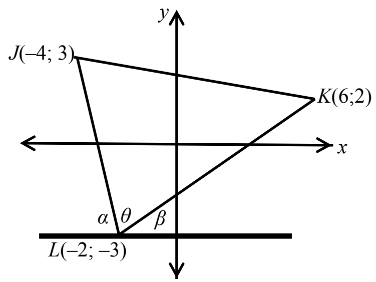
$\theta + \text{ref}\angle = 180°$ - m<0; ref. $\angle$ + θ = 180°
$\theta = 180° - 30{,}96°$
$\theta = 149{,}04°$



Finding an angle that is not in relation to a horizontal plane

Construct a horizontal plane, parallel to the x-axis. This will allow you to use the 'sum of adjacent angles on a straight line' in order to calculate the value of the angle.





$m_{JL} = -\frac{6}{2} = -3$

$m = \tan\alpha$

$3 = \tan\alpha$

$\tan^{-1}(3) = \alpha$

$71{,}6° = \alpha$

$m_{KL} = \frac{5}{8}$

$m = \tan\beta$

$\frac{5}{8} = \tan\beta$

$\tan^{-1}\left(\frac{5}{8}\right) = \beta$

$32° = \beta$

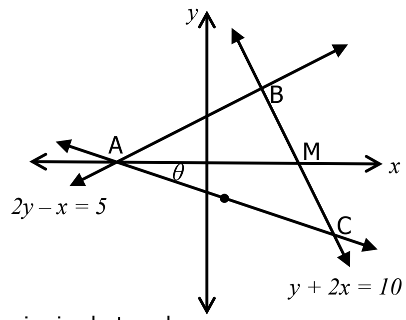
$\theta = 180° - (\alpha + \beta)$

$= 180° - (71{,}6° + 32°)$

$= 76{,}4°$

EXAMPLE

Given: In the diagram: Straight line with the equation $2y - x = 5$, which passes through A and B. Straight line with the equation $y + 2x = 10$, which passes through B and C. M is the midpoint of BC. A, B and C are vertices of ΔABC. $M\hat{A}C = \theta$. A and M lie on the x-axis.



Questions:

1. Determine the following:
   a. The co-ordinates of A
   b. The co-ordinates of M.
   c. The co-ordinates of B.
2. What type of triangle is ABC? Give a reason for your answer.
3. If $A(-5; 0)$ and $B(3; 4)$, show that $AB = BC$ (leave your answer in simplest surd form).
4. If $C(7; -4)$, determine the co-ordinate of N, the midpoint of AC.
5. Hence, or otherwise, determine the length of MN.
6. If $ABCD$ is a square, determine the co-ordinates of D.
7. Solve for θ correct to one decimal places.

Solutions:

a. $2y - x = 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}$

x – cut : $0 = \frac{1}{2}x + \frac{5}{2}$

$0 = x + 5$

$-5 = x$

$\therefore A(-5; 0)$

b. $y + 2x + 10$

$y = -2x + 10$

x – cut : $0 = -2x + 10$

$2x = 10$

$x = 5$

$\therefore M(5; 0)$

c. $\frac{1}{2}x + \frac{5}{2} = -2x + 10$

$x + 5 = -4x + 20$

$5x = 15$

$x = 3$

$y = -2(3) + 10$

$y = 4$

$\therefore B(3; 4)$

2. ABC is a right-angled triangle:

$m_{AD} \times m_{BC} = -1$

$\therefore b = 90°$

3. $d_{AB} = \sqrt{(-5-3)^2 + (0-4)^2}$

$= 4\sqrt{5}$

$d_{BC} = \sqrt{(3-7)^2 + (4-(-4))^2}$

$= 4\sqrt{5}$

$\therefore AB = BC$

4. $N(x; y) = \left(\frac{-5+7}{2}; \frac{0+(-4)}{2}\right)$

$N(1; -2)$

5. $MN = 2\sqrt{5}$ (Midpt theorem)

6. If $ABCD$ is a square, then AC is the diagonal, which makes N the midpoint for both diagonals $\therefore D(-3; -8)$

7. $m_{AC} = \frac{\Delta y}{\Delta x}$

$= \frac{0-(-4)}{-5-7}$

$= -\frac{1}{3}$

$m = \tan\theta$

$-\frac{1}{3} = \tan\theta$

$\tan^{-1}\left(-\frac{1}{3}\right) = \theta$

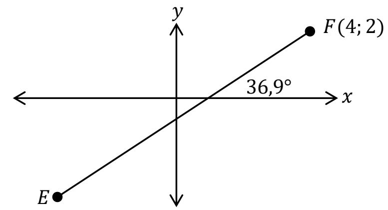
$\theta = 18{,}4°$

Converting an angle into a gradient

Sub. the ref. $\angle$ into $m = \tan\theta$.

Remember to add the – sign to answers for negative gradients.

Given: E and $F(4; 2)$ are points on a straight line with an angle of inclination of 36,9°. Determine the value of m correct to two decimal places.



$m = \tan\theta$

$m = \tan(36{,}9°)$

$m = 0{,}75$

HELPFUL HINTS:

1. Make a quick rough sketch if you are given co-ordinates without a drawing.
2. Always make y the subject if you are given straight line equations.
3. Know your types of triangles and quadrilaterals. Proving them or using their properties is a common occurrence.
4. The angle of inclination is ALWAYS in relation to the horizontal plane.

ANALYTICAL GEOMETRY

What is Analytical Geometry?

Analytical Geometry (also called Co-ordinate Geometry) is the study and application of straight line functions, trigonometry and Euclidean Geometry by using points on a Cartesian Plane. In Grade 12 we combine this knowledge and find the equation of circles.

Prior Knowledge:

1. All analytical formulae (distance, midpoint, gradients, and straight line functions)
2. Euclidean Geometry (types of triangles and quadrilaterals, circle geometry theorems)
3. Trigonometry (general applications, sine and cosine rules, sine area rule, double and compound angle identities)

Glossary of Terms:

Concentric
Two or more circles that share the same centre.

Median
Line from the vertex of a triangle to the midpoint of the opposite side.

Centroid
Point of intersection for all the medians.

Altitude
Perpendicular line drawn from a side of a triangle to the opposite vertex.

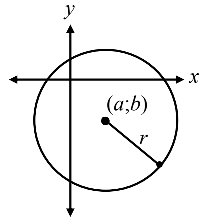
Orthocentre
Point of intersection for all the altitudes.

EQUATION OF A CIRCLE

The equation of a circle is given by the equation:

$$(x-a)^2+(y-b)^2=r^2,$$

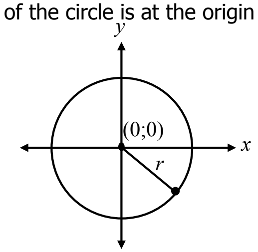
where $(a;b)$ is the centre of the circle, and x and y are coordinates of a point on the circle.



With radius as the subject:

$$r=\sqrt{(x-a)^2+(y-b)^2}$$

If the centre of the circle is at the origin (0;0):



$$(x-a)^2+(y-b)^2=r^2$$
$$(x-0)^2+(y-0)^2=r^2$$
$$\therefore x^2+y^2=r^2$$

OR

$$r=\sqrt{x^2+y^2}$$

The standard centre-radius equation can be expanded to give;

$$Ax^2+Bx+Cy^2+Dy+E=0$$

EXAMPLE 1

Find the equation of the circle with its centre through the origin and passing through the point (−5; 12).

$$r^2=(x-a)^2+(y-b)^2$$
$$r=\sqrt{(0-(-5))^2+(0-12)^2}$$
$$=13$$

$$(x-a)^2+(y-b)^2=r^2$$
$$(x-0)^2+(y-0)^2=13^2$$
$$\therefore x^2+y^2=169$$

EXAMPLE 2

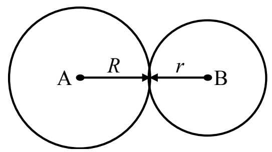
Determine the equation of the circle in the form $Ax^2+Bx+Cy^2+Dy+E=0$ that has its centre at $P(-3;1)$ and passes through $R(5;7)$.

$$PR=\sqrt{(x-a)^2+(y-b)^2}$$
$$PR=\sqrt{(5-(-3))^2+(7-1)^2}$$
$$=10$$

$$(x-a)^2+(y-b)^2=r^2$$
$$(x+3)^2+(y-1)^2=10^2$$
$$x^2+6x+9+y^2-2y+1=100$$
$$x^2+6x+y^2-2y-90=0$$

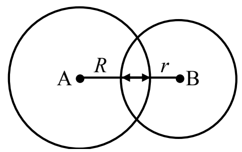
Given two circles with centres A and B respectively, it can be determined if the circles are:

a) Touching externally (one point of intersection)



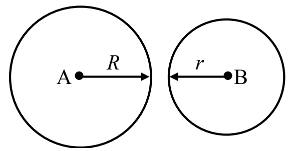
$AB=R+r$

b) Touching internally (Two points of intersection)



$AB<R+r$

c) Not touching at all



$AB>R+r$

EXAMPLE 3

Two circles are given:
$x^2-2x+y^2+6y-6=0$ with centre A, and
$(x+4)^2+(y-2)^2=9$ with centre B.

a) Express circle with Centre A in standard centre-radius form.
b) Determine whether the circles are touching externally, internally, or neither.

a)
$$x^2-2x+y^2+6y=6$$
$$x^2-2x+1+y^2+6y+9=6+1+9$$
$$(x-1)^2+(y+3)^2=16$$

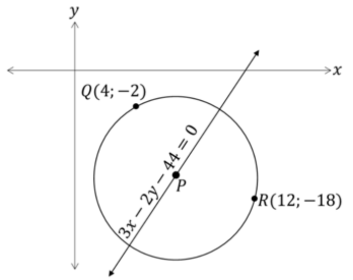
b) A(1;−3) and R=4 ; B(−4;2) and r=3

$$AB=\sqrt{(1-(-4))^2+(-3-2)^2}$$
$$=5\sqrt{2}$$
$$=7{,}07$$

AB > R + r $\therefore$ circles are NOT touching

EXAMPLE 4

In the diagram below:
Circle with Centre P is drawn. The line given by the equation $3x - 2y - 44 = 0$ passes through its centre. Points Q(4; −2) and R(12; −18) are points on the circumference.
Determine the co-ordinates of P.



$$3x - 2y - 44 = 0$$
$$-2 = -3x + 44$$
$$y = \frac{3}{2}x - 22$$

$\therefore P(x;y)$ is $P\left(x;\frac{3}{2}x - 22\right)$

$$PQ^2 = PR^2 \text{ (radii)}$$
$$(x-4)^2 + \left(\frac{3}{2}x - 22 - (-2)\right)^2 = (x-12)^2 + \left(\frac{3}{2}x - 22 - (-15)\right)^2$$
$$x^2 - 8x + 16 + \left(\frac{3x - 40}{2}\right)^2 = x^2 - 24x + 144 + \left(\frac{3x - 8}{2}\right)^2$$
$$x^2 - 8x + 16 + \frac{9x^2 - 240x + 1600}{4} = x^2 - 24x + 144 + \frac{9x^2 - 48x + 64}{4}$$
$$4x^2 - 32x + 64 + 9x^2 - 240x + 1600 = 4x^2 - 96x + 576 + 9x^2 - 48x + 64$$
$$-272x + 1664 = -144x + 640$$
$$1024 = 128x$$
$$8 = x$$

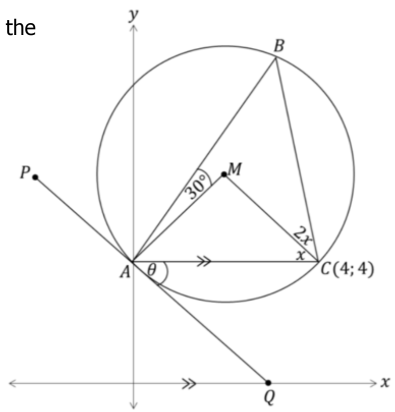
$$y = \frac{3}{2}x - 22$$
$$= \frac{3}{2}(8) - 22$$
$$= -10$$

EXAMPLE 5

Given: Circle with centre M passes through A, B and C(4; 4), PQ is a tangent to the circle at A. Q lies on the x-axis. A lies on the y-axis. AC is parallel to the x-axis.
$A\hat{C}M = x$; $B\hat{C}M = 2x$; $B\hat{A}M = 30°$; $C\hat{A}Q = \theta$

a) Determine the co-ordinates of A.
b) Prove that $x = 20°$.
c) Prove that $\theta = 70°$.

Round off your answers to TWO decimal places:
d) Determine the equation of the tangent PQ.
e) Determine the equation for radius AM.
f) Determine the co-ordinates of M.
g) Express the equation of the circle in centre-radius form.



a) (0;4)

b) $M\hat{A}C = x$ (∠'s opp equal sides; AM=CM)

$A\hat{M}C = 180° - 2x$ (int ∠'s of Δ)

$A\hat{B}C = 90° - x$ (∠ at centre = 2 x ∠ at circumference)

$$\hat{A} + \hat{B} + \hat{C} = 180° \text{ (int ∠ of Δ)}$$
$$30° + x + 3x + 90 - x = 180°$$
$$3x + 120° = 180°$$
$$3x = 60°$$
$$x = 20°$$

c) $M\hat{A}Q = 90°$ (tan ⊥ radius)

$M\hat{A}C = 20°$ (proven above)

$\therefore \theta = 70°$

d) $\tan 110° = m_{PQ}$

$= -2,75$

$y = -2,75x + 4$

e) $m_{AM} \times m_{PQ} = -1$

$m_{AM} = 0,36$

$y = 0,36x + 4$

f) $M(x;y)$ is $M(x;0,36x + 4)$

$$AM^2 = CM^2$$
$$(x-0)^2 + (0,36x + 4 - 4)^2 = (x-4)^2 + (0,36x + 4 - 4)^2$$
$$x^2 + 0,1296x^2 = x^2 - 8x + 16 + 0,1296x^2$$
$$8x = 16$$
$$x = 2$$

Construct perpendicular line from centre to chord.
∴M's x-value = 2 (⊥ line from centre to chord)

$y = 0,36(2) + 4$
$= 4,72$

$\therefore M(2;4,72)$

g) $MC = \sqrt{(2-4)^2 + (4,72-4)^2}$

$= 2,13$

$\therefore MC^2 = 4,52$

$(x+2)^2 + (y - 4,72)^2 = 4,52$

STATISTICS

REMINDER

Discrete data: Data that can be counted, e.g. the number of people.

Continuous data: quantitative data that can be measured, e.g. temperature range.

Measures of central tendency: a descriptive summary of a dataset through a single value that reflects the data distribution.

Measures of dispersion: The dispersion of a data set is the amount of variability seen in that data set.

Cumulative frequency: The total of a frequency and all frequencies so far in a frequency distribution

Variance: measures the variability from an average or mean. a Small change in the numbers of a data set equals a very small variance

Standard Deviation: the amount the data value or class interval differs from the mean of the data set.

Outliers: Any data value that is more than 1,5 IQR to the left of Q_1 or the right of Q_3, i.e.

Outlier < Q_1 – (1,5×IQR) or

Outlier > Q_3 + (1,5×IQR)

Regression: a measure of the relation between the mean value of one variable (e.g. output) and corresponding values of other variables (e.g. time and cost).

Correlation: interdependence of variable quantities

Causation: the action of causing something

Univariate: Data concerning a single variable

Bivariate: Data concerning two variables

Interpolation: an estimation of a value within two known values in a sequence of values.

Extrapolation: an estimation of a value based on extending a known sequence of values or facts beyond the area that is certainly known

REPRESENTING DATA

Ungrouped data = discrete
Grouped data = continuous

NB: Always arrange data in ascending order.

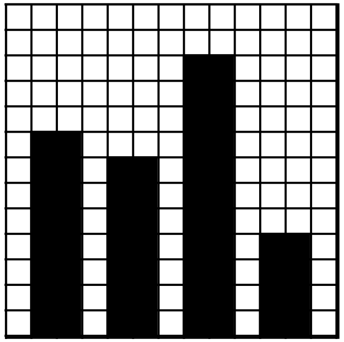
FREQUENCY TABLE

Mark	Tally	Frequency
4	||	2
5	||	2
6	||||	4
7	||||	5
8	||||	4
9	||	2
10	|	1

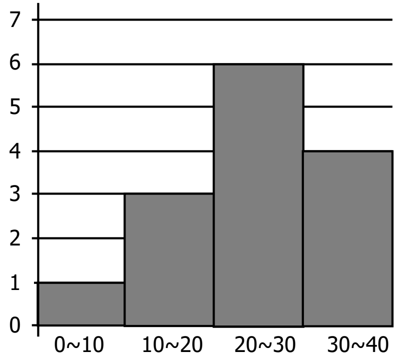
STEM AND LEAF PLOTS

Stem	Leaf
0	1, 1, 2, 2, 3, 4
1	0, 0, 0, 1, 1, 1
2	5, 5, 7, 7, 8, 8
3	0, 1, 1, 1, 2, 2
4	0, 4, 8, 9
5	2, 6, 7, 7, 8
6	3, 6

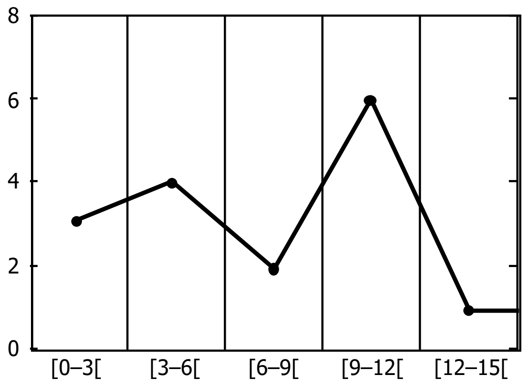
BAR GRAPH



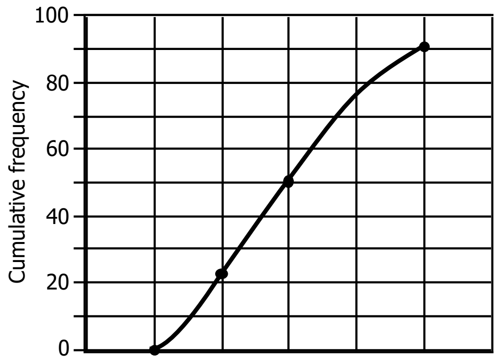
HISTOGRAM



FREQUENCY POLYGON



OGIVES



INDICATORS OF POSITION

Quartiles

The three quartiles divide the data into four quarters.

Q_1 = Lower quartile or first quartile
Q_2 = Second quartile or median
Q_3 = Upper quartile or third quartile

Percentiles

Indicates which percentage of data is below the specific percentile.

Q_1= 25th percentile
Q_2= 50th percentile
Q_3= 75th percentile

All other percentiles can be calculated using the formula:

$$i = \frac{p}{100}(n)$$

where;
i = the position of the p[th] percentile
p = the value of the i[th] position

MEASURES OF DISPERSEMENT

Range

$$\text{range} = \text{max value} - \text{min value}$$

Note: range is greatly influenced by outliers

Interquartile range

$$\text{IQR} = Q_3 - Q_1$$

Note: spans 50% of the data set

Semi-Interquartile range

$$\text{semi} - \text{IQR} = \frac{1}{2}(Q_3 - Q_1)$$

Note: good measure of dispersion for skewed distribution

STATISTICS

MEASURES OF CENTRAL TENDENCY FOR UNGROUPED DATA

Mean

$$\bar{x} = \frac{\text{sum of all values}}{\text{total number of values}}$$

$$\bar{x} = \frac{\Sigma x}{n}$$

where;

$\bar{x}$ = mean

Σx = sum of all values

n = number of values

Mode

The mode is the value that appears most frequently in a set of data points.

Bimodal: a data set with 2 modes

Trimodal: a data set with 3 modes

Median

The median is the middle number in a set of data points.

$$\text{position of median} = \frac{1}{2}(n+1)$$

Where;

n = number of values

If n = odd number, the median is part of the data set.

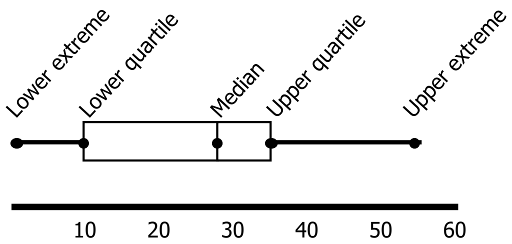
If n = even number, the median will be the average between the two middle numbers.

FIVE NUMBER SUMMARY

1. Minimum value
2. Lower quartile Q_1
3. Median
4. Upper quartile Q_3
5. Maximum value

BOX AND WHISKER PLOT

A box and whisker plot is a visual representation of the five number summary.



MEASURES OF DISPERSION AROUND THE MEAN

Variance

Variance measures the variability from an average or mean.

The variance for a population is calculated by:

1. Calculate the mean(the average).
2. Subtracting the mean from each number in the data set and then squaring the result. The results are squared to make the negatives positive. Otherwise negative numbers would cancel out the positives in the next step. It's the distance from the mean that's important, not positive or negative numbers.
3. Averaging the squared differences.

Standard deviation

Standard deviation is the amount the data value or class interval differs from the mean of the data set.

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{n}}$$

where;

n = number of data values

x = midpoint of interval

$\bar{x}$ = estimated mean

EXAMPLE:

Continuous data is grouped into class intervals which consist of an upper class boundary (maximum value) and lower class values (minimum value).

Class interval	frequency (f)	Midpoint $x = \frac{\text{uper class barrier} + \text{lower class barrier}}{2}$	$(f \times x)$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
$0 \le x \le 10$	3	$\frac{10+0}{2} = 5$	$3 \times 5 = 15$	$(5 \times \overline{15,71})^2 = 114,7$	$3(114,7) = 344,11$
$10 \le x \le 20$	7	$\frac{20+10}{15} = 15$	$7 \times 15 = 105$	$(15 \times \overline{15,71})^2 = 0,5$	$7(0,5) = 3,53$
$20 \le x \le 30$	4	$\frac{30+20}{2} = 25$	$4 \times 25 = 100$	$(25 \times \overline{15,71})^2 = 88,3$	$4(88,3) = 354,22$
total :	**14**	**14**	**220**		$\Sigma\mathbf{f}(\mathbf{x}-\bar{\mathbf{x}})^2 = \mathbf{692,86}$

Mean:

$$\text{mean}(\bar{x}) = \frac{\text{sum of all frequency} \times \text{mean value}}{\text{total frequency}}$$

$$= \frac{220}{14}$$

$$= 15,71$$

Standard deviation:

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{692,86}{14}}$$

$$= 7,03$$

A small Standard deviation tells us the numbers are clustered closely around the mean, a larger standard deviation indicates more scattered data.

STATISTICS

MEASURES OF CENTRAL TENDENCY FOR GROUPED DATA

Estimated mean

$$\text{mean}(\bar{x}) = \frac{\text{sum of all frequency} \times \text{mean value}}{\text{total frequency}}$$

where;

$\bar{x}$ = estimated mean

n = number of values

Modal class interval

The modal class interval is the class interval that contains the greatest number of data points.

Median class interval

The median class interval is the interval that contains the middle number in a set of data points.

$$\text{position of median} = \frac{1}{2}(n+1)$$

Where;

n = number of values

If n = odd number, the median is part of the data set.

If n = even number, the median will be the average between the two middle numbers.

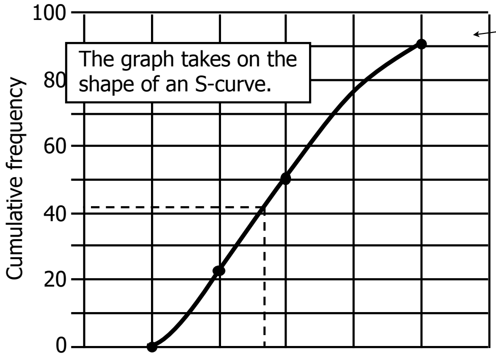
EXAMPLE:

Step 1: Determine cumulative frequencies form a frequency table.

We conduct a survey on the ages of people who visit the corner shop, 80 people partake in the survey.

Class interval	Frequency	Cumulative frequency	Interpretation	Graph points
$0 \leq x < 15$	0	0	0 participants are younger than 15.	(15;0)
$15 \leq x < 30$	14	0 + 14 = **14**	14 people were younger than 30.	(30;14)
$30 \leq x < 45$	22	14 + 22 = **36**	36 people were younger than 45.	(45;36)
$45 \leq x < 60$	30	36 + 30 = **66**	66 people were younger than 60.	(60;66)
$60 \leq x < 75$	14	66+ 14 = **80**	All participants were younger than 75.	(75;80)

Step 2: Represent information on a cummulative frequency/ogive curve



Coordinates (x;y)

The x-coordinate represents the upper boundary of the class interval.

y-coordinate represents the cumulative frequency.

Interpretations from the graph:

Median

There is an even nr of data items in our set (80) so the median liesmidway between the two middle values. The median is halfway between the 40th and 41st term. Find the value on the y-axis and draw a line from that point to determine the value on the x-axis.

Quartiles

Similar to the method used to find the median you can determine the upper or lower quartiles from the graph.

Percentiles

The median and quartiles divide the data into 50% and 25% respectively, should you need to calculate a different percentile this can be done by calculation or read from the graph.

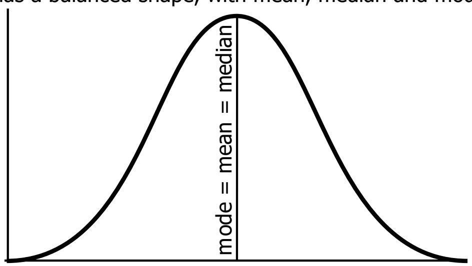
Calculation of the 90th percentile: $0,9 \times 80 = 72$

So 90% of the data is below the 72nd value which will be int the last class interval.

SYMMETRIC AND SKEWED DATA

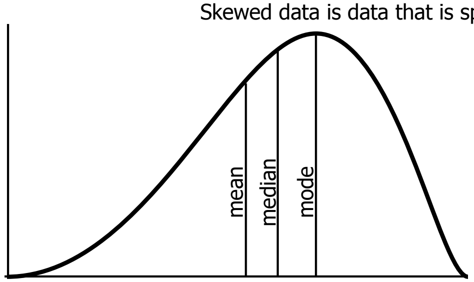
Symmetric

Symmetric data has a balanced shape, with mean, median and mode close together.

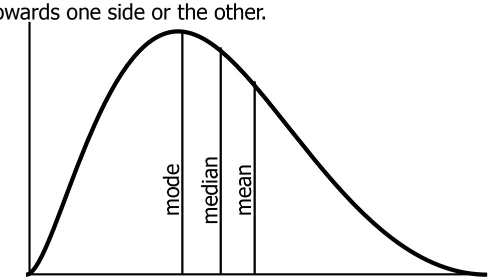


Skewed

Skewed data is data that is spread more towards one side or the other.



Skewed left: Negatively skewed if the tail extends to the left



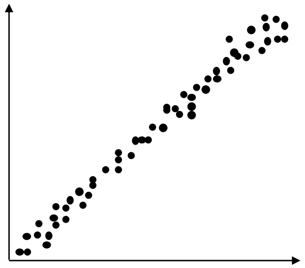
Skewed right: Positively skewed if the tail extends to the right

STATISTICS

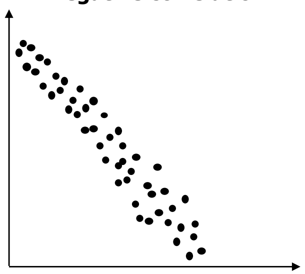
BIVARIATE DATA

Bivariate data can be represented by scatter plots:

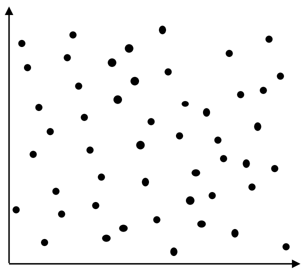
Positive linear correlation



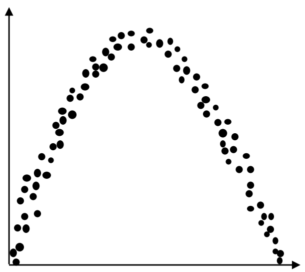
Negative correlation



No correlation



Non linear correlation



CORRELATION COEFFICIENT

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2} \times \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

Where r indicates the strength of the relationship between the two variables (x and y).

Properties:

- The correlation coefficient is a number between -1 and 1 ($-1 \leq r \leq 1$)
- Strong positive linear correlation, r is close to 1.
- Strong negative linear correlation, r is close to -1.
- No linear correlation or a weak linear correlation, $-0{,}3 < r < 0{,}3$

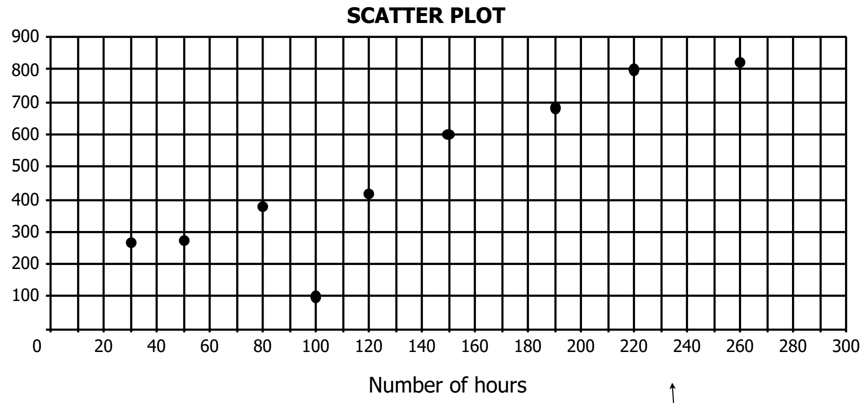
Value of r	Meaning
$r = 1$	Perfectly positive correlation
$0{,}9 \leq r < 1$	Very strong positive linear correlation
$0{,}7 \leq r < 0{,}9$	Significant positive linear correlation
$0{,}3 \leq r < 0{,}7$	Weak positive linear correlation
$-0{,}3 \leq r < 0{,}3$	No significant linear correlation
$-0{,}7 \leq r < -0{,}3$	Weak negative linear correlation
$-0{,}9 \leq r < -0{,}7$	Significant negative linear correlation
$-1 \leq r < -0{,}9$	Very strong negative linear correlation
$r = -1$	Perfect negative correlation

EXAMPLE:

Draw a scatter plot for the following data and calculate the correlation coefficient, then write down a conclusion about the type of correlation. Number of hours a sales person spends with his client vs the value of the sales for that client.

NUMBER OF HOURS	30	50	80	100	120	150	190	220	260
VALUE OF SALES (IN THOUSANDS OF RANDS)	270	275	376	100	420	602	684	800	820

Step 1: Scatter plot



Step 2: Using your calculator to find r

Once you understand the reasons for the process you can use your calculator to streamline the process.

r can be found by completing the following steps on your calculator:
(steps may vary slightly for different calculators)

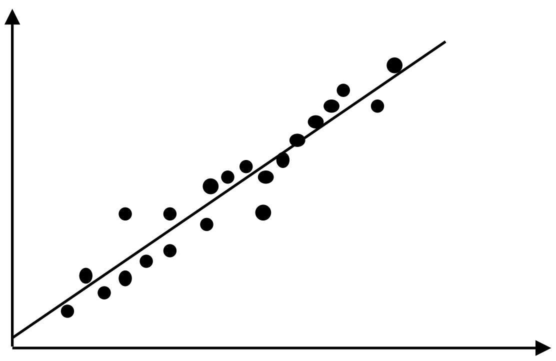
- Press [MODE] and [2:STAT] to enter stats mode
- [2:a+bx]
- Enter the number of hours in the x-column and the value of sales in the y-column
- [AC], this will clear the screen, but the data remains stored
- [SHIFT][1] to get the stats computation screen
- [5:REG] and then [3:r]

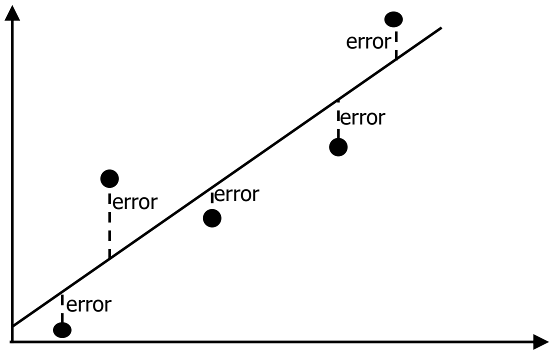
$r = 0{,}893$

∴ a significant positive linear correlation linear correlation exists between the number of hours a sales person as can be seen on the scatterplot- with the exception of the outlier.

THE LEAST SQUARES REGRESSION LINE

A line can be drawn through the points to determine if there is a significant negative or positive correlation on a scatter plot. A line of best fit will not represent the data perfectly, but it will give you an idea of the trend. The line of best fit is also called a regression line. This line will always pass through $(\bar{x};\bar{y})$ where $\bar{x}$ is the mean of the x-values and $\bar{y}$ is the mean of the y-values.





The aim of the least squares regression line is to make the total of the square of the errors as small as possible. The straight line minimizes the sum of squared errors. when we square each of those errors and add them all up, the total is as small as possible.

EXAMPLE:

Determine the equation of the least square regression line in the form $\hat{y} = a + bx$

Step 1: Complete the table as indicated by the column headings:

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(x-\bar{x})(y-\bar{y})$
1	2	–2	–2	4	4
2	4	–1	0	1	0
3	5	0	1	0	0
4	4	1	0	1	0
5	5	2	1	4	2
$\bar{x}$=3	$\bar{y}$=4			$\Sigma(x-\bar{x})^2$=10	$\Sigma(x-\bar{x})(y-\bar{y})$=6

Step 2: Calculate Slope/gradient b by substituting the totals of your table as needed:

$$b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

$$b = \frac{6}{10}$$

$$b = 0{,}6$$

Step 3: Calculate the y-intercept a by substituting the values of the point $(\bar{x};\bar{y})$:

$$a = \bar{y} - b\bar{x}$$

$$a = 4 - (0{,}6)(3)$$

$$a = 2{,}2$$

Step 4: Assemble the equation of a line

$$\hat{y} = a + bx$$

$$\hat{y} = 2{,}2 + 0{,}6x$$

Once you understand the process, these values can be found faster by completing the following steps on your calculator: (steps may vary slightly for different calculators)

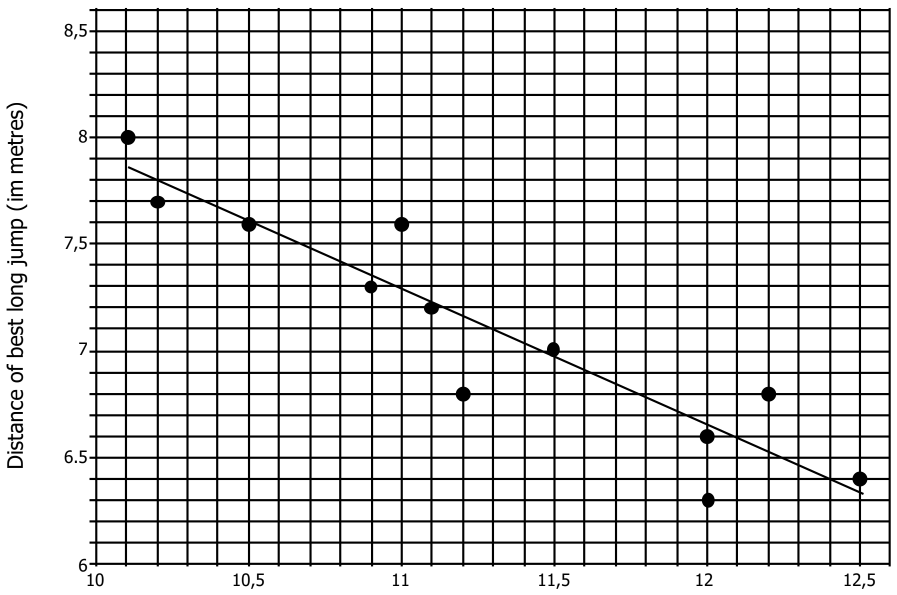
- Press [MODE] and [2:STAT] to enter stats mode
- [2:a+bx]
- Enter the values in the x-and y-column respectively
- [AC], this will clear the screen, but the data remains stored
- [SHIFT][1] to get the stats computation screen
- [5:REG]
- To find a choose the option [1:a] OR [2:B] to find the value of b

EXAMPLE:

The table below shows the time taken by 12 athletes to run 100m sprint and their best distance for log jump.

TIME FOR 100 M SPRINT (IN SECONDS)	10,1	10	11	11	11	11	11	12	12	12	12	13
DISTANCE OF BEST LONG JUMP (IN METRES)	8	7,7	7,6	7,3	7,6	7,2	6,8	7	6,6	6,3	6,8	6,4

SCATTER PLOT



1. Determine the a and b values for the least squares regression line.

This can be found by completing the following steps on your calculator:
(steps may vary slightly for different calculators)

- Press [MODE] and [2:STAT] to enter stats mode
- [2:a+bx]
- Enter the time values in the x-column and the distances in the y-column
- [AC], this will clear the screen, but the data remains stored
- [SHIFT][1] to get the stats computation screen
- [5:REG]
- To find a choose the option [1:a] OR [2:B] to find the value of b

$$a = 14{,}34\ ;\ b = -0{,}64$$

2. An athlete runs the 100m in 11,7 seconds, use the formula to predict the distance of this athlete's jump.

$\hat{y} = 14{,}34 - 0{,}64x$
$\hat{y} = 14{,}34 - 0{,}64(11{,}7)$
$\hat{y} = 6{,}852\ m$

3. Another athlete completes the 100m sprint in 12,3 seconds and his best jump is 7,6m. If this is included in the data will the gradient of the least squares regression line increase or decrease? Motivate your answer without using calculations.

The gradient will increase, the distance point will be much higher than the ones around that time.

4. Calculate the mean time and standard deviation for the data set.

This can be found by completing the following steps on your calculator:
(steps may vary slightly for different calculators)

- Press [MODE] and [2:STAT] to enter stats mode
- [1:Var]
- Enter the time values in the x-column
- [AC], this will clear the screen, but the data remains stored
- [SHIFT][1] to get the stats computation screen
- [4:Var]
- [2:$\bar{x}$] and = to find the mean time, which is 11,27 seconds OR
- [3:σx] and = to find the standard deviation for the sample. The standard deviation from the mean time is 0,755 seconds.

5. Find the correlation coefficient

- Press [MODE] and [2:STAT] to enter stats mode
- [2:a+bx]
- Enter the time values in the x-column and the distances in the y-column
- [AC], this will clear the screen, but the data remains stored
- [SHIFT][1] to get the stats computation screen
- [5:REG] and then [3:r]

r = –0.926

∴This is a significant negative linear correlation.