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Department:
Education
KWAZULU-NATAL

MATHEMATICS

LEARNER ASSISTANCE REVISION DOCUMENT

GRADE 11

2020

This document has been compiled by the FET Mathematics Subject Advisors together with Lead Teachers. It seeks to unpack the contents and to give more guidance to teachers.

TABLE OF CONTENTS

ALGEBRA EQUATIONS AND INEQUALITIES.

This section accounts for ±25 marks in Paper 1.

If understood properly, it also increases marks in other sections, e.g. Functions

- This section must be done really well when preparing for all the next examinations, namely **JUNE, PREPARATORY AND FINAL EXAMINATIONS**
- Learn **solving for *x* in a quadratic equation**, which must be in standard form.
  (use formula all the time if you struggle with inspection). ***Do this daily if you are not getting it right!***
- Basic understanding of **LAWS OF EXPONENTS** is important when manipulating equations with exponents.
- Simultaneous equations must be learnt over and over.
- Inequalities must be learnt (to the very least...find critical values).

ALGEBRA, EQUATIONS AND INEQUALITIES – CONCEPTS INVOLVED	
TYPES AND GROUPS OF QUESTIONS:	
• a × b = 0 standard form • factorization: any method • transposing • formula • rounding off to 2 decimal places	• surds: squaring both sides validity of the roots (checking and verifying) • inequalities: algebraically graphically table method • simultaneous equations: subject of the formula substitution

TOPIC	ACTIVITY
Factorisation	Common factor, solve
	Transpose, factorise, solve
	Remove brackets, transpose, factorise, solve
Quadratic formula	Formula, substitution, answers correct to 2 decimal places/surd form
	Remove brackets, transpose/standard form, correct to 2 decimal places/surd form
Surds	Square both sides, solve, validate
	Transpose, square both sides, solve, validate
Simultaneous equations	Subject of the formula, substitution, standard form, factorise, solve, substitution for the other variable

Exponents	Same base, equate exponents, solve
	Laws of exponents, write as a power, factorise, equate exponents, solve
	Split, factorise, simplify
	Same bases, laws of exponents, equate exponents, solve
Inequalities	Critical values, sketch, answer
	Standard form, factorise, critical values, sketch, solve
	Remove brackets, standard form, factorise, critical values, sketch, solve
Nature of the roots	Use $b^2 – 4ac \geq 0$, solve
	Formula, substitution, $b^2 – 4ac \geq 0$, solve
	Rational, irrational, real/non-real

SECTION	CONCEPT	EXAMPLE
Algebra, Equations & Inequalities (± 25)	**Factorisation**	1) $x^2 - 6x = 0$
		2) $(x-4)(x+2) = 0$
		3) $x - 3 = \frac{4}{x}$
		4) $3x^2 - 5x - 2 = 0$ $(where\ \ a\ is\ greater\ than\ 1)$
	Quadratic formula	1) $2x^2 + 3x - 1 = 0$ (ans corr to 2 decimal digits)
		2) $2x^2 + 3x - 1 = 0$ (ans in simplest surd form)
	Inequalities	1) $(x-4)(x+2) > 0$
		2) $(x+1)(2-x) < 0$
		3) $3^x(x-5) < 0$
		4) $x^2(x+5) < 0$
		5) $3x^2 - 5x - 2 \geq 0$ (for both a $>$ 0 and a $<$ 0)
	Exponential Equations	1) $2x^{-\frac{5}{3}} = 64$
		2) $2^{x+2} + 2^x = 20$
		3) $2.3^x = 81 - 3^x$
	Surds	1) $\sqrt{x+1} = x - 1$
		2) $2 + \sqrt{2-x} = x$
	Simultaneous Equations	1) $y = x^2 - x - 6$ and $2x - y = 2$
		2) $2x - y + 1 = 0$ and $x^2 - 3x - 4 = y^2$
		3) $3^{x-10} = 3^{3x}$ and $y^2 + x = 20$

EXAMPLES:

Example 1

$$(x-3)(x+5)=9$$
$$x^2+5x-3x-15-9=0$$
$$x^2+2x-24=0$$
$$(x+6)(x-4)=0$$
$$x=-6 \text{ or } x=4$$

Example 2

Solve for x:

$$\sqrt{x-2}+x=4$$
$$\sqrt{x-2}=4-x$$
$$x-2=(4-x)^2$$
$$x-2=16-8x+x^2$$
$$x^2-8x+16-x+2=0$$
$$x^2-9x+18=0$$
$$(x-3)(x-6)=0$$
$$x=3 \quad \text{or} \quad x=6$$

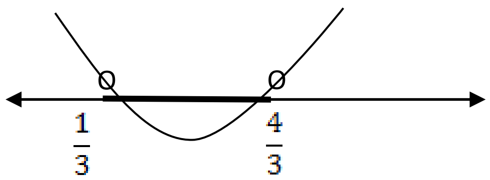
after checking both solutions

$x=3$ is the only solution

Example 3

Solve for x:

$$15x-4>9x^2$$
$$15x-4-9x^2>0$$
$$9x^2-15x+4<0$$
$$(3x-1)(3x-4)<0$$



$$\tfrac{1}{3}<x<\tfrac{4}{3}$$

PRACTICE EXERCISES

QUESTION 1

1.1. Solve for x

1.1.1. $x(x-4)=5$ (3)

1.1.2. $4x^2-20x+1=0$ (round off your answer correct to 2 decimal places) (4)

1.1.3. Solve simultaneously for x and y in the following system of equations:

$y-x+3=0$

$x^2-x=6+y$ (6)

QUESTION 2

2.1. Solve for x

2.1.1. $x^2-5x=-6$ (3)

2.1.2. $(3x+1)(x-4)<0$ (3)

2.1.3. $2x+\sqrt{x+1=1}$ (4)

2.1.4. $12^{5+3x}=1$ (2)

2.2. Solve for x and y

$2x-y=8$

$x^2-xy+y^2=19$ (7)

QUESTION 3

3.1. Solve for x

3.1.1. $(x+2)^2=3x(x-2)$ Giving your answer correct to one decimal digit (5)

3.1.2. $x^2-9x\geq 36$ (4)

3.1.3. $3^x-3^{x-2}=72$ (4)

3.2. Given $(2m-3)(n+5)=0$

Solve for:

3.2.1. n if $m=1$ (1)

3.2.2. m if $n\neq -5$ (1)

3.2.3. m if $n=-$ (2)

QUESTION 4

4.1. Solve for x

4.1.1. $(x-3)(x+1)=5$ (3)

4.1.2. $9^{2x-1} = \frac{3^x}{3}$ (3)

4.1.3. $2\sqrt{2-7x} = \sqrt{-36x}$ (4)

QUESTION 5

5.1. Solve for x:

5.1.1. $10x = 3x^2 - 8$ (3)

5.1.2. $x + \sqrt{x-2} = 4$ (5)

5.1.3. $x(2x-1) \geq 15$ (5)

5.2. Given: $P = \frac{4^{x+3} + 4^x}{8^{x+2} + 8^x}$

5.2.1. Simplify P (3)

5.2.2. Hence solve for x: P = 3 (2)

5.3. State whether the following numbers are rational, irrational or non-real.

5.3.1. $\sqrt{3}$ (1)

5.3.2. $\frac{22}{7}$ (1)

5.3.3. The roots of $x^2 + 4 = 0$ (1)

QUESTION 6

6.1 Solve for x:

6.1.1 $2x^2 + 11 = x + 21$ (3)

6.1.2 $3x^3 + x^2 - x = 0$ (5)

6.1.3 $2x + p = p(x + 2)$, stating any restriction (4)

6.1.4 $x^{-1} - x^{-\frac{1}{2}} = 20$

6.2. Solve for x and y simultaneously in the following equations

$2x^2 - 3xy = -4$ and $4^{x+y} = 2^{y+4}$ (6)

QUESTION 7

7.1. Solve for x. Leave the answer in the simplest surd form where necessary

7.1.1. $(2x + 5)(x^2 - 2) = 0$ (3)

7.1.2. $x^2 - 4 \geq 5$ (4)

7.1.3. $12^{2x} = 8.36^x$ (4)

7.2. Solve for x, correct to two decimal places:

$2(x+1)^2 = 9$ (4)

7.3. Solve for x and y simultaneously:

$y = -2x + 7$ and $\frac{y+5}{x-1} = \frac{1}{2}$ (4)

QUESTION 8

8.1. Given $x^2 + 2x = 0$

8.1.1. Solve for x (2)

8.1.2. Hence, determine the positive values of x for which $x^2 \geq -2x$ (3)

8.2. Solve for x:

$2x^2 - 3x - 7 = 0$ (correct to two decimal places) (4)

8.3. Given $k+5 = \frac{14}{k}$

8.3.1. Solve for k (3)

8.3.2. Hence, or otherwise, solve for x if $\sqrt{x+5} + 5 = \frac{14}{\sqrt{x+5}}$ (3)

8.4. Solve for x and y simultaneously if:

$x - 2y - 3 = 0$ *and*

$4x^2 - 5xy + y^2 = 0$ (7)

8.5. The roots of a quadratic equation is given by $x = \frac{-2\pm\sqrt{4-20k}}{2}$

Determine the values of k for which the equation will have real roots (2)

QUESTION 9

9.1 Solve for x

9.1.1 $2x^2 - 5x - 3 = 0$ (2)

9.1.2 $(x-3)(x-4) \geq 12$ (5)

9.2 Consider: $5x - \frac{3}{x} = 1$

9.2.1 Solve for x correct to two decimal places. (5)

1.2.2 Hence, determine the value of y if $5(2y+1) - \frac{3}{2y+1} = 1$. (3)

9.3 Solve simultaneously for x and y in the following set of equations:

$y = x - 1$ and $y + 7 = x^2 + 2x$ (5)

9.4 Calculate the value(s) of m if the roots of $3mx^2 - 7x + 3 = 0$ are equal. (4)

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QUESTION 10

10.1 Solve for x in each of the following:

10.1.1 $x(2x+5)=0$ (2)

10.1.2 $2x^2-3x=7$ (Give answer correct to TWO decimal places) (4)

10.1.3 $x-7-\sqrt{x-5}=0$ (4)

10.1.4 $\frac{1}{2}x(3x+1)<0$ (2)

10.2 Solve for x and y simultaneously:

$$2x+y=3 \quad and$$
$$x^2+y+x=y^2$$ (6)

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QUESTION 11

11.1 Solve for x:

11.1.1 $4x^2=81$ (2)

11.1.2 (a) $x^2-5x=2$, correct to TWO decimal places. (4)

(b) Hence, or otherwise, solve $(x^2-2)^2-5(x^2-2)-2=0$ (3)

11.1.3 $(2-x)(x+4)\geq 0$ (3)

11.1.4 $3^{x+1}-4+\frac{1}{3^x}=0$ (5)

11.2 Solve for x and y simultaneously:

$x+y=3$

$2x^2+2y^2=5xy$ (6)

QUESTION 12

12.1 Solve for x:

12.1.1 $3x^2+10x+6=0$ (correct to TWO decimal places) (3)

12.1.2 $\sqrt{6x^2-15}=x+1$ (5)

12.1.3 $x^2+2x-24\geq 0$ (3)

12.2 Solve simultaneously for x and y:

$5x + y = 3$ and $3x^2 - 2xy = y^2 - 105$ (6)

12.3 12.3.1 Solve for p if $p^2 - 48p - 49 = 0$ (3)

12.3.2 Hence, or otherwise, solve for x if $7^{2x} - 48(7^x) - 49 = 0$ (3)

QUESTION 13

13.1 Solve for x:

13.1.1 $x^2 + 9x + 14 = 0$ (3)

13.1.2 $4x^2 + 9x - 3 = 0$ (correct to TWO decimal places) (4)

13.1.3 $\sqrt{x^2 - 5} = 2\sqrt{x}$ (4)

13.2 Solve for x and y if:

$3x - y = 4$ and $x^2 + 2xy - y^2 = -2$ (6)

13.3 Given: $f(x) = x^2 + 8x + 16$

13.3.1 Solve for x if $f(x) > 0$. (3)

NUMBER PATTERNS

Quadratic Sequences

Examples

1. Consider the sequence: 5 ; 18 ; 37 ; 62 ; 93 ; …

1.1 If the sequence behaves consistently, determine the next TWO terms of the sequence.

1.2 Calculate a formula for the *n*th term of the sequence.

1.3 Use your formula to calculate *n* if the n^{th} term in the sequence is 1 278.

Worked Solution

1.1 130 ; 173

1.2 5 18 37 62 93

13 19 25 31 sequence of first difference

6 6 6 second difference is constant

The second difference is constant $\therefore\ T_n$ is quadratic $\therefore an^2+bn+c=T_n$

$2a=6$
$a=3$ $\quad\therefore\quad T_n=3n^2+bn+c$

$$5=3(1)^2+b(1)+c$$

$$b+c=2 \qquad \ldots(1)$$

$$18=3(2)^2+b(2)+c$$

$$2b+c=6 \qquad \ldots(2)$$

$(2)-(1)$: $b=4$

$$c=-2$$

$$\therefore\ T_n=3n^2+4n-2$$

1.3 $\quad 3n^2+4n-2=1278$

$$3n^2+4n-1280=0$$

$$(3n+64)(n-20)=0$$

$$n=\frac{-64}{3} \text{ or } n=20$$

$$n=\frac{-64}{3} \text{ is not valid } \therefore n=20$$

Exercise 4

1. Given the quadratic sequence: $-1\,;-7\,;-11\,;p\,;\ldots$
   - 1.1 Write down the value of p.
   - 1.2 Determine the n^{th} term of the sequence.
   - 1.3 The first difference between two consecutive terms of the sequence is 96. Calculate the values of these two terms.
2. Given the following quadratic sequence: $-2\,;0\,;3\,;7\,;\ldots$
   - 2.1 Write down the value of the next term of this sequence.
   - 2.2 Determine an expression for the n^{th} term of this sequence.
   - 2.3 Which term of the sequence will be equal to 322?
3. Look at the following sequence and answer the questions that follow:

   10 ; 21 ; 38 ; 61 ; ....
   - 3.1 Determine the type of sequence.
   - 3.2 Determine the general term.

3.3 Which term has a value of 1 245?

ANSWERS TO EXERCISES

Exercise 4

1.1 $p = 13$

1.2 $T_n = n^2 - 9n + 7$

1.3 $n = 52 \quad T_{52} = 2243$

2.1 The next term of the sequence is 12

2.2 $T_n = \frac{1}{2}n^2 + \frac{1}{2}n - 3 \; S_{500} = 31000$

2.3 The 25th term has a value of 322.

EXAMINATION QUESTIONS FROM PAST PAPERS

QUESTION 1

1.1 Given the sequence 3; 6; 13; 24; ….

1.1.1 Derive the general term of this sequence. (4)L2

1.1.2 Which term of this sequence is the first to be greater than 500. (5)L3

QUESTION 2

Given: 1; 11; 26; 46; 71; …….

2.1 Determine the formula for the general term of the sequence. (4)L2

2.2 Which term in the sequence has a value of 521? (4)L2

QUESTION 3

The sequence 3 ; 9 ; 17 ; 27 ; … is quadratic.

3.1 Determine an expression for the *n*-th term of the sequence. (4)L2

3.2 What is the value of the first term of the sequence that is greater than 269? (4)L3

QUESTION 4

Given the quadratic sequence: 5 ; 7 ; 13 ; 23 ; . . .

4.1 Calculate the nth term of the quadratic sequence. (4) L2

4.2 Determine between which two consecutive terms of the quadratic sequence the first difference will be equal to 2018. (3)L3

QUESTION 5

5.1 The above sequence 3; b; 19; 27; ... forms the first differences of a quadratic sequence. The first term of the quadratic sequence is 1.

5.1 Determine the fourth term(T_4) of the quadratic sequence. (2)L3

5.2 Determine the n^{th} term of the quadratic sequence. (4)L2

5.3 Calculate the value of n if $T_n - 1 = 7700$ (3)L2

QUESTION 6

Given the quadratic sequence: 4; 4; 8; 16; . . .

6.1 Calculate the n^{th} term of the quadratic sequence. (4)L1

6.2 Between which two consecutive terms of the quadratic sequence, will the first difference be equal to 28088? (4)L3

QUESTION 7

Given the quadratic sequence: 3; 5; 11; 21; x

7.1 Write down the value of x. (1)L1

7.2 Determine the value of the 48th term. (5)L2

7.3 Prove that the terms of this sequence will never consist of even numbers. (2)L3

7.4 If all the terms of this sequence are increased by 100, write down the general term of the new sequence. (2)L2

QUESTION 8

8.1 In a quadratic pattern, with $T_n = an^2 + bn + c$, the second term is equal to 8 and the first differences of the quadratic sequence are given as: $6; 12; 18; \ldots\ldots\ldots\ldots$

8.1.1 Write down the values of the first four terms of the quadratic sequence. (3)L2

8.1.2 Calculate the value of T_{40} of the quadratic sequence. (5)L3

QUESTION 9

9.1 A quadratic number pattern $T_n = an^2 + bn + c$ has a third term equal to -1, while the first differences of the quadratic sequence are given by: $-12; -8; -4$

9.1.1 Write down the values of the first four terms of the quadratic sequence. (2)L2

9.1.2 Calculate the value of a, b and c. (3)L2

QUESTION 10

The first four terms of a quadratic sequence are 9 ; 19 ; 33 ; 51 ;...

10.1 Write down the next TWO terms of the quadratic sequence. (2)L1

10.2 Determine the n^{th} term of the sequence. (4)L1

10.3 Prove that all the terms of the quadratic sequence are odd. (3)L3

QUESTION 11

Given the quadratic sequence 1; 6; 15; 28; …

11.1 Write down the second difference. (1)L1

11.2 Determine the *n*th term. (4)L1

11.3 Calculate which term of the sequence equals 2701. (3)L2

QUESTION 12

The first four terms of a quadratic sequence are $8 ; 15 ; 24 ; 35 ;...$

12.1 Write down the next TWO terms of the quadratic sequence. (1)L1

12.2 Determine the n^{th} term of the sequence. (4)L2

12.3.1 Write down the values of x and y in terms of k. (2)L2

12.3.2 Hence, calculate the value of $T_x + T_y$ in terms of k in simplest form. (4)L3

FUNCTIONS

1. FUNCTION AND MAPPING NOTATION

In Grade 10 learners were introduced to different ways of representing functions.

The different notations are summarised below:

- $y = \ldots..$ equation notation
- $f(x) = \ldots..$ function notation
- $f : x \rightarrow \ldots..$ mapping notation

2. INTERCEPTS WITH THE AXES

To determine the x – intercept(s), substitute $y = 0$.
For example: If $f(3) = 0$, then the function has an x intercept at $(3;\ 0)$.

To determine the y – intercept(s), substitute $x = 0$.
For example: If $f(0) = 4$, then the function has a y intercept at $(0;\ 4)$.

3. GRAPH INTERPRETATION

3.1 Axes of symmetry:

If a function has a line of symmetry, it means that the function is a mirror image of itself about that line. In other words, if the graph was folded along the line of symmetry, it would duplicate itself on the other side of the line.

3.2 Asymptotes:

Asymptotes are imaginary lines that a graph approaches, but never touches or cuts.

3.3 Domain and range:

Domain: The domain refers to the set of possible $\boldsymbol{x} - \textbf{values}$ for which a function is defined.

Range: The range refers to the set of possible $\boldsymbol{y} - \textbf{values}$ that the function can assume.

4. BASELINE ACTIVITY

For each of the following functions

(I) Sketch the graph of the function.

(II) Determine the domain and the range of each function.

(III) Determine the equation of the axes of symmetry and asymptotes, where applicable.

Straight Line

1.1 $y = x + 3$

1.2 $y = (x - 2)$

1.3 $y = -x$

1.4 $y = -2x - 3$

Parabola

2.1 $y = x^2 + 3$

2.2 $y = (x - 2)^2$

2.3 $y = -x^2$

2 .4 $y = -2x^2 - 3$

Hyperbola

3.1 $y = \frac{6}{x} + 3$

3.2 $y = \frac{6}{x-2}$

3.3 $y = -\frac{6}{x}$

3.4 $y = \frac{-2}{x} - 3$

Exponential graph

4.1 $y = 2^x + 3$

4.2 $y = 2^{x-2}$

4.3 $y = 2^{-x}$

4.4 $y = -(2)^x - 3$

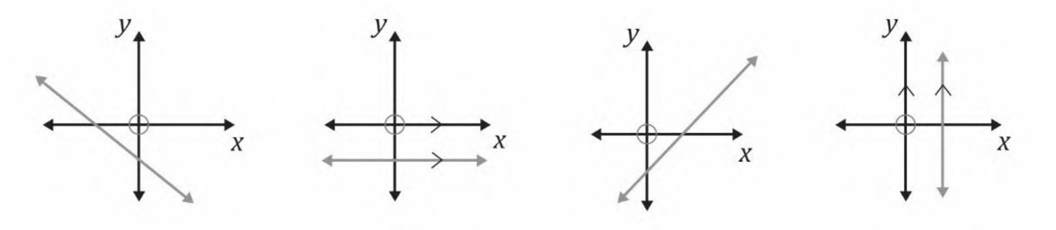
BASIC INFORMATION ON THE DIFFERENT TYPES OF GRAPHS

A. STRAIGHT LINE

General representation or equation:

$y = ax + q$ or $f(x) = mx + c$, a or m is the gradient, and q or c is the $y-$intercept .

Also note the shape of the following linear functions:



$a < 0$ $a = 0$ $a > 0$ a is undefined

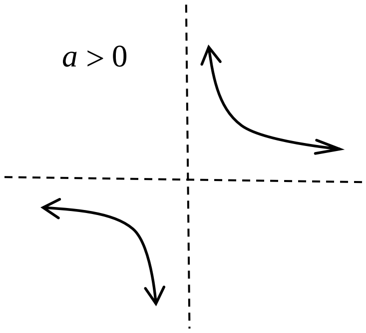
$q < 0$ $y = q$ $q < 0$ there is no q-value

For all the linear functions, except horizontal and vertical lines, the domain is $x \in R$, and the range is $y \in R$.

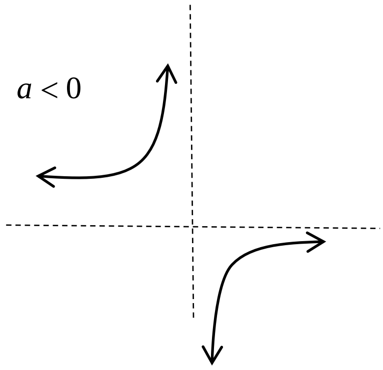
B. HYPERBOLA

General representation or equation:

$$y = \frac{a}{x + p} + q$$



Dotted lines are asymptotes



Dotted lines are asymptotes

- The value of q represents the vertical translation (shift) from the *x*-axis.
- The value of p represents the horizontal translation (shift) from the *y*-axis.
- In the case of $y=\frac{a}{x}$, $p=0$ and $q=0$. The **vertical** asymptote is the *y*-axis ($x=0$) and the **horizontal** asymptote is the *x*-axis ($y=0$). The **axes of symmetry** are $y=x$ (+ve gradient) and $y=-x$ (-ve gradient).
The **domain** is $x \in R,\ x \neq 0$; and the r**ange** is $y \in R, y \neq 0$.
- In the case of $y=\frac{a}{x}+q$, $p=0$. The **vertical** asymptote is the *y*-axis ($x=0$) and the **horizontal** asymptote is $y=q$. The **axes of symmetry** are $y=x+q$ (+ve gradient) and $y=-x+q$ (-ve gradient). The d**omain** is $x \in R,\ x \neq 0$; and the r**ange** is $y \in R,\ y \neq q$.
- In the case of $y=\frac{a}{x+p}+q$, the **vertical** asymptote is $x=-p$ and the **horizontal** asymptote is $y=q$.
The **axes of symmetry** are $y=\pm(x+p)+q$. The d**omain** is $x \in R,\ x \neq -p$ and the **range** is $y \in R, y \neq q$.
- Alternative method to determine the equations of the axes of symmetry:
In all cases the one axis of symmetry has a gradient of $+1$ and the other a gradient of -1. Therefore the equations of the axes of symmetry are $y=x+c$ and $y=-x+c$. In all cases the value of c may be determined by simply substituting the coordinates of the point of intersection of the two asymptotes into the above equations – since the axes of symmetry always pass through this point.

Example no. 1:

Given : $f(x)=\frac{3}{x-2}+1$

1.1. Write down the equations of the asymptotes of f.

1.2. Determine the coordinates of B, the x-intercept of *f*.

1.3. Determine the coordinates of D, the y-intercept of f.

1.4. Determine the domain and the range of f.

1.5. Determine the equations of the two axes of symmetry of f.

1.6. Draw a sketch graph of f.

Solution:

1.1 For the vertical asymptote:

$$x-2=0$$

$$x=2$$

Horizontal asymptote:

$$y=1$$

1.2 For the x – intercept, substitute $y=0$:

$$\frac{3}{x-2}+1=0$$

$$\frac{3}{x-2}=-1$$

$$-1(x-2)=3$$

$$x=-1$$

1.3 For the y – intercept, substitute $x=0$:

$$y=\frac{3}{-2}+1=\frac{3-2}{-2}=-\frac{1}{2}$$

1.4 Domain is $x \in R;\ x \neq 2$

Range is $y \in R;\ x \neq 1$

1.5 Point of intersection of asymptotes: (2 ; 1)

Axis of symmetry with positive gradient:
Substitute (2 ; 1) into $y=x+c$:

$$1=2+c$$

$$c=-1$$

$$y=x-1$$

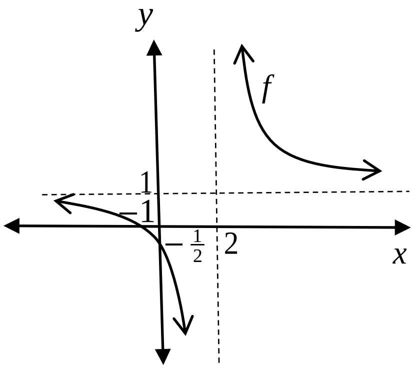
Axis of symmetry with negative gradient:
Substitute (2 ; 1) into $y=-x+c$:

$$1=-2+c$$

$$c=3$$

$$y=-x+3$$

1.6)



C. PARABOLA

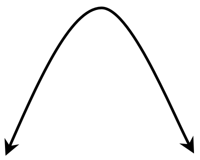
Defining Equation:

$y = a(x+p)^2 + q$ or $y = ax^2 + bx + c$ or $y = a(x - x_1)(x - x_2)$

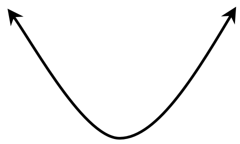
Sketching a parabola:

Shape

for $a < 0$



for $a > 0$



For $y = ax^2 + bx + c$, the **turning point** is $\left(\frac{-b}{2a} ; f\left(\frac{-b}{2a}\right)\right)$ and the **y-intercept** is $y = c$.

Given: $y = ax^2 + bx + c$

***y*-intercept** : $(0 ; c)$

Turning point (TP) :

$x = \frac{-b}{2a}$ (the axis of symmetry)

Substitute this value into the equation to find the y-coordinate of the TP, i.e. the minimum or maximum value.

Given: $y = a(x+p)^2 + q$

Multiply out the expression to get it in the form $y = ax^2 + bx + c$

***y*-intercept**:$(0 ; c)$

Turning Point (TP): $(-p ; q)$

If there are ***x*-intercepts**: Let $y = 0$ and solve for x (factorise or use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$).

- If $a < 0$, the function has a **maximum** value, represented by the ***y*** value of the turning point.
- If $a > 0$, the function has a **minimum** value, represented by the ***y*** value of the turning point.
- The equation of the axis of symmetry is given by $x = \frac{-b}{2a}$, (is the x value of the turning point)
- The **domain** is $x \in R$
- The **range: If $a > 0$** then $y \geq$ minimum value ; **If $a < 0$** then $y \leq$ maximum value.

To determine the equation of a parabola:

<u>**Given:**</u> **TP and one other point**

Use $y = a(x+p)^2 + q$

<u>**Given:**</u> ***x*-intercepts and one other point**

Use $y = a(x - x_1)(x - x_2)$

TP is $(-p;q)$; substitute that in above equation.
Substitute the other point for x and y.
Solve for a.
Rewrite the equation with the values for a, p and q.

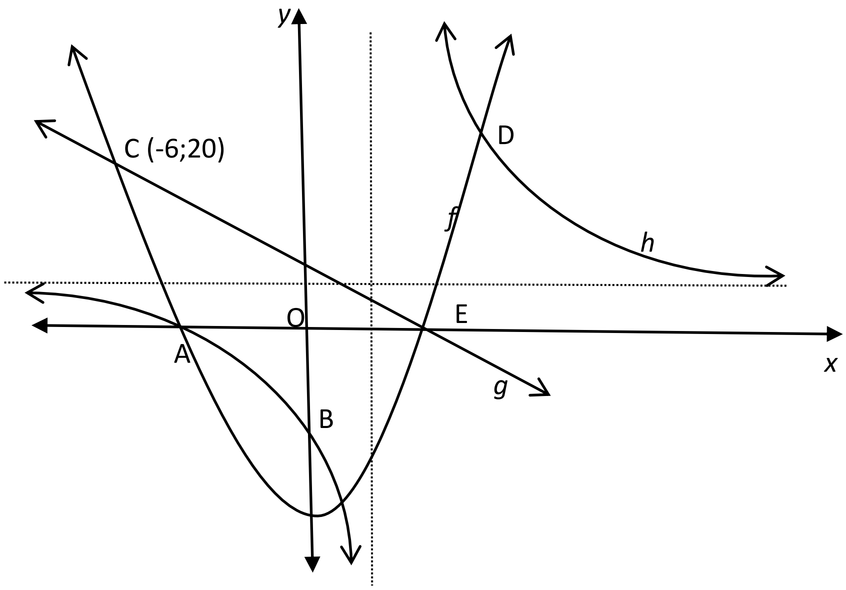
If required, rewrite in the form $y = ax^2 + bx + c$.

Substitute the x-intercepts for x_1 and x_2.
Substitute the other point for x and y.
Solve for a.
Rewrite the equation with the values for a, x_1 and x_2.
If required, rewrite in the form $y = ax^2 + bx + c$.

Example no. 2:

Sketched below are the graphs of: $g(x) = -2x + 8$; $f(x) = x^2 + k$; and $h(x) = \frac{6}{x-2} + 1$.

A is an x-intercept and B a y – intercept of h. C (−6; 20) and E are the points of intersection of f and g.



2.1 Determine the coordinates of A, B and E.

2.2 Show that the value of $k = -16$

2.3 Determine the domain and the range of f.

2.4 Write down the values of x for which $g(x) - f(x) \geq 0$.

2.5 Determine the equation of the axis of symmetry of h that has a negative gradient.

2.6 Write down the range of s, if $s(x) = f(x) + 2$.

2.7 Write down the range of t, if $t(x) = h(x) + 2$.

Solution:

2.1 At A, substitute $y=0$:

$$\frac{6}{x-2}+1=0$$

$$6=-x+2$$

$$\therefore x=-4$$

Thus: $A(-4;0)$

At B, substitute $x=0$:

$$y=\frac{6}{-2}+1$$

$$y=-3+1$$

$$\therefore y=-2$$

Thus: $B(0;-2)$

E is the *x*-intercept of the straight line and the parabola. It is easy and straight-forward to use the equation of the straight line to get the coordinates of E.

At E, substitute $y=0$, $\therefore 0=-2x+8$

$$x=4$$

Thus: $E(4;0)$

2.2 C(-6; 20) is on f and g.

Substitute C into $f(x)=x^2+k$

$$20=(-6)^2+k$$

$$k=-16$$

2.3 Domain is $x \in R$

Range is $y \geq -16;\ y \in R$

2.4 These are the values of *x* for which the graphs of *g* and *f* intersect **or** where *f* is below *g*.

It occurs from C(-6 ; 20) and E(4 ; 0).

That is $-6 \leq x \leq 4$.

2.5 Point of intersection of asymptotes: $(2;1)$

For axis of symmetry with negative gradient:

$$y=-x+c$$

Substitute $(2;1)$: $1=-2+c$

$$c=3$$

$$y=-x+3$$

2.6 The "+ 2" implies a shift vertically upwards by 2 units. The new minimum value will now be -14. The range of s is $y \geq -14$.

2.7 The " + 2" implies a shift vertically upwards by 2 units.

The range of t is $y \neq 1+2;\ y \in R$

$$y \neq 3;\ y \in R$$

D. EXPONENTIAL GRAPH

Defining equation: $y=ab^{x+p}+q$.

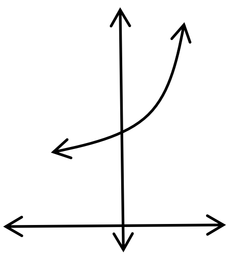
If q = 0 and p = 0 then $y=ab^x$.

If p = 0 then $y=ab^x+q$.

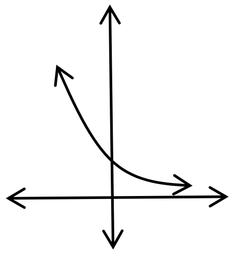
The restriction is $b>0;\ b \neq 1$

Shape:

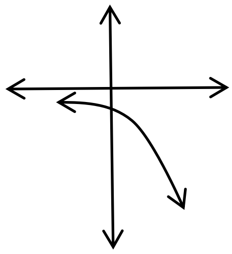
for $a > 0$ *and* $b > 1$



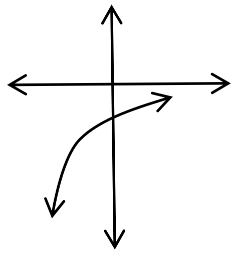
for $a > 0$ *and* $0 < b < 1$



for $a < 0$ *and* $b > 1$



for $a < 0$ *and* $0 < b < 1$



- For $y = ab^x$, the **asymptote** is $y = 0$ and the **y-intercept** is $y = a$.
- For $y = ab^x + q$, the **asymptote** is $y = q$ and the **y-intercept** is $y = a + q$.
- For $y = ab^{x+p} + q$, the **asymptote** is $y = q$ and the **y-intercept** is $y = ab^p + q$.

Example no. 3:

Given: $f(x) = 3^{-x+1} - 3$

3.1 Write $f(x)$ in the form $y = ab^x + q$

3.2 Draw the graph of f, showing all the intercepts with the axes and the asymptote.

3.3 Write down the domain and the range of f.

Solution:

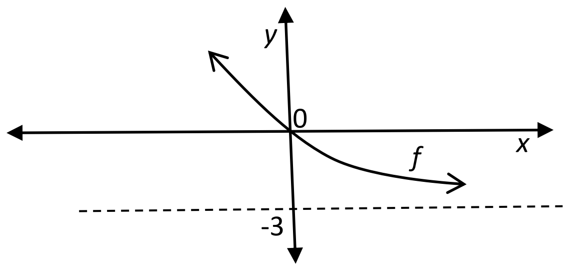
3.1 $y = 3^{-x+1} - 3 = 3^{-x}.3 - 3 = 3.3^x - 3 = 3\left(\frac{1}{3}\right)^x - 3$

3.2 The asymptote is $y = -3$.

For the x-intercept, let $y = 0$: $3\left(\frac{1}{3}\right)^x - 3 = 0$

$$\left(\frac{1}{3}\right)^x = 1$$

$$x = 0$$



3.3. The domain is $x \in R$, and the range is $y > -3$; $y \in R$.

QUESTIONS FROM PAST PAPERS ON FUNCTIONS AND GRAPHS

QUESTION 5 (GR. 12 DBE NOVEMBER 2010)

Consider the function $f(x) = 4^{-x} - 2$

5.1 Calculate the coordinates of the intercepts of *f* with the axes. (4)

5.2 Write down the equation of the asymptote of *f*. (1)

5.3 Sketch the graph of *f*. (3)

5.4 Write down the equation of g if *g* is the graph of *f* shifted 2 units upwards. (1)

5.5 Solve for x if $f(x) = 3$. (You need not simplify your answer.) (3)

[12]

QUESTION 5 (Gr. 12 DBE MARCH 2011)

Consider the function $f(x) = \dfrac{3}{x-1} - 2.$

5.1 Write down the equations of the asymptotes of *f*. (2)

5.2 Calculate the intercepts of the graph of *f* with the axes. (3)

5.3 Sketch the graph of *f*. (3)

5.4 Write down the range of $y = -f(x)$. (1)

5.5 Describe, in words, the transformation of f to g if $g(x) = \dfrac{-3}{x+1} - 2.$ (2)

[11]

QUESTION 5 (GR. 12 DBE MARCH 2010)

Given: $f(x) = \dfrac{2}{x-3} + 1$

5.1 Write down the equations of the asymptotes of *f*. (2)

5.2 Calculate the coordinates of the *x*- and *y*-intercepts of *f*. (3)

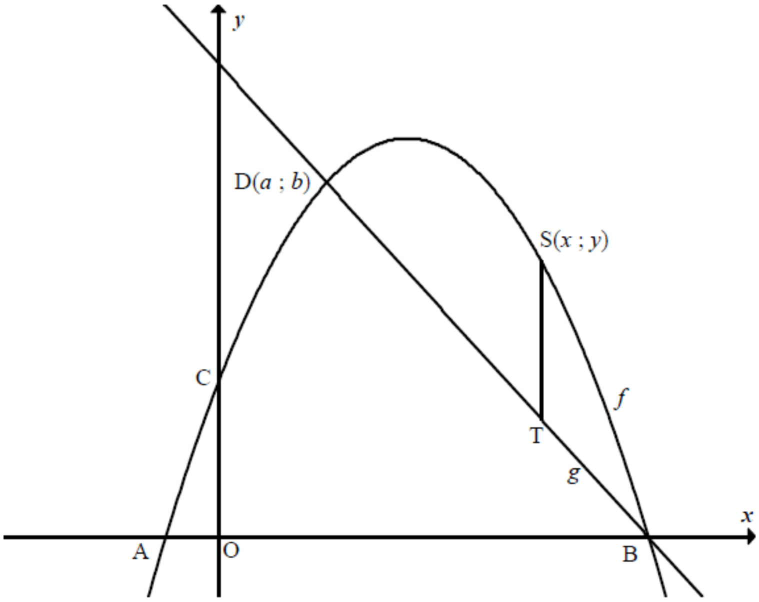
5.3 Sketch *f*. Show all intercepts with the axes and the asymptotes. (3)

[8]

QUESTION 6 (GR. 12 DBE MARCH 2010)

The graphs of $f(x) = -x^2 + 7x + 8$ and $g(x) = -3x + 24$ are sketched below.

f and g intersect in D and B. A and B are the x-intercepts of f.



6.1 Determine the coordinates of A and B. (4)

6.2 Calculate a, the x-coordinate of D. (4)

6.3 S(x ; y) is a point on the graph of f, where $a \leq x \leq 8$. ST is drawn parallel to the y-axis with T on the graph of g. Determine ST in terms of x. (2)

6.4 Calculate the maximum length of ST. (2)

[12]

QUESTION 4 (GR. 12 DBE MARCH 2015)

Given: $g(x) = \dfrac{6}{x+2} - 1$

4.1 Write down the equations of the asymptotes of g. (2)

4.2 Calculate:

4.2.1 The y – intercept of g (1)

4.2.2 The x – intercept of g (2)

4.3 Draw the graph of g, showing clearly the asymptotes and the intercepts with the axes. (3)

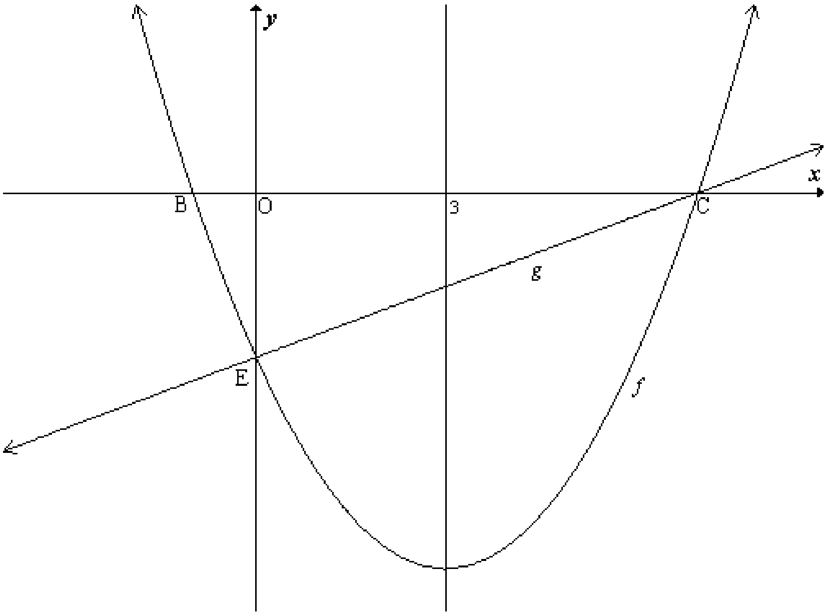
4.4 Determine the equation of the line of symmetry that has a negative gradient, in the Form $y = \ldots$ (3)

4.5 Determine the value(s) of x for which : $\dfrac{6}{x+2} - 1 \geq -x - 3$ (2)

[13]

QUESTION 6 (GR. 12 DBE MARCH 2011)

A parabola f intersects the x-axis at B and C and the y-axis at E. The axis of symmetry of the parabola has equation $x = 3$. The line through E and C has equation $g(x) = \dfrac{x}{2} - \dfrac{7}{2}$.



6.1 Show that the coordinates of C are (7 ; 0). (1)

6.2 Calculate the x-coordinate of B. (1)

6.3 Determine the equation of f in the form $y = a(x - p)^2 + q$. (6)

6.4 Write down the equation of the graph of h, the reflection of f in the x-axis. (1)

6.5 Write down the maximum value of $t(x)$ if $t(x) = 1 - f(x)$. (2)

6.6 Solve for x if $f(x^2 - 2) = 0$. (4)

[15]

QUESTION 4 (GR. 12 DBE NOVEMBER 2015)

Given: $f(x) = 2^{x+1} - 8$

4.1 Write down the equation of the asymptote of f. (1)

4.2 Sketch the graph of f. Clearly indicate ALL intercepts with the axes as well as the asymptote. (4)

4.3 The graph of g is obtained by reflecting the graph of *f* in the *y* – axis. Write down the equation of *g*. (1)

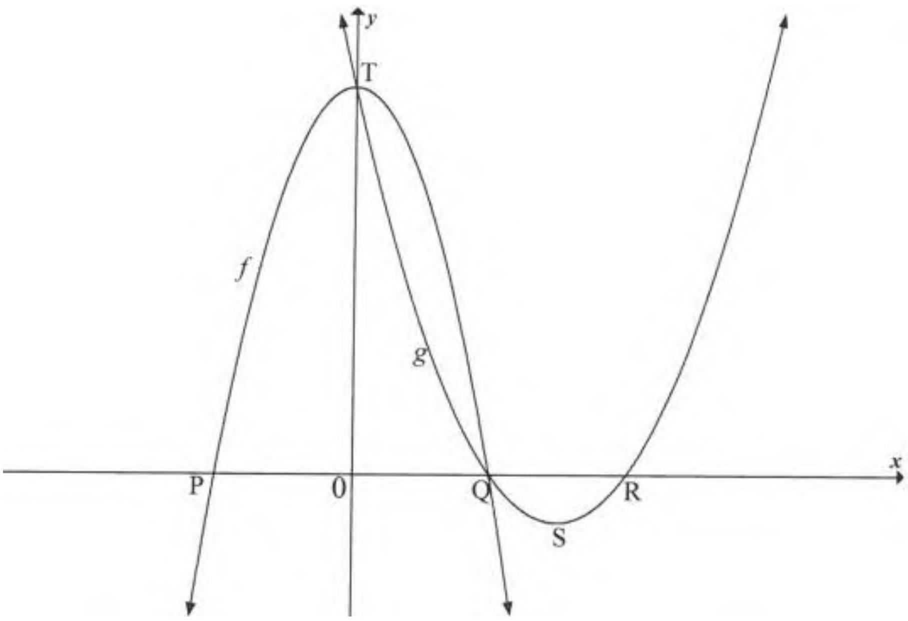
[6]

QUESTION 6 (GR. 12 DBE NOVEMBER 2015)

6.1 The graphs of $f(x) = -2x^2 + 18$ and $g(x) = ax^2 + bx + c$ are sketched below.

Points P and Q are the x – intercepts of *f*. Points Q and R are the x – intercepts of g.

S is the turning point of g. T is the y – intercept of both *f* and *g*.



6.1.1 Write down the coordinates of T. (1)

6.1.2 Determine the coordinates of Q. (3)

6.1.3 Given that $x = 4{,}5$ at S, determine the coordinates of R. (2)

6.2 The function defined as $y = \frac{a}{x+p} + q$ has the following properties:

- The domain is $x \in R,\ x \neq 2$.
- $y = x + 6$ is an axis of symmetry.
- The function is increasing for all $x \in R,\ x \neq 2$.

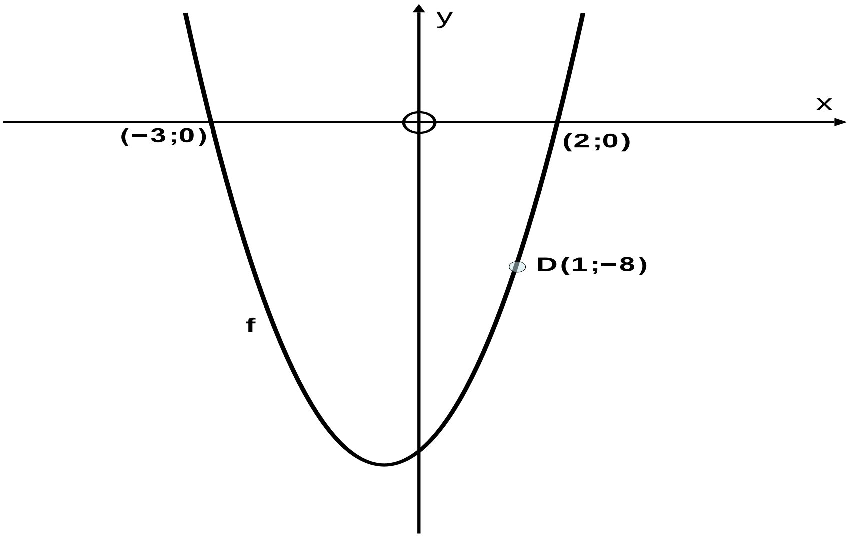
Draw a neat sketch graph of this function. Your sketch must include the asymptotes, if any. (4)

[10]

QUESTION 5 (GR. 12 DBE MARCH 2015 – adapted for gr. 11)

The graph of $f(x) = ax^2 + bx + c\,; a \neq 0$ is drawn below. D(1 ; – 8) is a point on f.

f intersects the $x-\text{axis}$ at (– 3 ; 0) and (2 ; 0).



6.1 For which value(s) of x is $f(x) \leq 0$? (2)

6.2 Determine the values of a, b and c. (5)

6.3 Determine the coordinates of the turning point of f. (3)

6.4 Write down the equation of the axis of symmetry of h if $h(x) = f(x-7) + 2$. (2)

[12]

QUESTION 5 (GR. 12 DBE NOVEMBER 2011)

5.1 Consider the function: $f(x) = \frac{-6}{x-3} - 1$

5.1.1 Calculate the coordinates of the $y-\text{intercept of } f$. (2)

5.1.2 Calculate the coordinates of the $x-\text{intercept of } f$. (2)

5.1.3 Sketch the graph of f in your ANSWER BOOK, showing clearly the asymptotes and the intercepts with the axes. (4)

5.1.4 For which values of x is $f(x) > 0$? (2)

5.1.5 Calculate the average gradient of f between $x = -2$ and $x = 0$. (4)

5.2 Draw a sketch graph of $y = ax^2 + bx + c$, where $a < 0,\ b < 0,\ c < 0$ and $ax^2 + bx + c = 0$ has only ONE solution. (4)

[19]

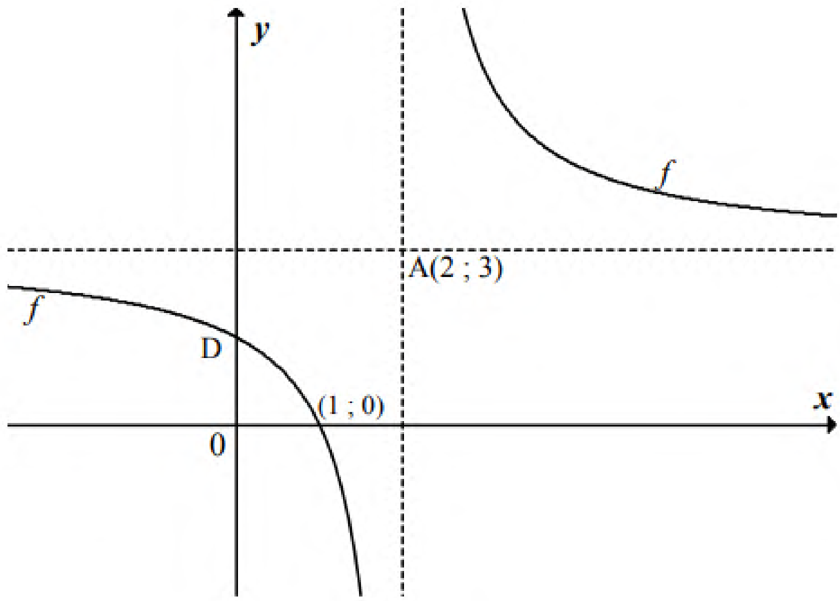
QUESTION 4 (GR. 12 DBE NOVEMBER 2010)

Given: $f(x) = \frac{a}{x-p} + q$.

The point A(2; 3) is the point of intersection of the asymptotes of f.

The graph of f intersects the $x - \text{axis}$ at (1; 0).

D is the $y - \text{intercept}$ of f.



4.1 Write down the equations of the asymptotes of f. (2)

4.2 Determine an equation of f. (3)

4.3 Write down the coordinates of D. (2)

4.4 Write down an equation of g if g is the straight line joining A and D. (3)

4.5 Write down the coordinates of the other point of intersection of f and g. (4)

[14]

ADDITIONAL EXERCISES

QUESTION 1

1.1 Given: $f(x) = x^2 - 2x - 3$

1.1.1 Calculate the intercepts with axes. (3)

1.1.2 Calculate the coordinates of the turning point. (2)

1.1.4 Draw a graph of **f** showing all the intercepts with axes and the turning point. (3)

1.1.5 Write down the range and domain of **f.** (2)

1.2 Given: $f(x) = (x-2)^2 - 9$

1.2.1 Write down the coordinates of the turning point of the graph of **f.** (1)

1.2.2 Calculate the x and the y intercept of the graph of **f.** (4)

1.2.3 Draw a neat graph of **f** and show the intercepts of the axes and the turning point. (3)

1.2.4 Hence write the range and the domain of the function. (2)

1.2.5 For which values of **x** is f(x) decreasing? (2)

1.2.6 Use your graph to solve the inequality: $f(x) \leq 0$. (2)

1.2.7 Write down the equation (in turning point form) of the graph obtained by

(a) shifting **f,** 2 units left and 9 units up. (2)

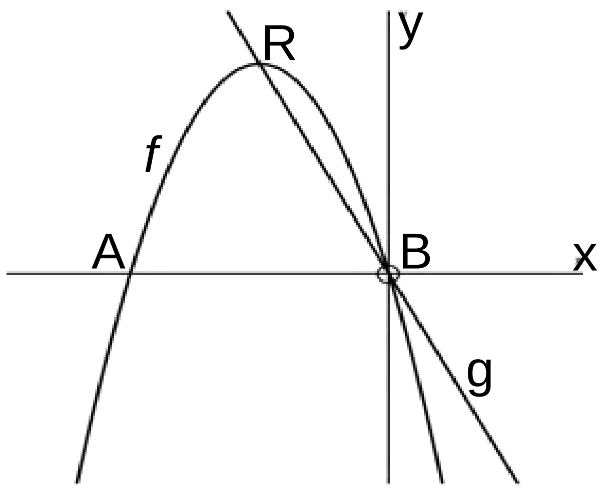
(b) reflecting **f** in the y – axis. (2)

(c) reflecting **f** in the x – axis. (2)

QUESTION 2.

2.1 Sketch below are the graphs of:

$f(x) = -(x+2)^2 + 4$ and $g(x) = ax + q$,R is the turning point of **f.**



2.2.1 Write down the coordinates of R. (2)

2.2.2 Calculate the length of AB. (2)

2.2.3 Determine the equation of **g**. (2)

2.2.4 For which values of **x** is g(x) > f(x). (2)

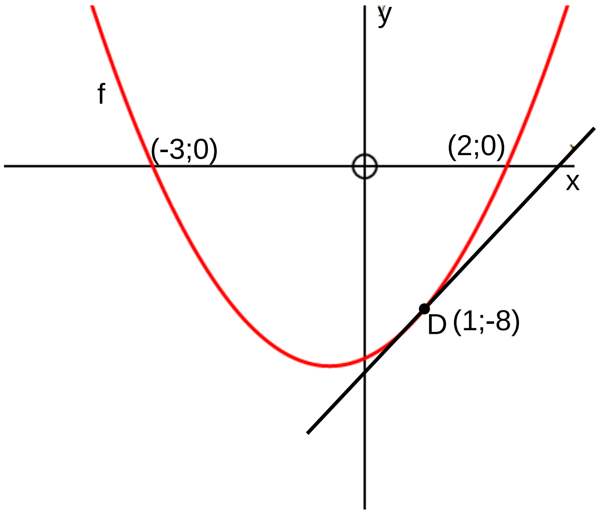
2.2.5 Write down the equation of the axis of symmetry of **h** if h(x) = f(− x). (2)

2.2.6 Write down the range of p if p(x) = −f(x) (2)

QUESTION 3

The graph of $f(x)=x^2+bx+c\,;a\neq 0$ and $g(x)=mx+k$

D(1;8) is a common point on **f** and **g.** **f** intersects the x –axis at (-3;4) and (2;0)
g is the tangent to f at D.



3.1 For which value(s) of **x** is f(x) ≤ 0? (2)

3.2 Determine the value of a, b and c. (5)

3.3 Determine the coordinates of the turning points of **f.** (3)

3.4 Write down the equation of the axis of symmetry of **h** if h(x) = f(x-7) + 2. (2)

3.5 Calculate the gradient of **g**. (3)

QUESTION 4 (DBE NOV. 2018)

Given: $f(x)=\dfrac{-1}{x-1}$

4.1 Write down the domain of **f.** (1)

4.2 Write down the asymptotes of **f.** (2)

4.3 Sketch the graph of f, clearly showing all intercepts with the axes and any asymptotes. (3)

QUESTION 5

Given: $f(x)=\dfrac{6}{x+2}-1$

5.1 Write down the equations of the asymptotes of **g.** (2)

5.2 Calculate:

5.2.1 The y – intercept of **g**. (1)

5.2.2 The x – intercept of **g**. (2)

5.3 Draw the graph of g, show clearly the asymptotes and the intercepts with the axes. (3)

5.4 Determine the equation of the line of symmetry, that has a negative gradient, in the form y = …… (3)

5.5 Determine the value(s) of x for which $\frac{6}{x+2}-1\geq -x-3$ (2)

QUESTION 6

Given: $f\left(x\right)=2^{x+1}-8$

6.1 Write down the equation of the asymptote of **f.** (1)

6.2 Sketch the graph of **f**, clearly indicate ALL the intercepts with the axes as well as the asymptote. (3)

6.3 The graph of g is obtained by reflecting the graph of y about the y – axis. Write down the equation of ***g***. (1)

Q UESTION 7

Given: $f\left(x\right)=2^{-x}+1$

7.1 Determine the coordinates of the y – intercepts (2)

7.2 Sketch the graph of **f**, clearly indicate ALL the intercepts with the axes as well as the asymptote. (3)

7.3 Calculate the average gradient of **f** between the points on the graph where x = −2 and x = 1. (3)

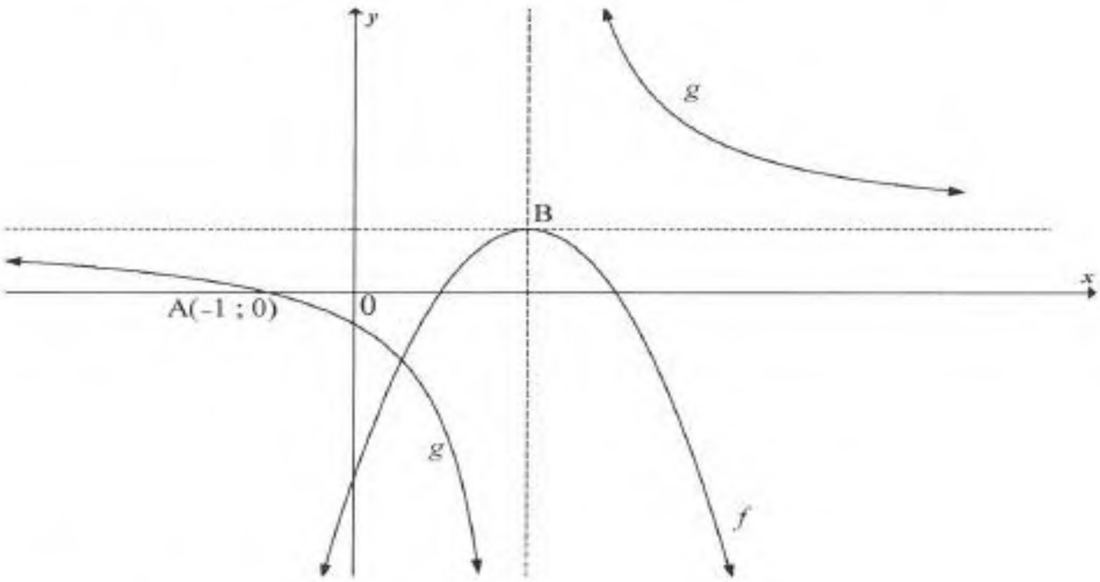
7.4 If $h(x) = 3f(x)$, write down the equation of the asymptote of **h**. (2)

QUESTION 5 (DBE November 2016)

QUESTION 5

Sketched below is the parabola f with equation $f(x) = -x^2 + 4x - 3$ and a hyperbola g with equation $(x - p)(y + t) = 3$.

- B, the turning point of f, lies at the point of intersection of the asymptotes of g.
- $A(-1\,;0)$ is the x-intercept of g.



5.1	Show that the coordinates of B are $(2\,;1)$	(2)
5.2	Write down the range of f.	(1)
5.3	For which value(s) of x will $g(x) \geq 0$?	(2)
5.4	Determine the equation of the vertical asymptote of the graph of h if $h(x) = g(x+4)$	(1)
5.5	Determine the values of p and t.	(4)

ANALYTICAL (COORDINATE) GEOMETRY

LEARNING HINTS

Mathematical language and terminology must be learnt in more detail

1. *Learners should then follow the method laid out below:*

   - Select the correct formula from the data sheet
   - Label the ordered pairs using the correct two points, $e.g.\quad A \text{ and } B.$
   - Substitute correctly and accurately into your chosen formula and **use brackets** where necessary to avoid operations that require isolated expressions and negative signs.
   - Emphasis on the application of distributive law of multiplication.
     $e.g - (x+1) = -x - 1.$

2. Often Analytical Geometry questions *follow on*, (scaffolding). Look out for that, as you might have already calculated or proven an aspect before, that you will require for the next sub-question :
   - Even if you failed to show or prove in the previous question, accept that as true in the follow up questions.
3. *Use the diagram more effectively.*
   $e.g.$ Highlight the sides you are going to use for proving perpendicular lines, so you can see clearly which points you are going to use for the substitution.

   You must answer the question, and remember to conclude, exactly what you were asked to show / prove / conclude.

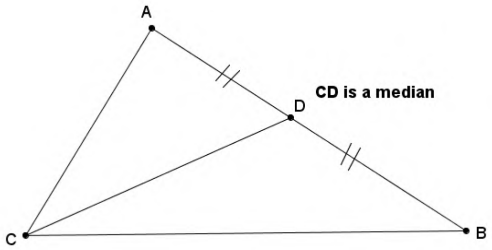
4. *Learners need to know the properties of all geometric figures e.g. triangles and quadrilaterals*
5. Learners need to be able to determine whether a particular point is inside, outside or on the circle by comparing that distance and the radius.
6. Practice exercises are often required to teach the above points.
7. Grade 11 work must ***NOT*** be ignored, *e.g.*

NOTE:

Always **refer to a diagram** when doing problems involving Analytical Geometry. A diagram helps you to visualise the problem accurately.

Median: The median of a triangle bisects the opposite side of a triangle.

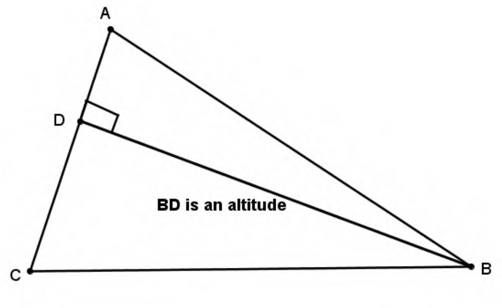
The median of a triangle bisects the area of a triangle.



To find the equation of the median:

- Determine the coordinates of D using the formula for the midpoint
- Use the coordinates of C and D to find the equation.

Altitude: The altitude of a triangle is perpendicular to the opposite side of a triangle.



To find the equation of an altitude:

- Determine the gradient AC= m_{AC}
- Determine the m_{BD} using the fact that $m_{BD} \times m_{AC} = -1$
- Use the coordinates of B and m_{BD} to find the equation.

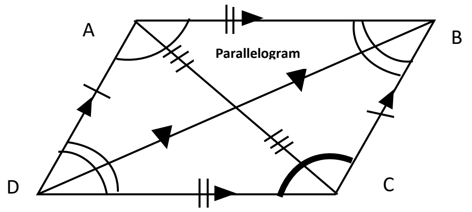
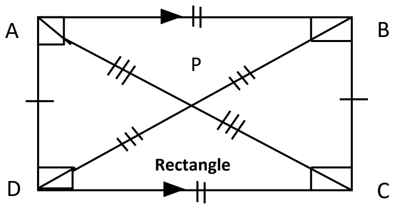
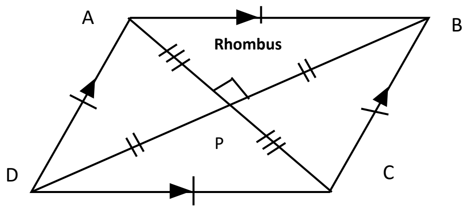
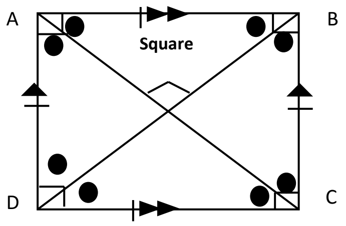
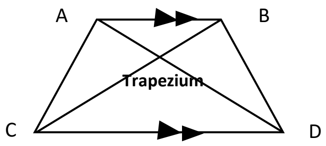
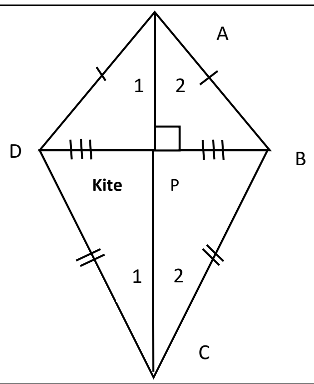
Perpendicular bisector: The perpendicular bisector of a line segment is also found in a triangle.

To find the equation of the perpendicular bisector:

- Determine the coordinates of M using the midpoint formula.
- Find the gradient AB i.e. m_{AB}
- Determine m_{DP} using the fact that DP⊥AB $\therefore m_{DP} \times m_{AB} = -1$
- Use the coordinates of M and m_{DP} to find the equation.

- Determine the gradient AB, m_{AB}
- Determine the gradient CD, using AB⊥CD
- Use the gradient CD and the point B to find the equation CD

PROPERTIES OF QUADRILATERALS

	1. Opposite sides parallel. 2. Opposite sides equal. 3. Opposite angles are equal. 4. Co-interior angles on same side of a transversal are supplementary. 5. Diagonals bisect each other.
	1. All properties of parallelogram 2. Has 4 right angles. 3. Diagonals are equal.
	1. All properties of parallelograms. 2. Has 4 equal **sides,** 3. Diagonals bisect opposite angles. 4. Diagonals each other at right angle.
	1. All properties of parallelogram, rectangle, and rhombus 2. 4 equal sides and 4 equal (right) angles.
	1. One pair of parallel sides.
	1. 2 pairs of adjacent sides equal. 2. 1 pair of opposite angles equal 3. The long diagonal bisects the short one at right angle. 4. Diagonals bisect opposite angles. $\hat{A}_1 = \hat{A}_2$ $\hat{C}_1 = \hat{C}_2$

FUNDAMENTAL COMPETENCE

Represent geometric figures on a Cartesian co-ordinate system by understanding that a point is determined by coordinates in the form of $(x_1;y_1),(x_2;y_2)$, $(x_3;y_3)$, etc.

Use a Cartesian co-ordinate system to apply the following formulae:

(a) the **distance** between the two points:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

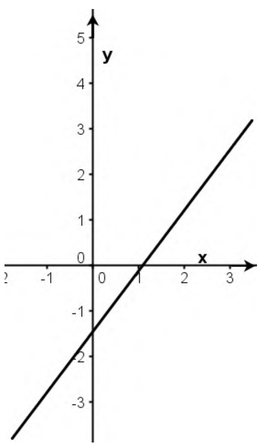
(b) the **gradient of the line segment** joining the points (including collinear points) or the **inclination** of a line.

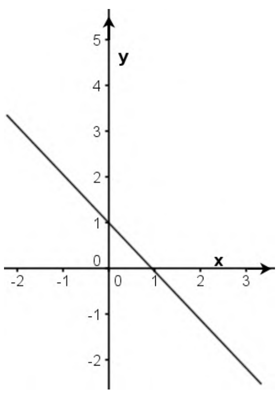
$m=\dfrac{y_2-y_1}{x_2-x_1}$ or $\boldsymbol{m=\tan\theta}$ where $\boldsymbol{\theta}$ **is the anlge that a line makes with a positive direction** of the x $axis$.

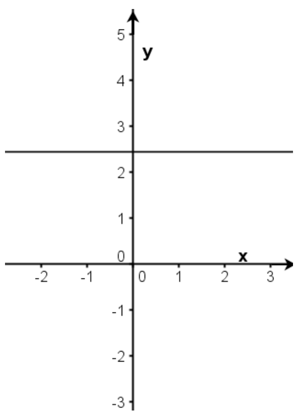
- *The lines joining **collinear points** have the **same gradient**.*

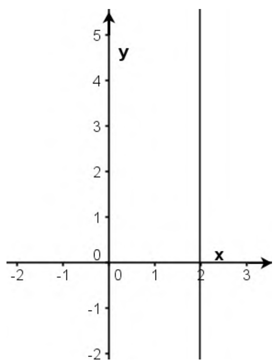
- The gradient or slope of a straight line through the points A$(x_1;y_1)$ and B$(x_2;y_2)$ in which $x_1 \neq x_2$ is given by: Gradient of AB= $m_{AB}=\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_1}{x_2-x_1}=\frac{increase\ in\ y}{increase\ in\ x}$

$\boldsymbol{m}>0$	$\boldsymbol{m}<0$	$\boldsymbol{m=0}$	$\boldsymbol{m}$**- undefined**
$\boldsymbol{tan\theta}>0$	$\boldsymbol{tan\theta}<0$	$\boldsymbol{tan\theta=0}$	$\boldsymbol{tan\theta}$ **is undefined**
		Horizontal line	**Vertical line**









Perpendicular and Parallel Lines

- The product of the gradients of perpendicular lines is **-1,** i.e. $m_1 \times m_2=-1$
- $m_1 \times m_2=-1$ cannot be applied when one of the lines is parallel to the y −axis
- When two lines are parallel, then $m_1=m_2$

If AB∥PQ, then $m_{AB}=m_{PQ}$ (AB and PQ not parallel to the$y-axis$)

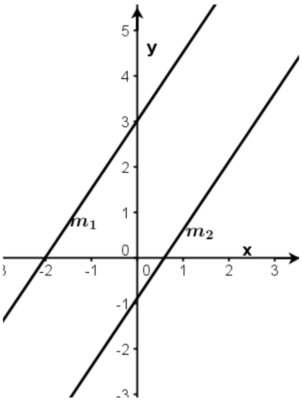
If $m_{AB}=m_{PQ}$,then AB∥PQ

If AB⊥AQ, then $m_{AB} \times m_{AQ} = -1$ (AB and PQ not parallel to the $y-axis$)

If $m_{AB} \times m_{AQ} = -1$,then AB⊥AQ

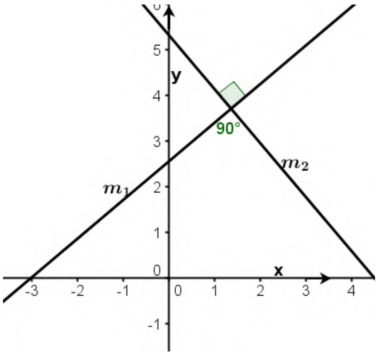
Parallel Lines

$\boldsymbol{m_1 = m_2}$



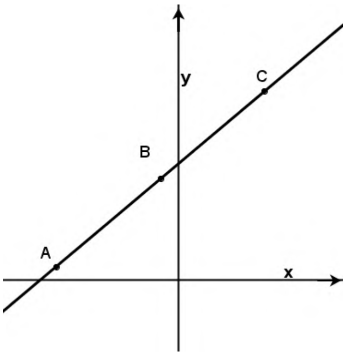
Perpendicular Lines

$\boldsymbol{m_1 \times m_2 = -1}$



Collinear Points

$\boldsymbol{m_{AB} = m_{BC}}$



(c) the co-ordinates of the **mid-point** of the line segment joining the points:

$$\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$$

(d) the equation of a line **through two given points** and the equation of a line through **one point and parallel or perpendicular** to a given line:

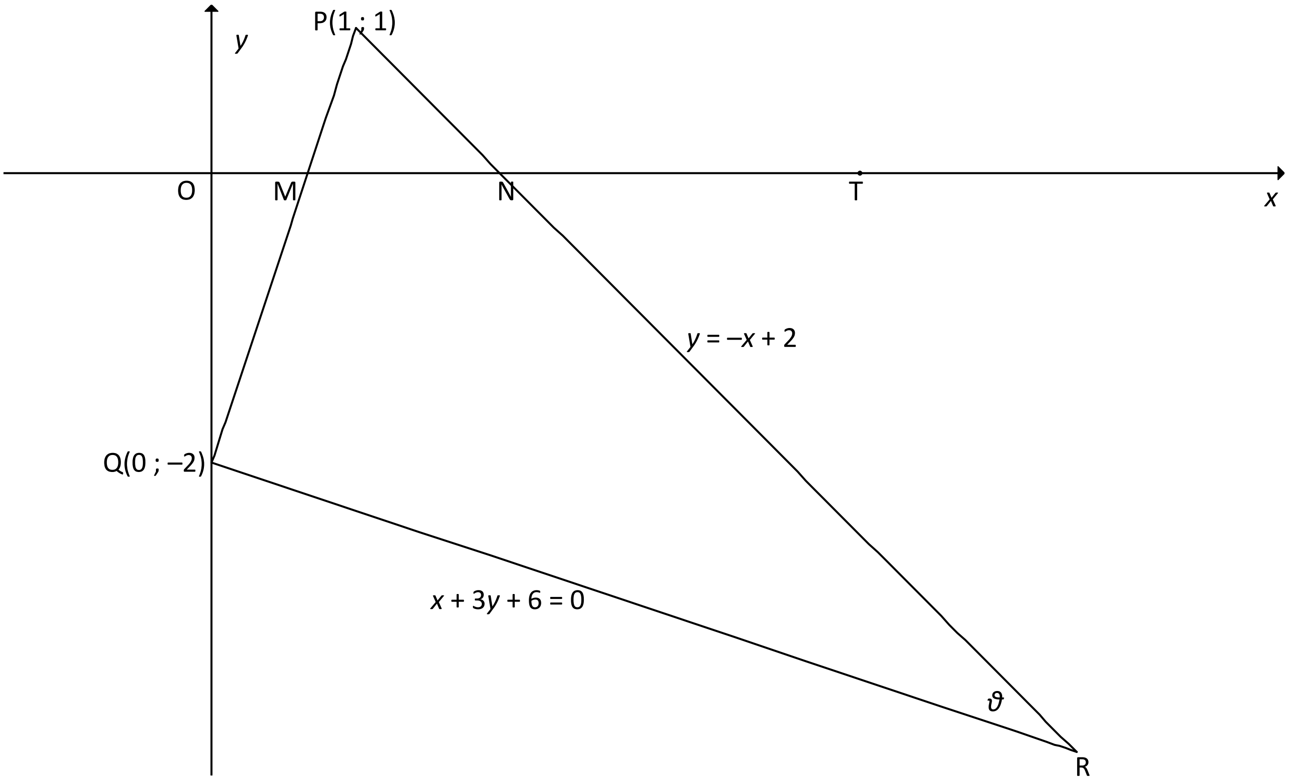
$y - y_1 = m(x - x_1)$ or $y = mx + c$

viz. : If $AB \parallel CD$, then $m_{AB} = m_{CD}$

If $AB \perp CD$, then m_{AB} x $m_{CD} = -1$

WORKED EXAMPLES (METHODOLOGY)

In the diagram below, P(1 ; 1), Q(0 ; –2) and R are the vertices of a triangle and $P\hat{R}Q = \theta$. The x-intercepts of PQ and PR are M and N respectively. The equations of the sides PR and QR are $y = -x + 2$ and $x + 3y + 6 = 0$ respectively. T is a point on the x-axis, as shown.



1.1 Determine the gradient of QP.

Solution:

$$m_{PQ} = \frac{1-(-2)}{1-0}$$
$$= 3$$

1.2 Prove that $P\hat{Q}R = 90°$.

Solution:

QR: $y = -\frac{1}{3}x - 2$

$$\therefore m_{QR} = -\frac{1}{3}$$

$$m_{PQ} \times m_{QR} = 3 \times -\frac{1}{3}$$
$$= -1$$

$\therefore PQ \perp QR \qquad \therefore P\hat{Q}R = 90°$

1.3 Determine the coordinates of R.

Solution:

$$-\frac{1}{3}x-2=-x+2$$

$$\frac{2}{3}x=4$$

$$x=6$$

$$y=-4$$

$\therefore$ R$(6\,;-4)$

1.4 Calculate the length of PR. Leave your answer in surd form.

Solution

$$\text{PR}=\sqrt{(1-6)^2+(1-(-4))^2}$$
$$=\sqrt{50}=5\sqrt{2}$$

1.5 Determine the equation of a circle passing through P, Q and R in the form $(x-a)^2+(y-b)^2=r^2$.

Solution

PR is a diameter [chord subtends 90°]

Centre of circle$l$: $\left(\frac{1+6}{2}\,;\frac{1-4}{2}\right)$

$$=\left(3\frac{1}{2}\,;-1\frac{1}{2}\right)$$

$$r=\frac{\sqrt{50}}{2}\quad\text{OR}\quad\frac{5\sqrt{2}}{2}\quad\text{OR}\quad 3{,}54$$

$$\therefore\left(x-\frac{7}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{50}{4}\quad\text{OR}\quad\frac{25}{2}\quad\text{OR } 12{,}5$$

1.6 Determine the equation of a tangent to the circle passing through P, Q and R at point P in the form $y=mx+c$.

Solution

m of radius = –1

$\therefore m$ of tangent = 1

Equation of tangent:

$$y - y_1 = (x - x_1)$$
$$y - 1 = x - 1$$

$\therefore y = x$

1.7 Calculate the size of θ.

Solution:

$\tan \text{P}\hat{\text{N}}\text{T} = m_{\text{PR}} = -1$

$\therefore \text{P}\hat{\text{N}}\text{T} = 135°$

$\tan \text{P}\hat{\text{M}}\text{T} = m_{\text{PQ}} = 3$

$\therefore \text{P}\hat{\text{M}}\text{T} = 71{,}57°$

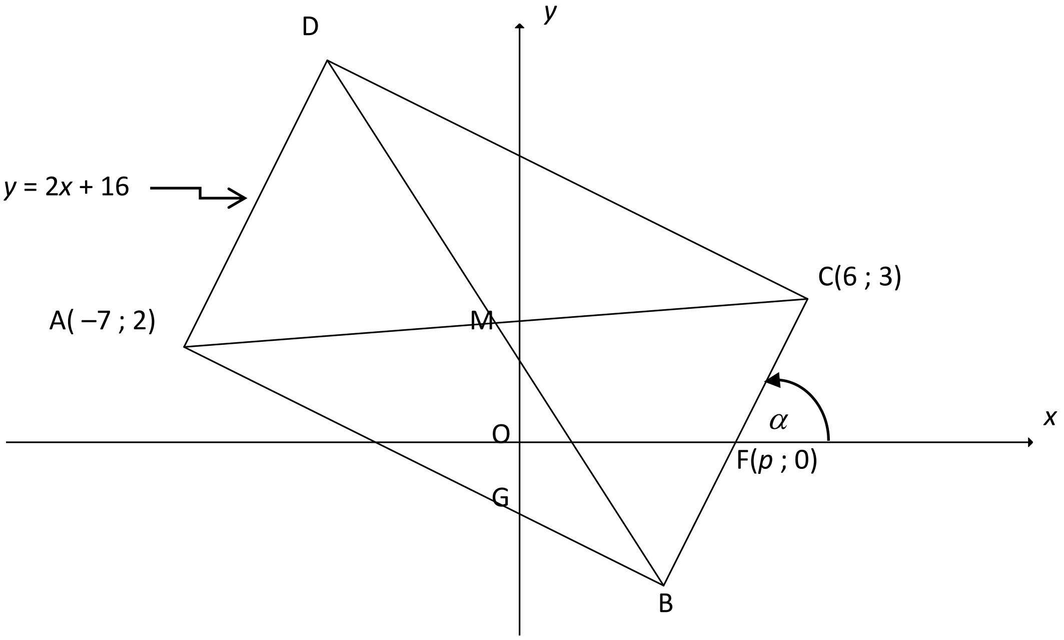
$\hat{\text{P}} = 63{,}43°$ [ext ∠ of Δ]

$\therefore \theta = 26{,}57°$ [sum of ∠s in Δ]

QUESTION 2

In the diagram, A (–7 ; 2), B, C(6 ; 3) and D are the vertices of rectangle ABCD.

The equation of AD is y = 2x + 16. Line AB cuts the y-axis at G. The x-intercept of line BC is F(p ; 0) and the angle of inclination of BC with the positive x-axis is α. The diagonals of the rectangle intersect at M.



2.1 Calculate the coordinates of M.

2.2 Write down the gradient of BC in terms of p.

2.3 Hence, calculate the value of p.

2.4 Calculate the length of DB.

2.5 Calculate the size of α.

2.6 Calculate the size of $O\hat{G}B$

2.1 $\left(-\frac{1}{2};\frac{5}{2}\right)$

2.2 $-\frac{3}{p-6}$

2.3 $p=4\frac{1}{2}$

2.4 $\sqrt{170}$

2.5 $\alpha=63{,}43^0$

2.6 $116{,}57^0$

ACTIVITIES

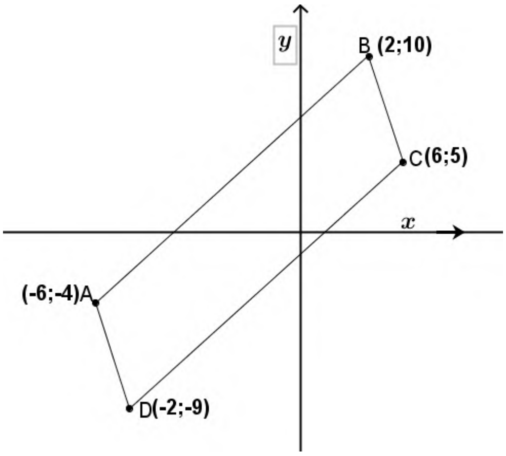
1. Determine the distance between the points given below. Leave answers in surd form.

(a) $(3;0)$ and $(0;3)$

(b) $(4;0)$ and $(2;3)$

2. The distance between A $(x; 15)$ and B $(-7; 3)$ is **13 units**. Calculate the **possible values** of x.

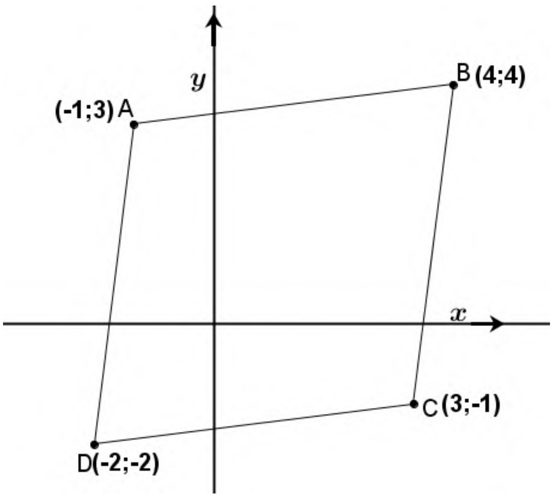
3. Given the following diagram:



(a) Prove that **ABCD** is a parallelogram using the lengths of the sides.

(b) Prove that **ABCD** is a parallelogram using the diagonals.

4. In the diagram below, ABCD is rhombus:

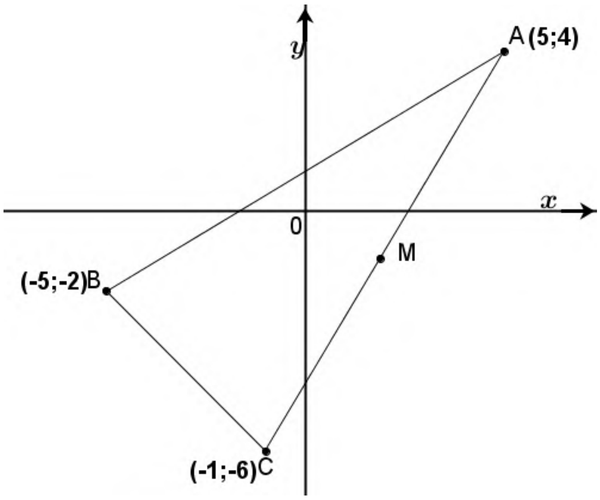


4.1 Determine the gradients of AC and BD.

4.2 Show that AB = DC and AD = BC

5. Determine the 4th vertex of a parallelogram PQRT, if the three given vertices are P $(6; -3)$, Q$(3; 3)$ and C $(-2; 1)$

6. Use the diagram below to answer the questions that follow:



(a) Calculate the coordinates of M the midpoint of AC.

(b) Determine the gradient BC.

(c) Determine the equation of the line parallel to BC that passes through M.

(d) Give the coordinates of P, the midpoint of AB.

(e) Calculate the length of BC.

(f) Prove that: $BC = 2PM$.

7. The points A $(-5; 9)$, B $(-3; y)$ and C $(2; -5)$ are given.

(a). Determine the value of y if A; B and C are collinear.

(b). Determine the value of y if BC⊥ AC

8. Calculate the value of t, if AB is parallel to the line which passes through the points

C$(2; 3)$ and

D$(-2; -5)$ if A$(-3; t)$ and B$(0; -2)$

9. Prove that the following points are on the same line.

N$(-7; -8)$, A$(-1; 2)$ and G$(2; 7)$

10. Find the missing coordinates if these points are collinear.

H $(-3; 3)$, O$(-1; y)$, B$(2; -7)$

11. Determine the equation of a straight line passing through the points:

(a) $(3; 7)$ and $(-6; 1)$

(b) $(8; t)$ and $(t; 8)$

12. Determine the equation of the straight line:

(a) passing through the point $\left(-1; \frac{10}{8}\right)$ and with gradient $m = \frac{2}{3}$

(b) parallel to the x –axis and passing through the point $(0; 11)$

(c) perpendicular to the x –axis and passing through the point $\left(-\frac{3}{2}; 0\right)$

(d) with undefined gradient and passing through the point $(4; 0)$

(e) with $m = 3a$ and passing through the point $(-2; -6a + b)$

13. Determine whether or not the following two lines are parallel:

(a) $y + 2x = 1$ and $-2x + 3 = y$

(b) $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$

(c) $y = 2x - 7$ and the line passing through $(1; -2)$ and $\left(\frac{1}{2}; -1\right)$

(d) $y + 1 = x$ and $x + y = 3$

14. Determine the equation of the straight line that passes through the point $(1; -5)$ and is parallel to the line $y + 2x - 1 = 0$

15. Determine the equation of the straight line that passes through the point $\left(-2; \frac{2}{5}\right)$ and is parallel to the line with the angle of inclination $\theta = 145°$

16. Determine whether or not the following two lines are perpendicular.

(a) $y - 1 = 4x$ and $4y + x + 2 = 0$

(b) $10x = 5y - 1$ and $5y - x - 10 = 0$

(c) $x = y - 5$ and the line passing through $\left(-1; \frac{5}{4}\right)$ and $\left(3; -\frac{11}{4}\right)$

(d) $y = 2$ and $x = 1$

(e) $\frac{y}{5} = x$ and $3y + x = 9$

17. Determine the equation of a straight line that passes through the point $(-2; -4)$ and is perpendicular to the line $y + 2x = 1$

18. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with an angle of inclination $\theta = 135°$.

For tan$\theta = m$ where $m \geq 0$,$\theta = tan^{-1}m$

19. Determine the gradient correct to one decimal place of each of the following straight lines, given that the angle of inclination is equal to:

(a) 60° (b) 135° (c) 0° (d) 54° (e) 90°

20. Calculate the angle of inclination correct to one decimal place for each of the following:

(a) a line with $m = \frac{3}{4}$

(b) $2y - x = 6$

(c) the line passes through the points A $(-4; -1)$ and B $(2; 5)$

(d) $y = 4$

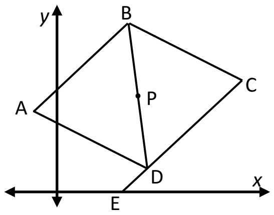
21. Determine the equation of a straight line passing through the point $(3; 1)$ and with an angle of inclination equal to 135°

22. Determine the acute angle between the line passing through the points A$\left(-2; \frac{1}{5}\right)$ and B $(0; 1)$ and the line passing through the points C $(1; 0)$ and D $(-2; 6)$

23. Determine the angle between the line $y + x = 3$ and the line $x = y + \frac{1}{2}$

24. Find the angle between the line $y = 2x$ and the line passing through the points P$\left(-1;\frac{7}{3}\right)$ and Q $(0; 2)$

25. ABCD is a parallelogram with $A(-1;4)$, $B(3;6)$, $C(x;y)$ and $D(4;1)$.



Determine:

(a) the gradient of AB.

(b) the midpoint P of BD.

(c) the coordinates of C.

(d) the equation of CD.

(e) the coordinates of E if E is the x-intercept of the line CD.

(f) the inclination of line AE

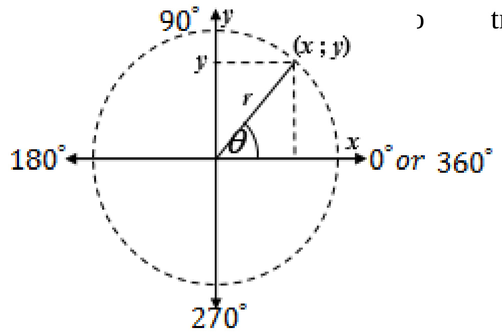
(g) the size of $A\hat{E}D$.

(h) the length of BC

TRIGONOMETRY

Cartesian plane

For each point $(x;y)$ on the terminal arm of θ, the following



trigonometric functions are defined :

$\sin\theta = \frac{y}{r}$ (sine)

$\cos\theta = \dfrac{x}{r}$ (cosine)

$\tan\theta = \dfrac{y}{x}$ (tangent)

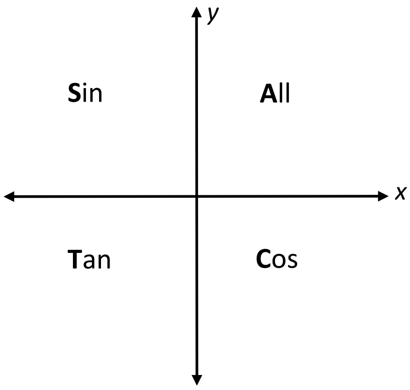
CAST DIAGRAM

The Cartesian plane is divided into four quadrants. The angle formed will determine the sign of each trigonometric functions. Complete the table by putting appropriate sign of variable and ratios.

	Interval				$\sin\theta$	$\cos\theta$	$\tan\theta$
Quad	θ	r	x	y	$\dfrac{y}{r}$	$\dfrac{x}{r}$	$\dfrac{y}{x}$
1	$(0°;90°)$						
2	$(90°;180°)$						
3	$(180°;270°)$						
4	$(270°;360°)$						

Conclusion

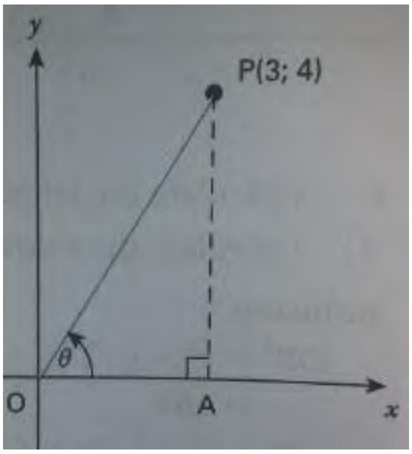
- All trig functions are _________________ in the ________quadrant
- $\sin\theta$ is _______________in the _________quadrant and the other two are __________
- $\tan\theta$ is _______________ in the _________quadrant and the other two are ___________
- $\cos\theta$ is ________________ in the _________quadrant and the other two are _____________



This is also referred to as the **CAST** diagram

Grade 10 Revision Exercise

PART A: Complete this exercise without using a calculator.



1. P(3;4) is a point in the Cartesian plane.

   OP makes an angle θ with the positive x-axis.

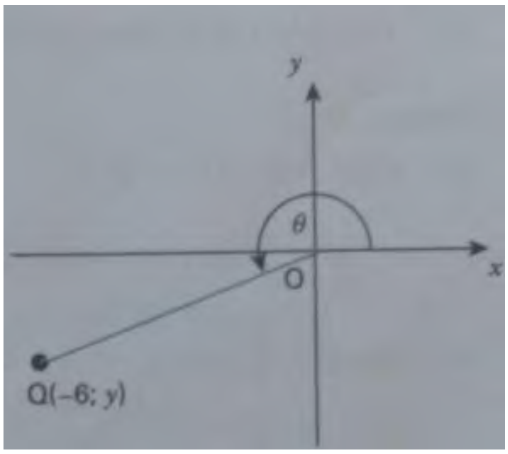
   Determine:

   a) OP

   b) $\sin\theta$

   c) $\cos\theta$

   d) $\tan\theta$



2. $OQ = r = 10$ and $Q(-6; y)$

   Determine the value of:

   a) y

   b) $\sin\theta + \tan\theta$

   c) $\sin^2\theta + \cos^2\theta$

3. If $13\cos\theta - 12 = 0$ and $180° < \theta < 360°$, calculate with the aid of a diagram the value of:

   (a) $\tan\theta$ (b) $(\sin\theta + \cos\theta)^2$ (c) $1 - \sin^2\theta$

4. If $\sin\alpha = \frac{3}{5}$ with $\alpha \in [90°;270°]$ and $\cos\beta = -\frac{12}{13}$ with $\beta \in [0°;180°]$, calculate the aid of the diagram the value of $\cos\alpha + \tan\beta$.

PART B: You may use a calculator to answer these questions. Leave your answer rounded off to TWO decimal places.

1. Find the values of θ in the equations below:

(a) $\sin\theta = 0,78$

(b) $\tan\theta = \cos 21°$

(c) $\sin\theta = \tan 12° + \cos 72°$

(d) $5\cos\theta + 2 = 3$

Exercise 1 (Reduction Formulae)

1. If $\sin 40° = p$ write the following in terms of p.

(a) $\sin 50°$ (b) $\sin 140°$ (c) $\cos 50°$

(d) $\sin(-40°)$ (e) $\tan 320°$

2. If $\tan 202° = t$ write the following in terms of t.

(a) $\tan(-202°)$ (b) $\cos 518°$ (c) $\sin 338°$

(d) $\frac{\cos 68°}{\cos 22°}$ (e) $\frac{\cos(-202°)}{\tan 22°}$

IDENTITIES

- $\tan\theta = \frac{\sin\theta}{\cos\theta};\ \theta \neq k.90^{\circ}$, k is an odd integer

Use the definition of trigonometric ratios in the above table

LHS $= \tan\theta =$ ______

RHS $= \frac{\sin\theta}{\cos\theta} =$ _______

$= \_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_\_$

$\therefore$

- $\sin^2\theta + \cos^2\theta = 1$

$\text{LHS} = \sin^2\theta + \cos^2\theta$

$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$

$\frac{y^2 + x^2}{r^2} ......hint:.x^2 + y^2 = r^2$

$\therefore \frac{r^2}{r^2} = 1$

$\therefore L.H.S = R.H.S$

$\sin^2\theta + \cos^2\theta = 1$

This identity can be written in the following ways:

a) $\sin^2\theta = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$ b) $\cos^2\theta = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Examples:

Factorise the following:

1) $2\sin\theta\cos\theta - \cos\theta$

$= \cos\theta(2\sin\theta - 1)$

2) $2\sin^2\theta - 3\sin\theta - 2$

$= (2\sin\theta + 1)(\sin\theta - 2)$

Alternative Method

$\text{let } k = \sin\theta$

$2\sin^2\theta - 3\sin\theta - 2$

$= 2k^2 - 3k - 2$

$= (2k + 1)(k - 2)$

$= (2\sin\theta + 1)(\sin\theta - 2)$

3) $\sin^2 x - 2\sin x\cos x + \cos^2 x$

$= (\sin x - \cos x)(\sin x - \cos x)$

Alternative Method

Let $\sin x = \mathrm{k}$ and $\cos x = \mathrm{p}$

$\sin^2 x - 2\sin x\cos x + \cos^2 x$

$= \mathrm{k}^2 - 2\mathrm{pq} + \mathrm{p}^2$

$= (\mathrm{k} - \mathrm{p})(\mathrm{k} - \mathrm{p})$

$= (\sin x - \cos x)(\sin x - \cos x)$

4) $1 - \tan^2 \mathrm{y}$

$= (1 - \tan y)(1 + \tan y)$

Proving Identities

To prove identities follow these steps:

- Make $\tan\theta = \frac{\sin\theta}{\cos\theta}$ or $\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ so that all the ratios are in terms of sin and cos.
- Consider the LHS or RHS as an algebraic expression and use algebraic manipulations to simplify further.
- If there are fractions, find the LCD and add.
- If there are fractions over fractions, simplify as you would in algebra.
- Factorise where possible.
- Use the square identity where possible in any of the forms:

  $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$
- Simplify both sides of the identity as far as possible.

Examples

1. **Prove the identity**

$$\frac{\tan x + \sin x}{1 + \frac{1}{\cos x}} = \sin x$$

Solution

$$\textbf{LHS} = \frac{\frac{\sin x}{\cos x} + \sin x}{1 + \frac{1}{\cos x}}$$

$$= \frac{\frac{\sin x + \sin x.\cos x}{\cos x}}{\frac{\cos x + 1}{\cos x}}$$

$$= \frac{\sin x\,(1+\cos x)}{\cos x} \times \frac{\cos x}{1+\cos x}$$

$$= \sin x$$

$\therefore$ **LHS = RHS**

Prove:

$$\sin^2\alpha - (\tan\alpha - \cos\alpha)(\tan\alpha + \cos\alpha) = \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}$$

State restrictions where applicable.

SOLUTION

Step 1: Use trigonometric identities to simplify each side separately

Simplify the left-hand side of the identity:

$$\begin{aligned}\text{LHS} &= \sin^2\alpha - (\tan\alpha - \cos\alpha)(\tan\alpha + \cos\alpha)\\ &= \sin^2\alpha - (\tan^2\alpha - \cos^2\alpha)\\ &= \sin^2\alpha - \tan^2\alpha + \cos^2\alpha\\ &= (\sin^2\alpha + \cos^2\alpha) - \tan^2\alpha\\ &= 1 - \tan^2\alpha\end{aligned}$$

Simplify the right-hand side of the identity so that it equals the left-hand side:

$$\begin{aligned}\text{RHS} &= \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}\\ &= \frac{\cos^2\alpha}{\cos^2\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha}\\ &= 1 - \tan^2\alpha\end{aligned}$$

$\therefore$ LHS = RHS

Alternative method: we could also have started with the left-hand side of the identity and substituted $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and simplified to get the right-hand side.

Restrictions

We need to determine the values of α for which any of the terms in the identity will be undefined:

$$\cos^2\alpha = 0$$
$$\therefore \cos\alpha = 0$$
$$\therefore \alpha = 90^\circ \text{ or } 270^\circ$$

We must also consider the values of α for which $\tan\alpha$ is undefined. Therefore, the identity is undefined for $\alpha = 90^\circ + k\,.\,180^\circ$.

Exercise 2 (Identities)

1. Simplify as far as possible:

(a) $\frac{\cos\alpha}{\sin\alpha}$ (b) $\frac{1-\sin^2 x}{\cos x}$

(c) $\frac{\cos^2\beta - 1}{1-\sin^2\beta}$ (d) $\frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$

(e) $\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$ (f) $\frac{\sin^3\theta + \sin\theta.\cos^2\theta}{\cos\theta}$

2. Prove that:

(a) $\sin x.\tan x + \cos x = \frac{1}{\cos x}$ (b) $\frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} = 2\tan x$

(c) $\tan x + \frac{1}{\tan x} = \frac{1}{\sin x.\cos x}$ (d) $\frac{\tan x}{\cos x(1+\tan^2 x)} = \sin x$

(e) $\frac{(1-\sin^2 x)^2}{\sin^2 x} = \frac{\cos^2 x}{\tan^2 x}$ (f) $\frac{1-\sin^2 x}{\sin^2 x + 2\sin x + 1} = \frac{1-\sin x}{1+\sin x}$

(g) $\sin^2 x + \sin^2 x.\tan^2 x = \tan^2 x$

REDUCTION FORM AND CO-FUNCTIONS

1. The reduction formulae hold for any angle θ. For convenience, we assume θ is an acute angle $(0° < \theta < 90°)$.
2. When determining function values of $(180° \pm \theta)$, $(360° \pm \theta)$ and $(-\theta)$ the function does not change.
3. When determining function values of $(90° \pm \theta)$ and $(\theta \pm 90°)$ the function changes to its co-function.

Second Quadrant:

- **sine function is positive**
- $\sin(180° - \theta) = \sin\theta$
- $\cos(180° - \theta) = -\cos\theta$
- $\tan(180° - \theta) = -\tan\theta$
- $\sin(90° + \theta) = \cos\theta$
- $\cos(90° + \theta) = -\sin\theta$

First Quadrant:

- **all functions are positive**
- $\sin(360° + \theta) = \sin\theta$
- $\cos(360° + \theta) = \cos\theta$
- $\tan(360° + \theta) = \tan\theta$
- $\sin(90° - \theta) = \cos\theta$
- $\cos(90° - \theta) = \sin\theta$

Third Quadrant:

- **tangent function is positive**
- $\sin(180° + \theta) = -\sin\theta$
- $\cos(180° + \theta) = -\cos\theta$
- $\tan(180° + \theta) = \tan\theta$

Fourth Quadrant:

- **cosine function is positive**
- $\sin(360° - \theta) = -\sin\theta$
- $\cos(360° - \theta) = \cos\theta$
- $\tan(360° - \theta) = -\tan\theta$

Negative angles

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$(\theta - 360°) = -(360° - \theta)$

$(\theta - 180°) = -(180° - \theta)$

$\sin(\theta - 180°) = \sin -(180° - \theta) = -\sin\theta$

Example:

$$\frac{\sin(A - 360°).\cos(90° + A)}{\cos(90° - A).\tan(-A)}$$

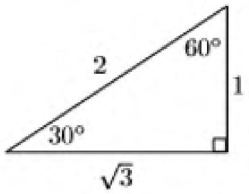
$$= \frac{\sin A(-\sin A)}{\sin A(-\tan A)}$$

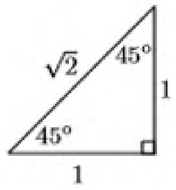
$$= \frac{\sin A}{\left(\dfrac{\sin A}{\cos A}\right)}$$

$$= \cos A$$

Special angle triangles

These values are useful when we need to solve a problem involving trigonometric functions without using a calculator. Remember that the lengths of the sides of a right-angled triangle obey the theorem of Pythagoras.





θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Determine the value of the expression, without using a calculator:

$$\frac{\cos 420° - \sin 225° \cos(-45°)}{\tan 315°}$$

SOLUTION

Step 1: Use reduction formulae to express each trigonometric ratio in terms of an acute angle

$$\begin{aligned}&\frac{\cos 420° - \sin 225° \cos(-45°)}{\tan 315°}\\ &= \frac{\cos(360° + 60°) - \sin(180° + 45°)\cos(-45°)}{\tan(360° - 45°)}\\ &= \frac{\cos 60° - (-\sin 45°)(\cos 45°)}{-\tan 45°}\\ &= \frac{\cos 60° + \sin 45° \cos 45°}{-\tan 45°}\end{aligned}$$

Now use special angles to evaluate the simplified expression:

$$\begin{aligned}&= \frac{\cos 60° + \sin 45° \cos 45°}{-\tan 45°}\\ &= \frac{\frac{1}{2} + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{-1}\\ &= -\left(\frac{1}{2} + \frac{1}{2}\right)\\ &= -1\end{aligned}$$

Co-function

Use the calculate to find the values of the following, Rounding off to TWO decimal places:

1) $\cos 20^{\circ} =$ ________________
2) $\sin 70^{\circ} =$ ________________
3) $\cos 45^{\circ} =$ ________________
4) $\sin 45^{\circ} =$ ________________
5) $\cos 60^{\circ} =$ ________________
6) $\sin 30^{\circ} =$ ________________
7) $\cos 0^{\circ} =$ ________________
8) $\sin 90^{\circ} =$ ________________
9) $\sin 15^{\circ} =$ ________________
10) $\cos 75^{\circ} =$ ________________
11) $\tan 23^{\circ} =$ ____________________
12) $\dfrac{\cos 67}{\sin 67} =$ ______________________

Conclusion____
____
____
____
____

Example:

Determine the value of $\dfrac{\tan 210^{\circ} \sin 160^{\circ} \cos 300^{\circ}}{\sin 150^{\circ} \cos 290^{\circ}}$

Solution:

$$\frac{\tan 210^{\circ} \sin 160^{\circ} \cos 300^{\circ}}{\sin 150^{\circ} \cos 290^{\circ}}$$

$$= \frac{\tan(180^{\circ} + 30^{\circ}) \sin(180^{\circ} - 20^{\circ}) \cos(360^{\circ} - 60^{\circ})}{\sin(180^{\circ} - 30^{\circ}) \cos(360^{\circ} - 70^{\circ})}$$

$$= \frac{\tan 30^{\circ} \sin 20^{\circ} \cos 60^{\circ}}{\sin 30^{\circ} \cos 70^{\circ}}$$

$$= \frac{\tan 30^{\circ} \sin 20^{\circ} \cos 60^{\circ}}{\cos 60^{\circ} \sin 20^{\circ}}$$

$$= \frac{\sqrt{3}}{3}$$

Exercise 1

1. Simplify the following:

(a) $\dfrac{\cos(180°+\theta).\cos(90°-\theta)}{\sin(90+\theta).\sin(180-\theta)}$

(b) $\tan(180°+\theta).\cos(90°+\theta)+\sin(360-\theta).\tan(180°-\theta)$

(c) $\sin^2(180°+\theta)-\cos^2(90°-\theta)$

(d) $\dfrac{\sin^2(360°-\theta)}{\cos(90°+\theta).\sin(540°-\theta)}$

(e) $\dfrac{\sin(-\theta-900°).\tan(180°+\theta).\cos(\theta-360°)}{\sin(-\theta).\cos\theta.\tan1485°}$

2. Evaluate without the use of a calculator.

(a) $\tan^2 135°$

(b) $\tan(-300°).\sin 600°$

(d) $\tan 315°-2\cos 60°+\sin 210°$

(d) $\tan120°.\cos 210°-\sin^2 315°$

(e) ` $\dfrac{\tan150°}{\tan 240°}-\dfrac{\sin 300°}{\sin120°}$

(f) $\dfrac{\sin 315°.\cos(-315°).\sin 210°}{\tan 225°}$

(g) $\sqrt{4^{\sin150°}.2^{3\tan 225°}}$

(h) $\dfrac{\tan 330°\sin120°\sin 260°}{\cos 225°\sin^2 315°\cos 350°}$

3. Calculate :

(a) $\cos 30°\times\sin 60°$

(b) $\sin 60°\cos 30°-\cos 60°\sin 30°$

(c) $\dfrac{\sin 45°.\sin 30°.\tan 60°}{\cos 60°.\tan 30°.\cos 45°}$

4. (a) Prove without using a calculator that:

(i) $\dfrac{\cos180°.\sin 225°.\cos 80°}{\sin170°.\tan135°}=-\dfrac{\sqrt{2}}{2}$

(ii) $\dfrac{\cos 315°+1}{\sin 315°-1}=-1$

(iii) $\theta=30°$ is a solution to $(\sin\theta)^{\sin\theta}=\dfrac{1}{\sqrt{2}}$

Equations and General Solutions

Equations

Example 1

Solve for x if $\sin x = 0,5 \quad 0° \leq x \leq 360°$

Solution:

$$\sin x = 0,5$$
$$x = \sin^{-1}(0,5)$$
$$x = 30°$$

With an understanding of the CAST diagram, (+'ve) sine can be in the 1st and 2nd Quadrants.

$x = 30°$ in the first quadrant

$x = 180° - 30°$ in the second quadrant
$x = 150°$

Example 2

Solve for x if $\cos(2x) = 0,87 \quad 0° \leq x \leq 360°$

Solution:

$$\cos(2x) = 0,87$$
$$2x = \cos^{-1}(0,87)$$
$$2x = 29,54°$$

With an understanding of the CAST diagram, (+'ve) cosine can be in the 1st and 4th Quadrant.

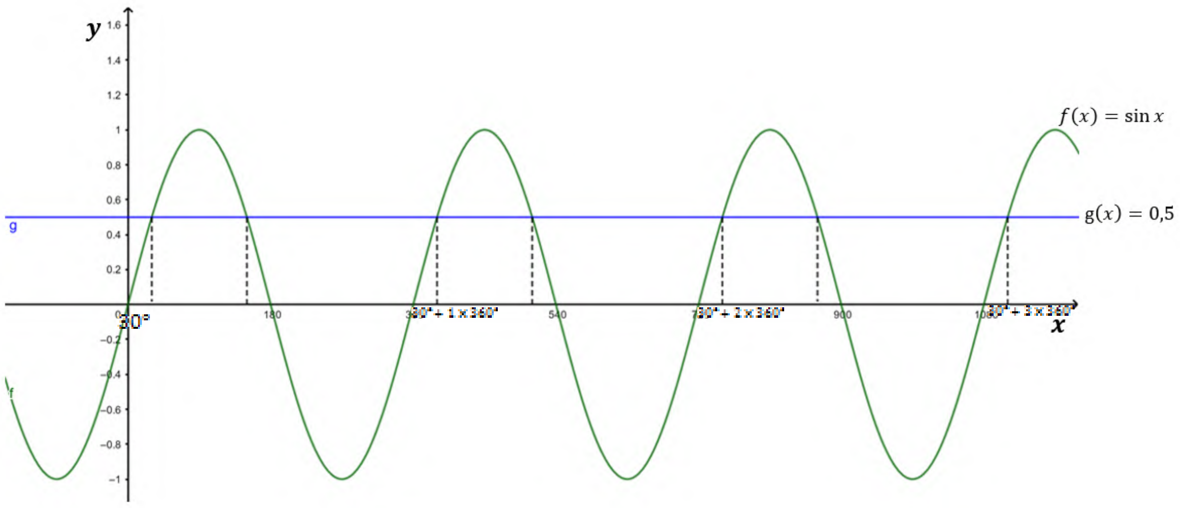
$2x = 29,54°$ in the first quadrant
$$x = \frac{29,54°}{2}$$
$$x = 14,77°$$

$2x = 360° - 29,54°$ in the fourth quadrant
$$2x = 330,46°$$
$$x = \frac{330,46°}{2}$$
$$x = 165,23°$$

Note that the examples above are for one revolution. We can then work out general solutions to find solutions for any number of revolutions.

General Solutions

Consider the graph of the functions of $f(x) = \sin(x)$ and $g(x) = 0,5$ below:



Notice that the graph of $f(x) = \sin(x)$ repeats itself every $360°$ and also intercepts the line of $g(x) = 0,5$ at $x = 30°$ and then $x = 30° + 1 \times 360°$ and also $x = 30° + 3 \times 360°$ and so on such that $x = 30° + k \times 360°$ where $k \in Z$. The same applies to the second quadrant solution. $x = 150°$ is a solution as well as $x = 150° + 1 \times 360°$ and $x = 150° + 2 \times 360°$ and so on such that $x = 150° + k \times 360°$ where $k \in Z$.

So, the general solution for the equation given in <u>Example 1</u> above would be solved as follows:

$$
\begin{aligned}
&\sin x = 0,5 \\
&\quad x = \sin^{-1}(0,5) \\
&\quad x = 30° + k \times 360° \quad k \in Z \\
&\quad\quad \text{and} \\
&\quad x = 150° + k \times 360° \quad k \in Z
\end{aligned}
$$

The cosine function also repeats itself every $360°$. The general solution for <u>Example 2</u> above would be laid out as follows:

$\cos(2x) = 0,87$

$2x = \cos^{-1}(0,87)$

$2x = 29,54° + k \times 360°$

$x = \dfrac{29,54° + k \times 360°}{2}$

$x = 14,77° + k \times 180° \qquad k \in Z$

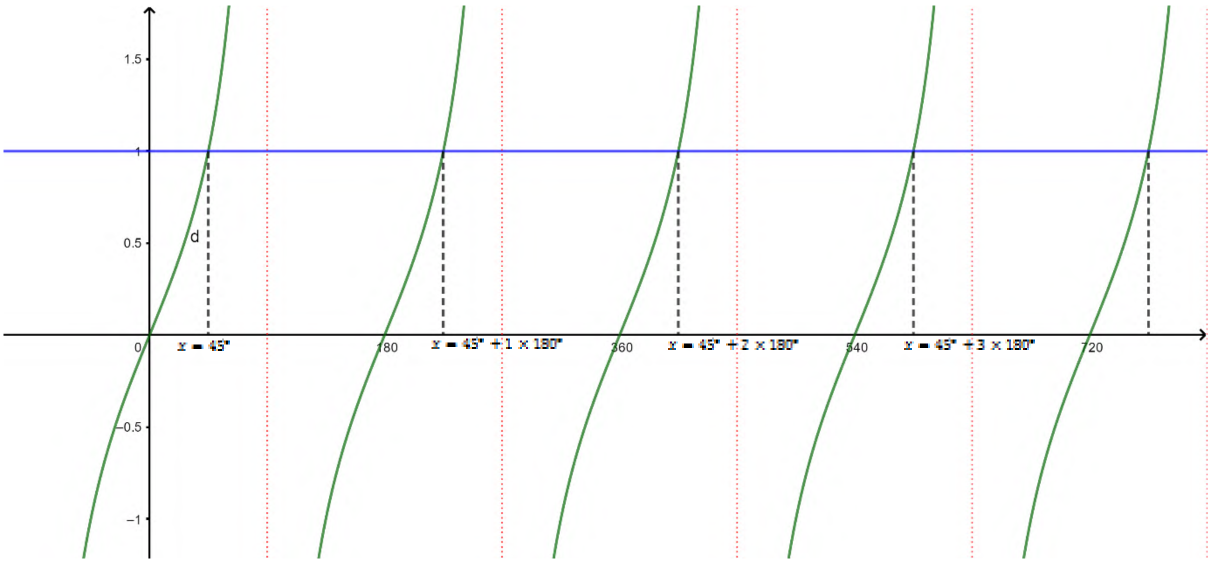
and

$2x = 330,46° + k \times 360°$

$x = \dfrac{330,46° + k \times 360°}{2}$

$x = 165,23° + k \times 180° \qquad k \in Z$

Consider the function of $f(x) = \tan(x)$ and how it intersects the line $y = 1$ below.



Because the tangent function repeats itself every $180°$ the general solution to the equation $\tan(x) = 1$ will be presented as follows:

Solve for x if $\tan(x) = 1$

<u>Solution</u>

$\tan(x) = 1$

$x = \tan^{-1}(1)$

$x = 45° + k \times 180°$

$x = 45° + 180°k \qquad k \in Z$

the above solution includes all possible values of *x*.

Trigonometric Equations That Use Identities and Require Factorisation

Solve for x, finding the general solution in each case:

1 $2\sin^2 x + \sin x = 0$

Solution

$$2\sin^2 x + \sin x = 0$$
$$\sin x(2\sin x + 1) = 0$$
$$\sin x = 0$$
$$x = 0° + 360°k$$
and
$$x = 180° + 360°k$$

or $2\sin x + 1 = 0$
$$2\sin x = -1$$
$$\sin x = -\frac{1}{2}$$
$$x = 180° + 30° + k360°$$
$$= 210° + k360°$$
$$OR$$
$$x = 360° - 30° + k360°$$
$$= 330° + k360°, \kappa \in Z$$

2 $2\cos^2 x - 2 - \sin x = 0$

.

Solution

$$2\cos^2 x - 2 - \sin x = 0$$
$$2(1 - \sin^2 x) - 2 - \sin x = 0$$
$$2 - 2\sin^2 x - 2 - \sin x = 0$$
$$-2\sin^2 x - \sin x = 0$$
$$2\sin^2 x + \sin x = 0$$
$$\sin x(2\sin x + 1) = 0$$
$$\sin x = 0$$
$$x = 0° + 360°k$$
and
$$x = 180° + 360°k$$

or $2\sin x + 1 = 0$
$$2\sin x = -1$$
$$\sin x = -\frac{1}{2}$$
$$x = 180° + 30° + k360°$$
$$= 210° + k360°$$
and
$$x = 360° - 30° + k360°$$
$$= 330° + k360°, \kappa \in Z$$

3. $\tan^2 x - 4 = 0$

Solution

$$\begin{aligned}\tan^2 x - 4 &= 0\\ (\tan x - 2)(\tan x + 2) &= 0\\ \tan x - 2 &= 0 \quad \text{or} \quad \tan x + 2 = 0\\ \tan x &= 2 \qquad\qquad \tan x = -2\\ x &= \tan^{-1}(2) \qquad x = \tan^{-1}(-2)\\ x &= 63,43° + 180°k \qquad x = -63,43° + 180°k, k \in Z\end{aligned}$$

4. $\tan^3 x + 8 = 0$

Solution

$$\begin{aligned}\tan^3 x - 8 &= 0\\ (\tan x - 2)(\tan^2 x + 2\tan x + 4) &= 0\\ \tan x - 2 &= 0 \quad \text{or} \quad \tan^2 x + 2\tan x + 4 = 0\\ \tan x &= 2 \qquad \text{no solution}\\ x &= \tan^{-1}(2)\\ x &= 63,43° + 180°k\end{aligned}$$

5. $2\sin^2 x - 7\sin x - 4 = 0$

Solution

$$\begin{aligned}2\sin^2 x - 7\sin x - 4 &= 0\\ 2\sin^2 x - 8\sin x + 1\sin x - 4 &= 0\\ 2\sin x(\sin x - 4) + 1(\sin x - 4) &= 0\\ (\sin x - 4)(2\sin x + 1) &= 0\\ \sin x - 4 &= 0\\ \sin x &= 4\\ &\text{no solution}\end{aligned}$$

or

$$\begin{aligned}2\sin x + 1 &= 0\\ 2\sin x &= -1\\ \sin x &= -\frac{1}{2}\\ x &= 180° + 30° + k360°\\ &= 210° + k360°\\ &\text{and}\\ x &= 360° - 30° + k360°\\ &= 330° + k360°, \kappa \in Z\end{aligned}$$

Mixed Exercise

1. Solve for x if $\sqrt{3}\tan x - 3 = 0$ and $x \in [180°;\ 360°]$

2. Find the general solution for x given that $4\sin^2 x - 1 = 0$

3. Find the general solution for x in each equation below:

   a) $\sin^2 x - \sin x \cos x = 0$

   b) $\cos^2 x - \sin x - 1 = 0$

   c) $-\cos x = \sin(x - 10°)$

   d) $\cos x = \tan(-x)$

   e) $2\sin x = \sqrt{2 + 3\cos x}$

   f) $2\cos^2 x + \cos x - 1 = 0$

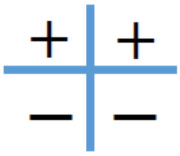
   g) $6\sin^2 x - \cos x = 5$

   h) $\tan^3 x + 8 = 0$

TRIGONOMETRIC GRAPHS

Complete the table and draw the following graphs on the same set of axis:

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
$y = \sin x$													
$y = 2\sin x$													
$y = 3\sin x$													
$y = \frac{1}{2}\sin x$													
$y = \cos x$													
$y = -2\cos x$													
$y = 3\cos x$													
$y = \frac{1}{2}\cos x$													



Signs in 4 quadrants

+ means graph is above x- axis

− means graph is above x- axis

Shape: Wave-like shape, starting at the origin

Intercepts: y-intercept = 0

x-intercept **(zeros)** = $0°;180°;360°$ (every $180°$ starting at $0°$)

Domain: The domain is usually limited to the interval $[0°;360°]$ or $[-360°;360°]$

Infinite angles are possible as a line centred at the origin on the Cartesian plane can be rotated many times. Rotating the line anti-clockwise gives positive angles, and rotating clockwise gives negative angles.

Period: The period of the function is the number of degrees the function needs to complete one cycle. This corresponds to one rotation. The sine function repeats itself every $360°$.

Trigonometric Functions are periodic functions and the graph form $0°$ to $360°$ is exactly the same as the graph from $360°$ to $720°$

Range: Minimum value: -1 when the angle x is $270°$ or $-270°$

Maximum value: 1 when the angle x is $90°$ or $-90°$

$[-1; 1]$

Amplitude: The amplitude of a trigonometric graph is the greatest distance the function moves above or below the x-axis.

(This only applies for sin or cos graphs). For the basic graph (also known as parent function) the amplitude is 1. It is half the range $(i.e., \frac{1+1}{2} = 1)$.

Strategies for Graphical Interpretations: Typical interpretation questions

Type of question	Interpretation
$f(x) < 0$	$f(x)$ must be less than zero $f(x)$ lies below the x-axis value is excluded
$f(x) < g(x)$	$f(x)$ must be less than $g(x)$ $f(x)$ lies below $g(x)$ e.g. $\sin(x)$ lies below $\cos(x)$
$f(x) \geq 0$	$f(x)$ must be greater than or equal to zero $f(x)$ lies above x-axis value is included
$f(x) > g(x)$	$f(x)$ must be greater than $g(x)$ $f(x)$ lies above $g(x)$ value is excluded
$f(x).\ g(x) < 0$	One of the two graphs must be above x-axis while the other must be below x-axis. Value is excluded
$f(x).g(x) \geq 0$	Both the two graphs are above the x-axis or both the two graphs are below the x-axis. Value is included.
$f(x)\ .\ g(x) = 0$	Either $f(x)=0$ or $g(x)=0$ x-intercepts of f(x) and g(x)
$f(x) - g(x) = 0$	$f(x) = g(x)$ Points of intersection
(	Excluded because of restriction
]	Included because of restriction
(;]	*first*-value excluded and *last*-value is included because of restriction
(;) or $<$ $>$ or ○	Values excluded
[;] or $\geq$ $\leq$ or •	Values included

Summary of the Basic characteristics of trigonometric graphs

Finding of	$y=a\sin bx$	$y=a\cos bx$	$y=a\tan bx$
Domain	$[-360^\circ;360^\circ]$	$[-360^\circ;360^\circ]$	$(-90^\circ;0^\circ],(-90^\circ;-270^\circ)$, $(-270^\circ;360^\circ],[0^\circ;90^\circ)$, $(90^\circ;270^\circ),(270^\circ;360^\circ]$
Range	$[-1;\ 1]$	$[-1;\ 1]$	$[-\infty;\ \infty]$
Period	$\frac{360^\circ}{\lvert b\rvert}=\frac{360^\circ}{\lvert 1\rvert}=360^\circ$	$\frac{360^\circ}{\lvert b\rvert}=\frac{360^\circ}{\lvert 1\rvert}=360^\circ$	$\frac{180^\circ}{\lvert b\rvert}=\frac{180^\circ}{\lvert 1\rvert}=180^\circ$
Amplitude	$a=1$	$a=1$	No
Minimum value	-1	-1	No
Maximum value	1	1	No
Asymptote	NO	NO	$x=-90^\circ, x=-270^\circ$ $x=90^\circ, x=270^\circ$

NOTE: the value of *b* affects the period only and the amplitude (maximum & minimum) remains the same.

1. The Worked Out Examples/Activities

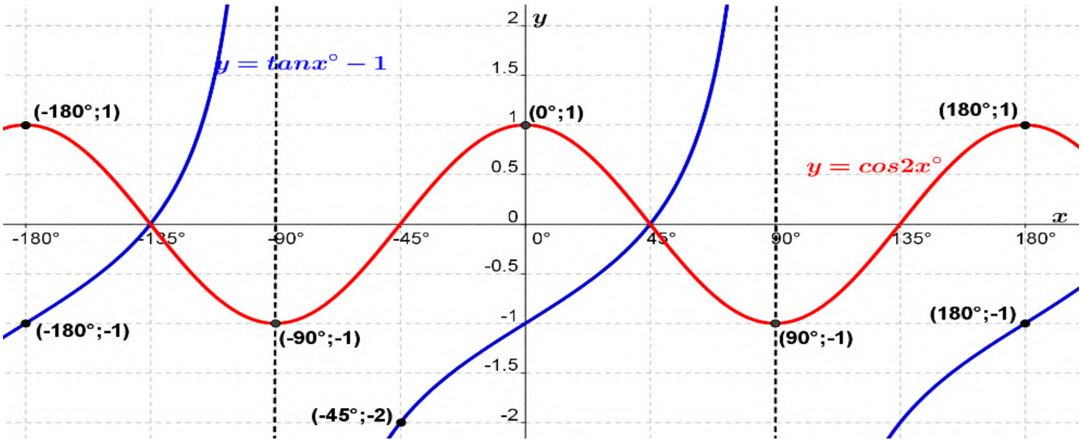
Example 1.

On the same system of axes, draw the sketch graphs of:

$y = tanx° - 1$ and $y = cos2x°$, for the interval[-180°; 0°]. Show all the intercepts with the axes and the co-ordinates of the turning points. Show the asymptotes of $y = tanx° - 1$.

(a) Use the sketch graphs to the value of x if:$cos2x° + 1 \leq tanx°$, in the interval [-180°; 0°].
(b) If the curve of $y = tanx° - 1$is moved upwards by 3 units, what will the new equation be?
(c) Write down the period of $y = cos2x°$

Solution

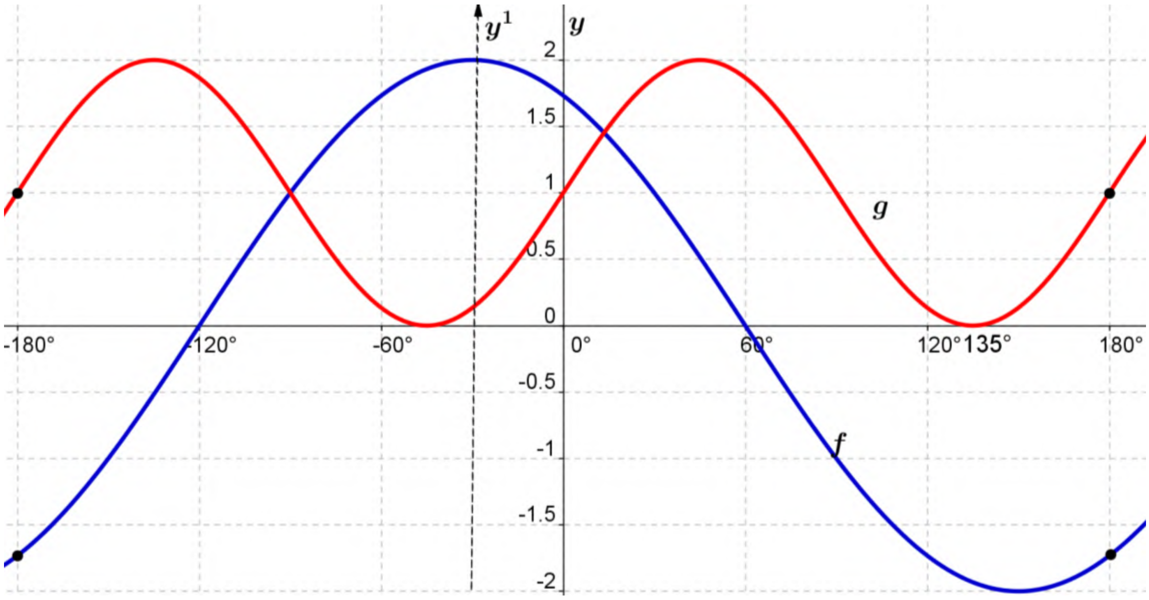


(a) $cos2x° + 1 \leq tanx°$
$cos2x° \leq tanx° - 1$
$-135° \leq x < 90°$
(b) $y = tanx° + 2$
(c) Period of $y = cos2x°$: 180°

Example 2

In the given figure the graphs of:

$f = \{(x; y)|y = \text{a}cos(x° + b)\}$ and $g = \{(x; y)|y = c + sindx°\}$ if $x \in [-180°; 180°]$ are given.



(a) the values of a, b, c and d by using the graphs, and write down the two equations in the form of: $y = \cdots$

(b) $f(0°)$ without using a calculator.

(c) x from the graphs if:

(i) $g(x) = 2$ and $x \in [-180°; 180°]$

(ii) $f(x) < g(x)$ and $x \in [-180°; 0°]$

(d) The y −axis is moved to the y^1 position in order to pass through a turning point of f.

the equation of f in the form: $y = \cdots$ with respect to the new system of axes.

Solution

(a) $a = 2$; $b = 30°$, $c = 1$ and $d = 2$

$\therefore f(x) = 2\cos(x° + 30°)$ and $g(x) = 1 + sin2x°$

(b) $f(0°) = 2cos30°$

$= 2\frac{\sqrt{3}}{2}$

$= \sqrt{3}$

(c) (i) $x = 45°; -135°$

(iii) $-180° \leq x < -90°$

(d) $y = 2\cos(x° + 30° - 30°)$

$y = 2cosx°$

2. Activities
(Sketching and Interpreting Graphs)

QUESTION 1

Given the functions $f(x) = 2sinx$ and $g(x) = tanx - 1$.

1.1 Sketch the graphs of f and g on the same system of axes **on the diagram sheet**,

for $x \in [-45°; 180°]$, clearly labelling endpoints, turning points, intercepts with the axes and asymptotes. (7)

1.2 Use your graphs to determine the value(s) of x for which $f(x).\ g(x) < 0$ (2)

1.3 If $h(x) = f(x) + 2$, write down the range of h. (2)

QUESTION 2

2.1 Draw the graphs of $y = \sin 4x$ and $y = \sin 2x$ on the same system of axes for $x \in [0°;180°]$ (4)

2.2 For which values of x ε [0°; 180°] is $\sin 4x \geq \sin 2x$? (2)

QUESTION 3

3.1 Use the system of axes on **DIAGRAM SHEET 3** to sketch the graphs of:

$$f(x) = -\frac{1}{2}\sin(x + 30°) \text{ and } g(x) = \cos 2x \text{ if } -180° \leq x \leq 180°$$

(6)

3.2 Write down the period of g. (1)

3.3 Graph h is obtained when the y-axis for f is moved 120° to the left.

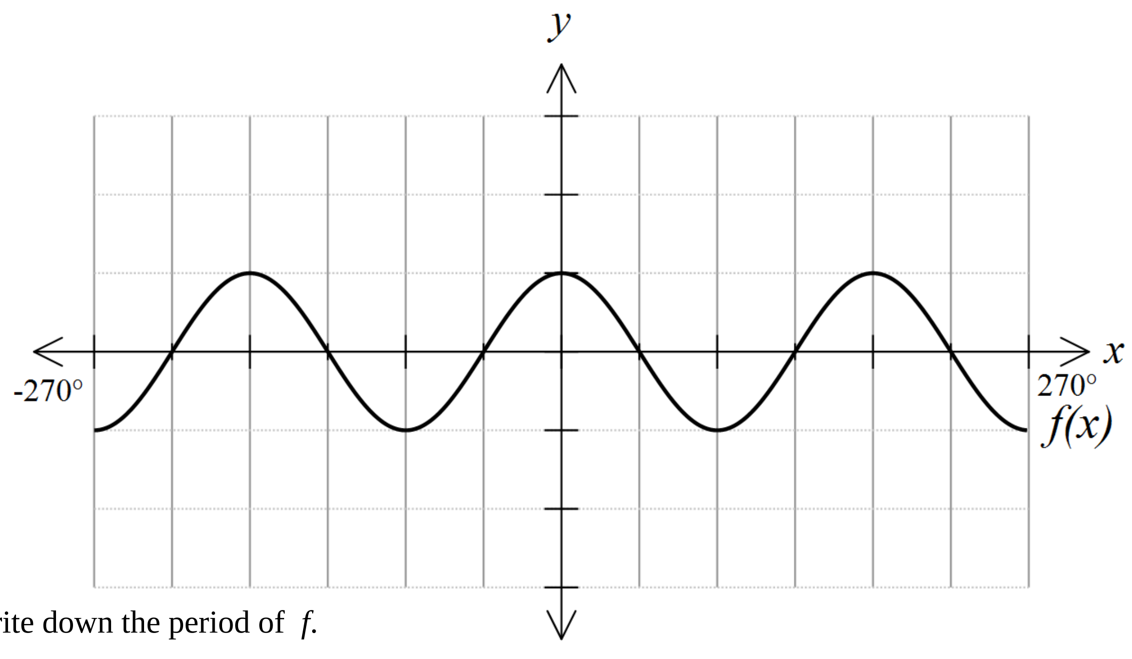
Give the equation of h in the form $h(x) =$.......... (2)

3.4 Determine the general solution of: $\cos 2x = 1$ **(2)**

(Sketching and Interpreting trig graphs/general solution)

QUESTION 4

The graph of $f(x) = \cos 2x$ for $-270° \leq x \leq 270°$ is drawn below



4.1 Write down the period of f. (1)

4.2 Write down the range of $\frac{f(x)}{2}$. (2)

4.3 Draw the graph of $g(x)=1+\sin x$ on the same set of axes as $f(x)$.
Show all turning points and intercepts with the axes. (3)

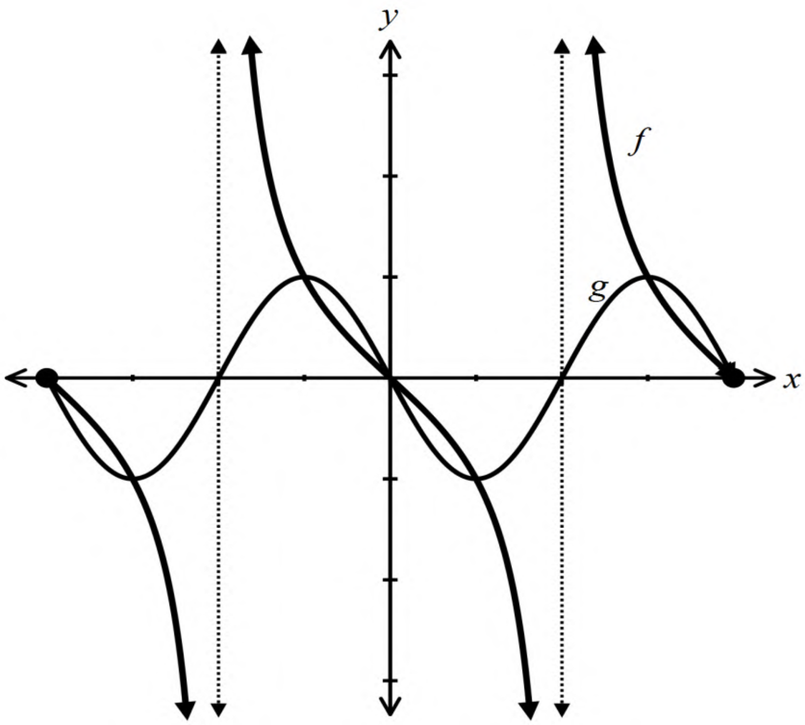
4.4 Use the graphs to determine the value(s) of x for which:

4.4.1 $g(x)-f(x)=3$ in the interval $0°<x<180°$ (1)

4.4.2 $f(x).g(x)\leq 0$ in the interval $0°<x<180°$ (2)

QUESTION 5

The graphs of $f(x)=a\tan x$ and $g(x)=-\sin bx$ for $x\in\left[-180^{\circ};180^{\circ}\right]$ are drawn below



Use the graphs to answer the following:

5.1 Determine the values of a and b. (2)

5.2 Solve for $x\in\left[0^{\circ};180^{\circ}\right]$ if $2\sin x\cos x-\tan x=0$ (3)

5.3 For which values of $x\in\left[-90^{\circ};180^{\circ}\right]$ is $g(x)\geq f(x)$? (4)

5.4 If $g(x)$ is shifted 90° to the right to form a new graph $h(x)$, write down the equation of $h(x)$ in its simplest form (2)

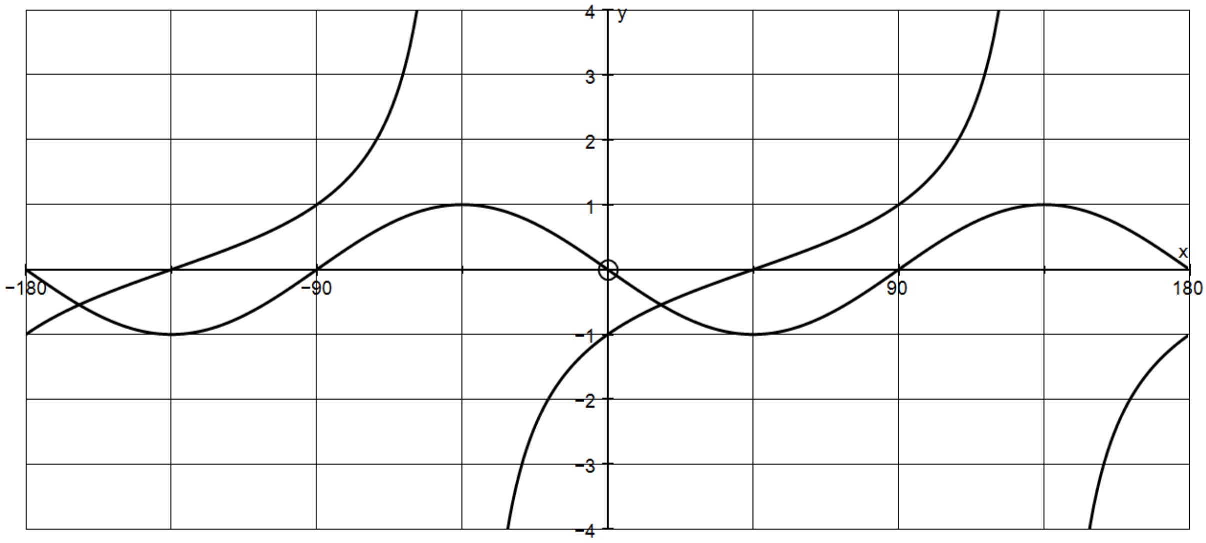
QUESTION 6

A function is defined as $f(x) = a \sin bx + c$

The function satisfies the following conditions:

- The period is $120°$
- The range is $y \in [-2; 6]$
- The co-ordinates of a maximum point are $(210°; 6)$

Write down the values of a, b and c [4]



Question 7

The diagram shows the graphs of $f(x) = \tan(x + d)$ and $g(x) = e\sin qx$ for the interval $-180° \leq x \leq 180°$.

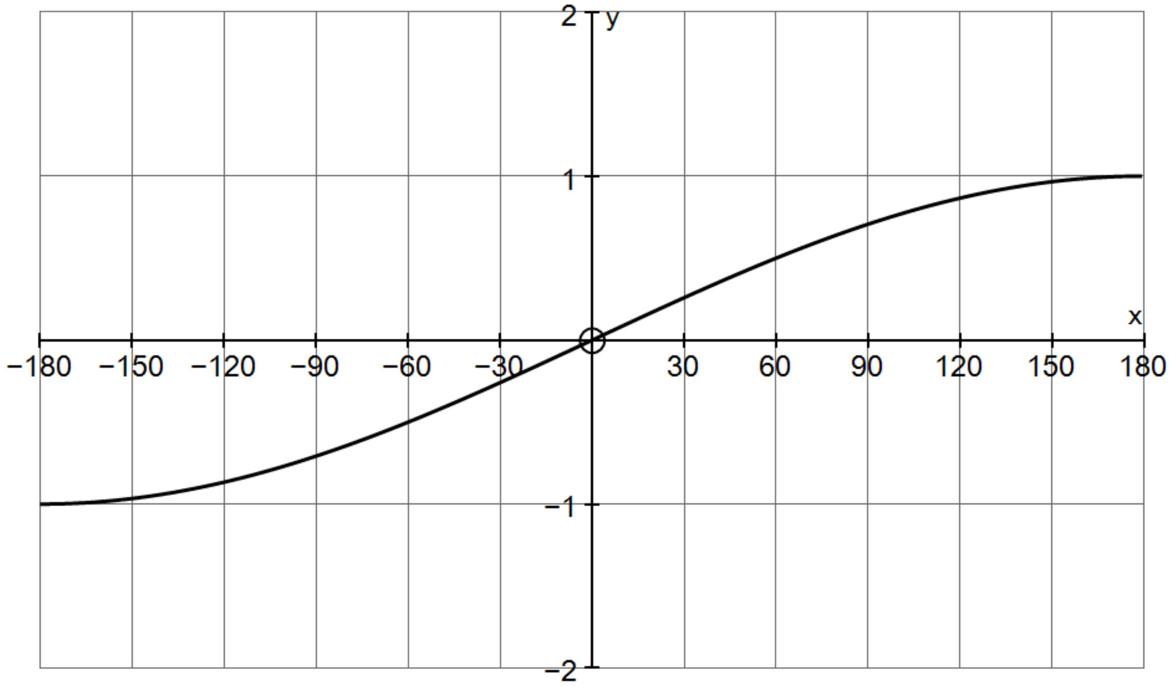
The graphs intersect at A$(x_a; y_a)$ and B$(x_b; y_b)$.

7.1 Write down the values of d, e and q. [3]

7.2 Use the graphs to solve for x if

7.2.1 $g(x) < 0$ [3]

7.2.2 $f'(x)$ and $g'(x)$ are both greater than or equal to zero but $x < 0$. [3]



QUESTION 8

Sketched below is $f(x)=\sin\frac{x}{2}$

8.1 For $f(x)$, write down the

8.1.1 range (1)

8.1.2 period (1)

8.2 Draw $g(x) = \cos(x + 60°)$ for $x \in [-180°; 180°]$.Clearly draw all intercepts with axes, turning points and starting and ending points. (3)

8.3 For which values of $x \in [-180°; 180°]$ is $f(x).g(x) \geq 0$. (4)

FORMULAE AND THE SOLUTION OFTRIANGLES

HIGHTS AND DISTANCES

The Prior Knowledge The Background Knowledge

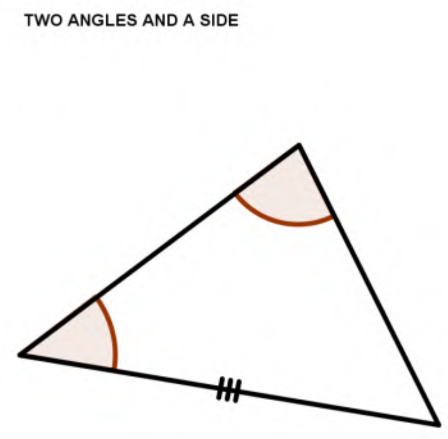
The Assumed Knowledge The Previous Knowledge

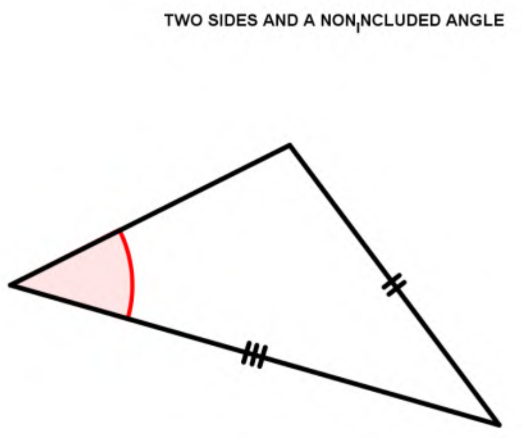
The Perceived Knowledge

1. THE SINE FORMULA

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \textbf{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Remember: The sine formula is used if **two angles and a side** are given in a triangle, or if **two sides and a non-included angle** are given.





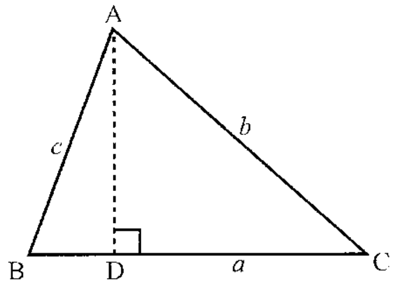
Remember: Opposite the longest side is the largest angle.

∴ When the triangle is obtuse-angled, the longest side is opposite the obtuse angle.

Given: An acute-angled ΔABC

Required to prove: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Construction: $AD \perp BC$



Proof:

$$\sin B = \frac{AD}{c}$$

$$\therefore AD = c.\sin B$$

$$\sin C = \frac{AD}{b}$$

$$\therefore AD = b.\sin C$$

$$\therefore c.\sin B = b.\sin C$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly by drawing CE ⊥ AB, it can be proved that:

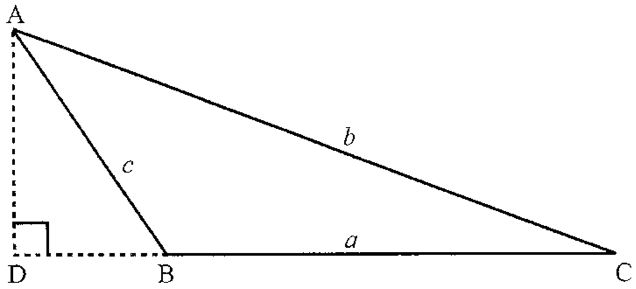
$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Given: ΔABC with B > 90°

Required to prove: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Construction: AD ⊥ BC with CB produced.



Proof:

$$\sin A\hat{B}C = \sin A\hat{B}D = \frac{AD}{c}$$

$$\therefore AD = c.\sin B$$

$$\sin C = \frac{AD}{b}$$

$$\therefore AD = b.\sin C$$

$$\therefore c.\sin B = b.\sin C$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly by drawing CE ⊥ AB with AB produced, it can be proved that:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

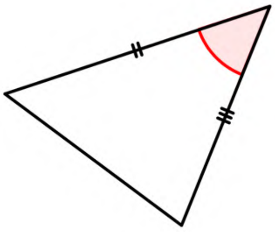
$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

2. THE COSINE FORMULA

$$a^2 = b^2 + c^2 - 2bcCosA$$

$$b^2 = a^2 + c^2 - 2acCosB$$

$$c^2 = a^2 + b^2 - 2abCosC$$



Remember: We use the cosine formula in this form to the third side of a triangle when two sides and the included angle are given.

$CosA = \frac{b^2+c^2-a^2}{2bc}$ $\quad CosB = \frac{a^2+c^2-b^2}{2ac}$ $\qquad CosC = \frac{a^2+b^2-c^2}{2ab}$

Remember: We use this formula in this form if **three sides** of a triangle are given and an angle must be calculated.

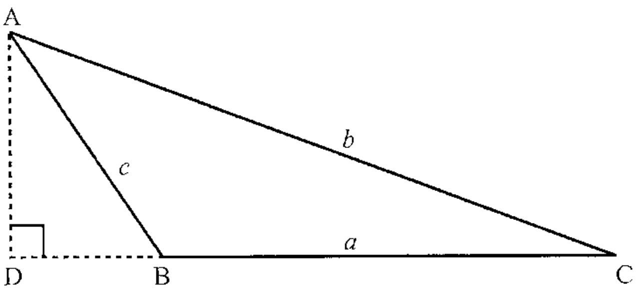
NB:

The Cosine-formula is used when the information in the triangle entails: S, S, S or S, A, S.

Given: ΔABC with $B > 90°$

Required to prove: $b^2 = a^2 + c^2 - 2ac \cos B$

Construction: $AD \perp BC$ with CB produced.



Proof:

$$\begin{aligned} b^2 &= AD^2 + DC^2 \text{ (Pythagoras in } \Delta ADC) \\ &= AD^2 + (a + BD)^2 \\ &= AD^2 + a^2 + 2a \cdot BD + BD^2 \\ &= (AD^2 + BD^2) + a^2 + 2a \cdot BD \\ &= c^2 + a^2 + 2a \cdot BD \end{aligned}$$

$$\cos A\hat{B}C = -\cos A\hat{B}D$$
$$= -\frac{BD}{c}$$
$$\therefore BD = -c\cos B$$
$$\therefore b^2 = a^2 + c^2 - 2ac\cos B$$

3. THE AREA FORMULA

Area of a ΔABC= $\frac{1}{2}abSin\hat{C}$

Area of a ΔABC = $\frac{1}{2}bcSin\hat{A}$

Area of a ΔABC = $\frac{1}{2}acSin\hat{B}$

Remember: The area formula is used to calculate the area of a triangle.

An unknown side can also be d if the area, a side and an angle are given. Note that this formula actually means: The area of a triangle= $\frac{1}{2}$(product of two adjacent sides) multiplied by the sine of an included angle.

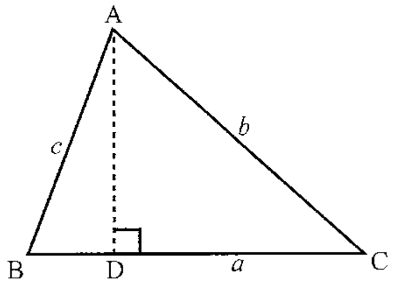
∴ In order to apply this formula, you only need: S, A, S in the triangle.

Proof :

Given: Acute-angled ΔABC.

Required to prove: Area $\Delta ABC = \frac{1}{2}ab\sin C$

Construction: $AD \perp BC$

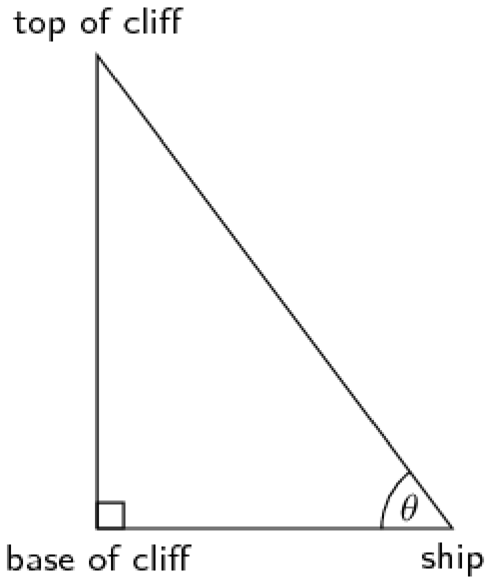


Proof: Area of $\Delta ABC = \frac{1}{2}a \cdot AD$

$$\sin C = \frac{AD}{b}$$
$$AD = b.\sin C$$
$$\therefore \text{Area } \Delta ABC = \frac{1}{2}ab\sin C$$

Two-dimensional

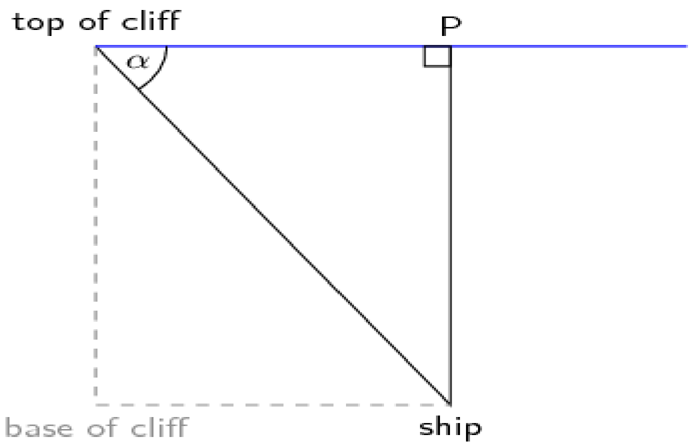
In two-dimensional problems we will often refer to the angle of elevation and the angle of depression. To understand these two angles let us consider a person sailing alongside some cliffs. The person looks up and sees the top of the cliffs as shown below:



In this diagram θ is the angle of elevation.

Angle of elevation

> The angle of elevation is the angle formed by the line of sight and the horizontal plane for an object above the horizontal plane.



In this diagram α is the angle of depression.

Angle of depression

> The angle of depression is the angle formed by the line of sight and the horizontal plane for an object below the horizontal plane.

4. SOLVING PROBLEMS IN TWO DIMENSIONS

Always in problems involving two or more triangles, the same method as for a single triangle is used.

HINTS, CLUES

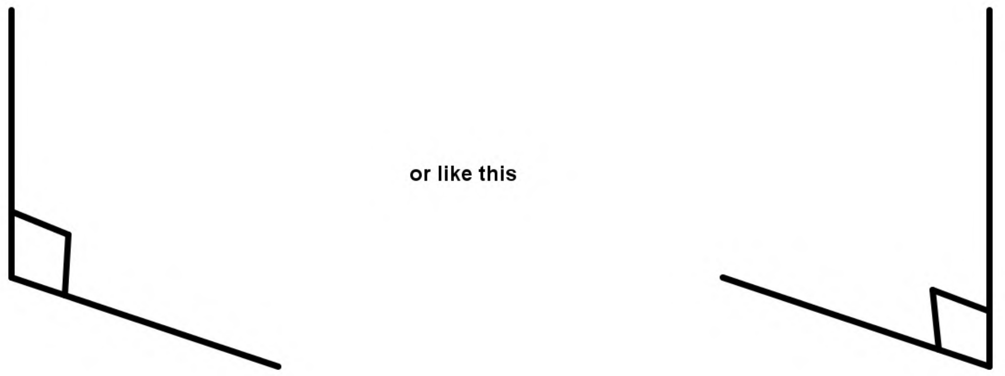
- The problem usually involves two triangles with a common side
- Often, one of the triangles is right-angled
- Use Geometry to obtain additional information, e.g. exterior angle of a triangle, corresponding and alternate angles
- Decide in which triangle the required side occurs. Start with the other triangle and calculate the common side using the sine or cosine formula
- Then use the sine formula or the cosine formula or trigonometrical ratios to solve the problem.

5. PROVING A FORMULA

- Sometimes we are required to prove some sort of a formula, before calculating a side or an angle.
- We use the same procedure as in solving a problem
- Trigonometrical identities such as: $\boldsymbol{Sin}(90° - \theta) = \boldsymbol{Cos}\theta$, $\boldsymbol{Cos}(90° - \theta) = \boldsymbol{Sin}\theta$ $\boldsymbol{Sin}[180° - (x + y)] = \boldsymbol{Sin}(x + y)$, etc. are used.

PROBLEMS IN THREE DIMENSIONS

- In three-dimensional problems right angles often don't look like right angles.

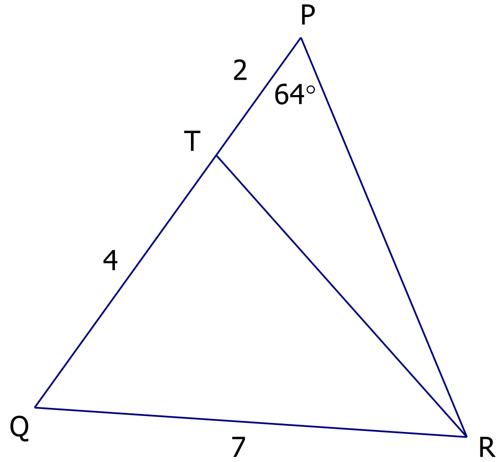


- Draw all vertical lines, vertical, so that a right angle may look like this:

- Always shade the horizontal plane roughly.
- Where you encounter problems with three triangles, you must work from the one with the most information via the second to the third.
- The cosine formula is used more often than in problems in two dimensions.

8. Examples/Activities

QUESTION 7



In the diagram, ΔPQR is drawn with T on PQ.

$\angle P = 64°$

QR = 7 units

PT = 2 units

QT = 4 units

1.1.1 Calculate the size of $\angle Q$, correct to the nearest degree. (4)

1.1.2 If $\angle Q = 66°$, determine the following:

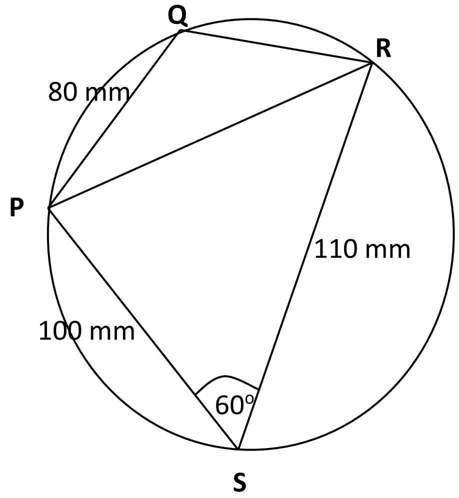
1.1.2.1 the area of ΔTQR. (3)

1.1.2.2 the length of TR. (3)

1.2 In the figure below PQ = 80 mm, PS = 100 mm, SR = 110 mm. $P\hat{S}R = 60°$.

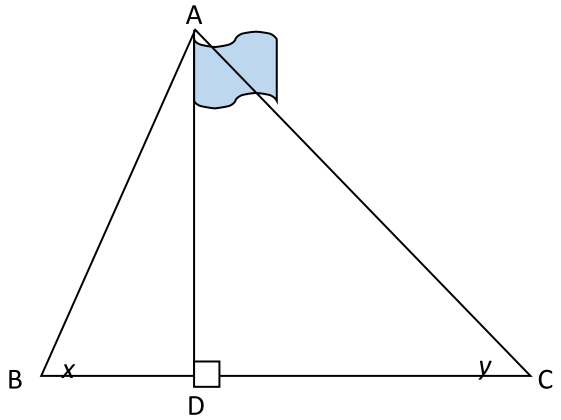
1.2.1 Show, by calculation that PR = 105,36 m (2)

1.2.2 Find the area of ΔPRS. (2)



1.2.1 B and C are in the same horizontal plane as D, the foot of flagpole AD.

The angles of elevation to A (the top of the flagpole) are x and y degrees from B and C respectively. The sketch below illustrates the situation.

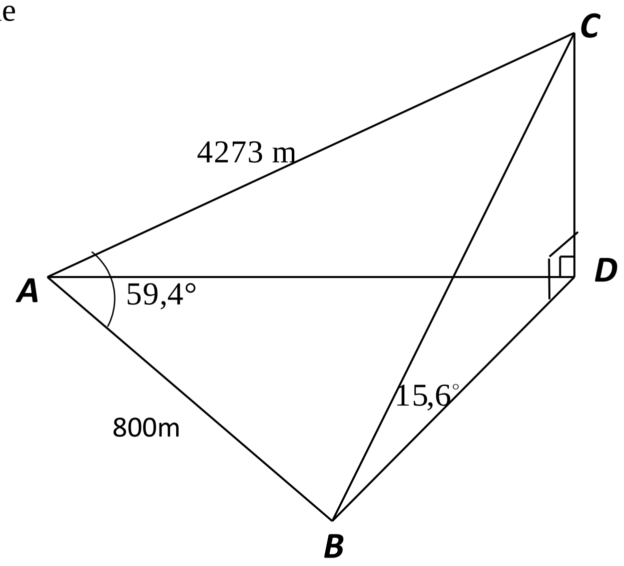


1.4.1 Show that: $AD = \frac{BC.\sin x\ .\sin y}{\sin(x+y)}$ (Hint: First find the length of AB) (5)

QUESTION 2

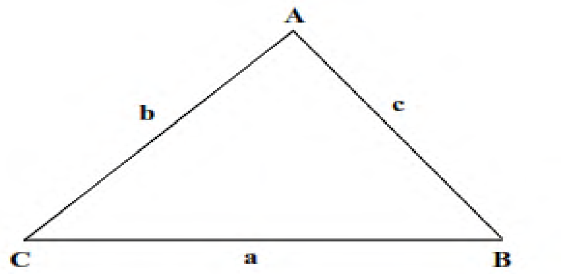
2.1 C is the top of a tower CD. A, B and D are in the same horizontal plane. The distance between A and B is 800m. CA is 4273m, $C\hat{A}B = 59,4°$ and the angle of elevation of C from B is $15,6°$.

Calculate the height of the tower, CD.



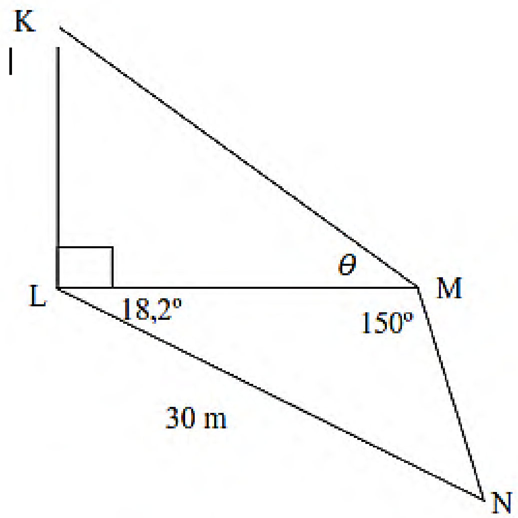
QUESTION 3

3.1 Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove that: $c^2 = a^2 + b^2 - 2\,ab.\cos C$



3.2 In the diagram below, KL is a building. Points L, M and N are in the same horizontal plane. The angle of elevation from M to the top of the building is .

$L\hat{M}N = 150°$ and $M\hat{L}N = 18{,}2°$. $LN = 30\ metres$.



3.2.1 Show that $KL = 60\tan\theta\ .\sin 11{,}8°$. (5)

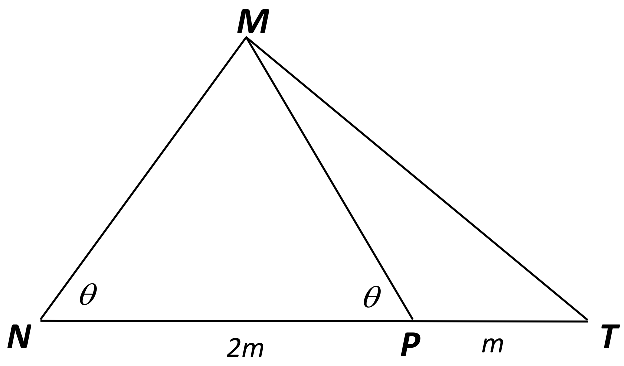
3.2.2 Calculate the height of the building, KL, if $\theta = 52{,}7°$. (2)

3.2.3 Calculate the area of ΔLMN. (2)

3.4 MNP is a triangle and P is a point on NT. MP is joined.

PT = m, NP = $2m$ and

$M\hat{P}N = \hat{N} = \theta$.



Prove :

Area $\Delta MTP = \frac{1}{2}m^2 \tan\theta$.

(4)

EUCLIDEAN GEOMETRY

GRADE 10

- Revise basic results established in earlier grades.
- Investigate line segments joining the mid-points of two sides of a triangle.
- Define the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and the trapezium. Investigate and make conjectures about properties of the sides, angles, diagonals and areas of these quadrilaterals. Prove these quadrilaterals.
- Solve problems and prove sides using the properties of parallel lines, triangles and quadrilaterals.

GRADE 11

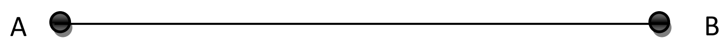
- Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles.
  - The line drawn from the centre of a circle perpendicular to a chord bisects the chord;
  - The perpendicular bisector of a chord passes through the centre of the circle;
  - The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);
  - Angles subtended by a chord of the circle, on the same side of the chord, are equal;
  - The opposite angles of a cyclic quadrilateral are supplementary;
  - Two tangents drawn to a circle from the same point outside the circle are equal in length;
  - The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.
- Solve circle geometry problems, providing reasons for statements when required.
- Prove riders.

TERMINOLOGY

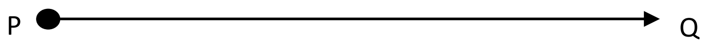
6.1 **POINT:** It is a location that can be described by giving its coordinates. It has no length or width. It is usually represented by a DOT (.). A capital letter (alphabet) is used to denote a point .e.g. P.

6.2 **LINE:** A set of points joined together. It can be straight or curved.

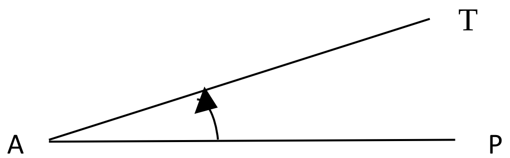
6.2.1 Line Segment: is obtained when two points are joined from one point **directly** to another. It has a fixed length. e.g.



1.2.2 **RAY**: a portion of a line which starts at a point and continue infinitely. It has no measurable length because it goes forever.



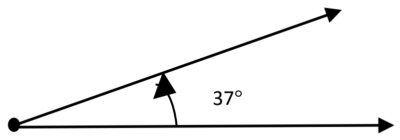
ANGLE: It is formed when two line segments meet at a point called a vertex. A shape formed by two lines or rays diverging from a common point (vertex).



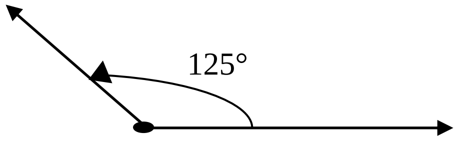
AP is a fixed arm and AT is a rotating arm. A protractor is an instrument used to measure an angle. The unit of measurement is degrees and is denoted by

TYPES OF ANGLES

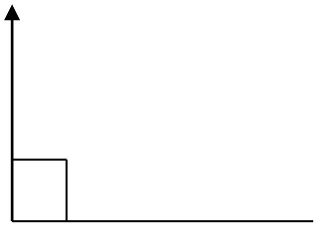
2.1 **ACUTE** ANGLE: angle between 0° and 90°



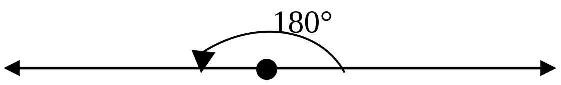
2.2 **OBTUSE** ANGLE: angle between 90° and 180°



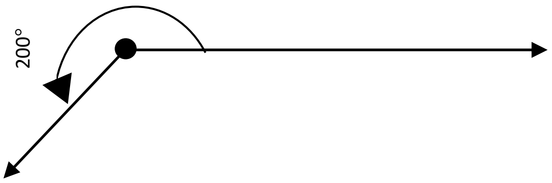
2.3 **RIGHT** ANGLE: angle which is of the size 90°



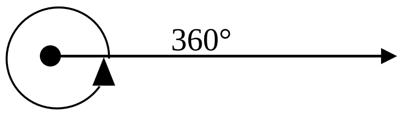
2.4 **STRAIGHT** ANGLE: angle of which the size is 180°



2.5 **REFLEX** ANGLE: angle between 180° and 360°

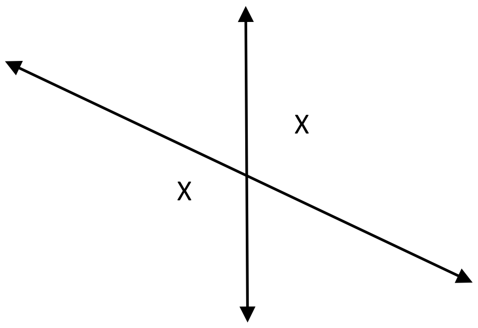


2.6 **REVOLUTION**: (FULL ANGLE) angle of which its magnitude is 360°

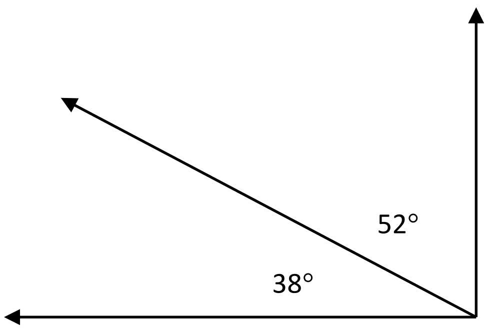


ANGLE RELATIONSHIP

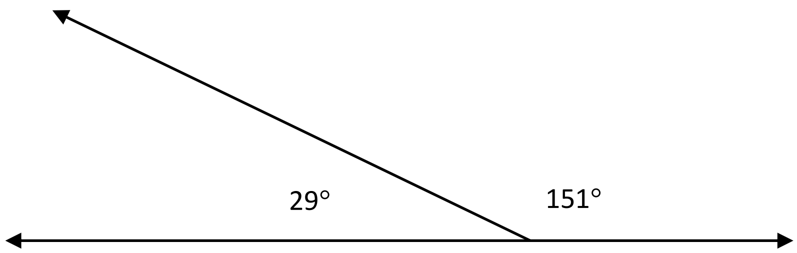
- Vertically Opposite: formed by intersection of two straight lines. Its "vertical" because they share the same vertex not that they are upright. They are equal.



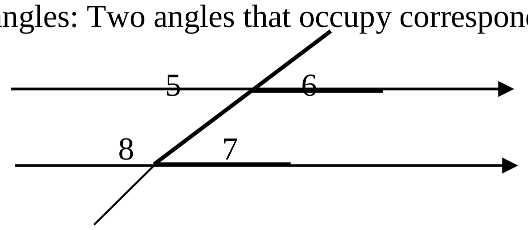
- Complementary angles: they add up to 90°



- Supplementary: they add up to 180°



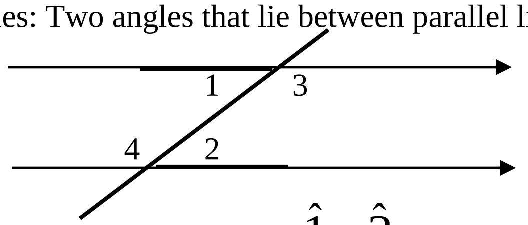
- Corresponding angles: Two angles that occupy corresponding positions (They form **F shape**).



$$\hat{6} = \hat{7}$$

$$\hat{5} = \hat{8}$$

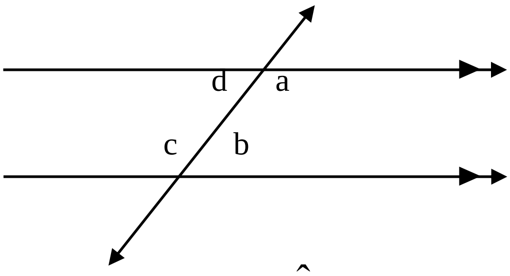
- Alternate angles: Two angles that lie between parallel lines on opposite sides of the transversal.



$\hat{1} = \hat{2}$

$\hat{3} = \hat{4}$

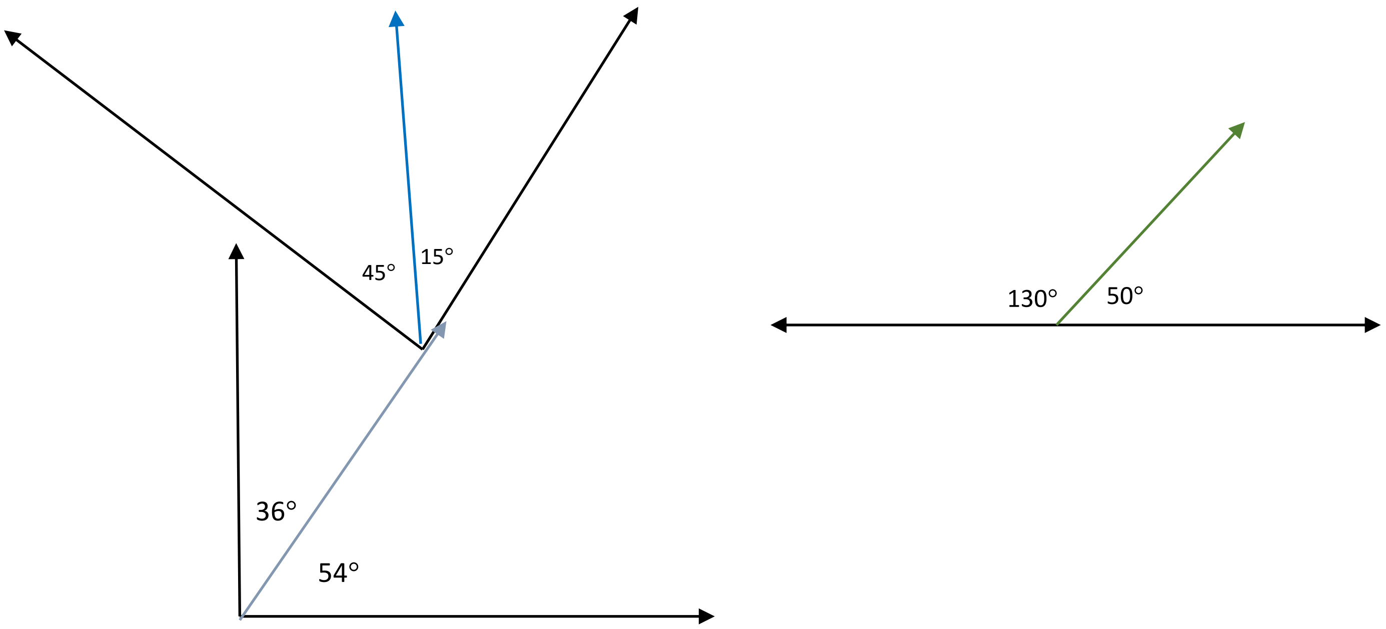
- Co-interior angles: Two angles that lie between parallel lines on the same side of the transversal. They add up to 180°.



$\hat{a} + \hat{b} = 180°$

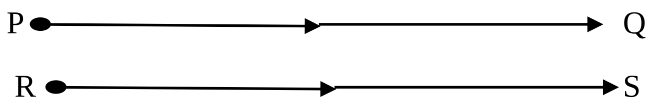
$\hat{d} + \hat{c} = 180°$

- Adjacent angles: Are "side by side" and share a common ray.



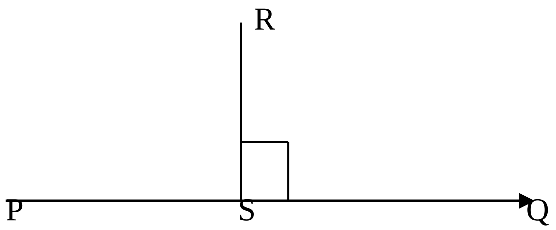
TYPES OF LINES

3.1 **PARALLEL** LINES: Lines which will never meet. They are denoted by sign //**.** They are always the same distance apart.

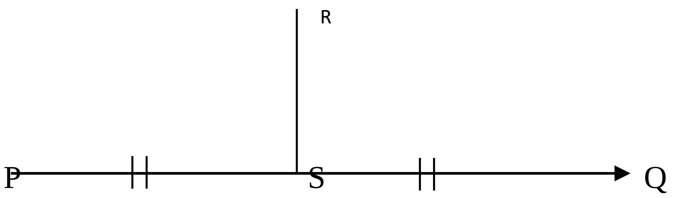


PQ // RS

3.2 **PERPENDICULAR** LINES**:** Lines that form an angle of 90 at their point of contact.

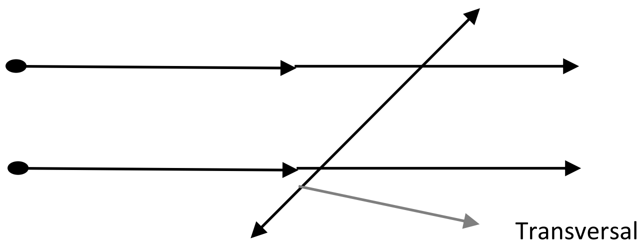


3.3 **BISECTOR**: A line, ray or line segment which cuts another line into two equal parts.



PS = SQ

3.3 **TRANSVERSAL** LINE: A line that cuts across the parallel lines.

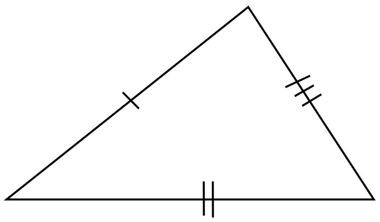


TRIANGLES

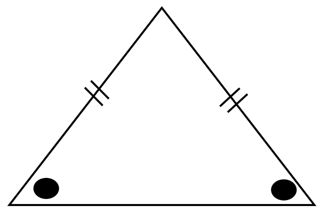
It is a closed geometrical figure with three sides and three interior angles. The three angles always add up to 180°.

TYPES OF TRIANGLES

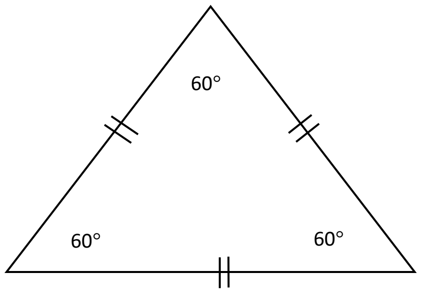
a) Scalene: no sides are equal and no angles are equal.



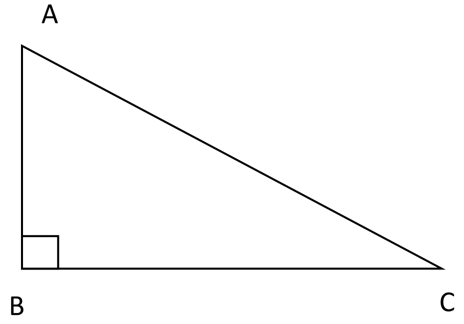
b) Isosceles: two sides are equal and two angles opposite equal sides.



c) Equilateral: three sides are equal and three angles are equal, each equal to 60°.



d) Right-angle: one angle is equal to 90°.

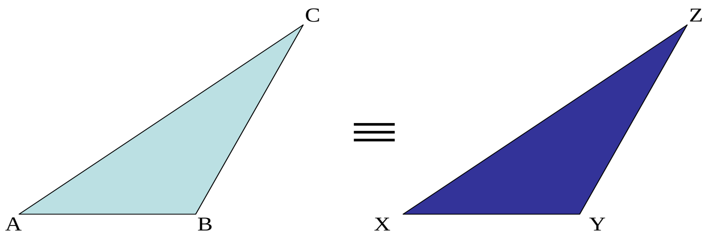


CONGRUENCY

CONGRUENT TRIANGLES

Congruent triangles are triangles that have the same shape and size. .i.e. corresponding sides are equal and corresponding angles are equal.

ΔABC is congruent to ΔXYZ

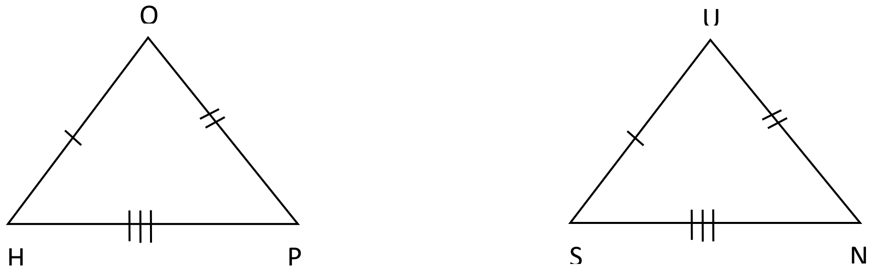


Corresponding parts of these triangles are equal.

Corresponding parts are angles and sides that "match."

CONDITIONS FOR CONGRUENCY

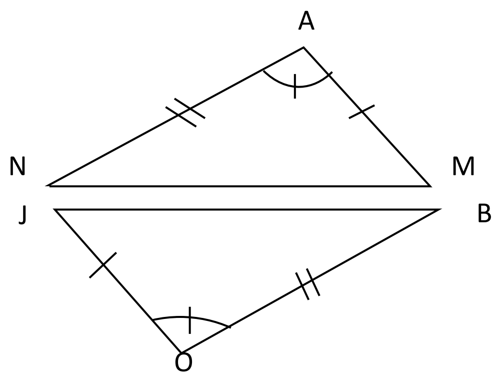
1



Δ HOP ≡ Δ SUN (SSS)

i.e. corresponding sides (SSS) of the two triangles are equal.

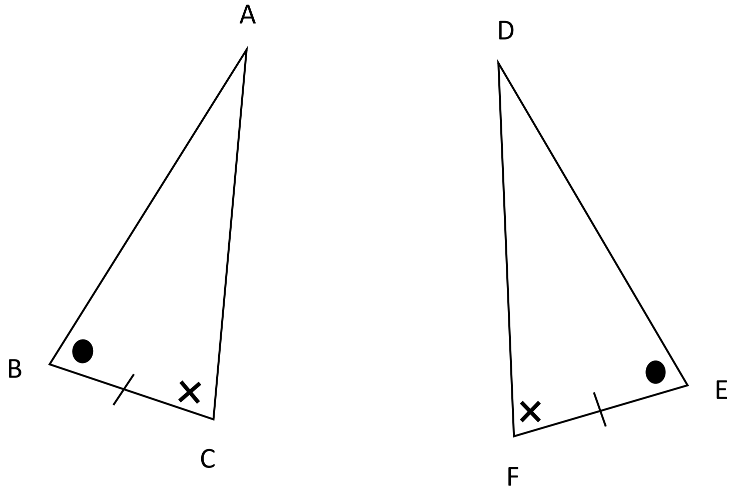
2



Δ MAN ≡ Δ JOB (SAS)

i.e. if two sides and the angle between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

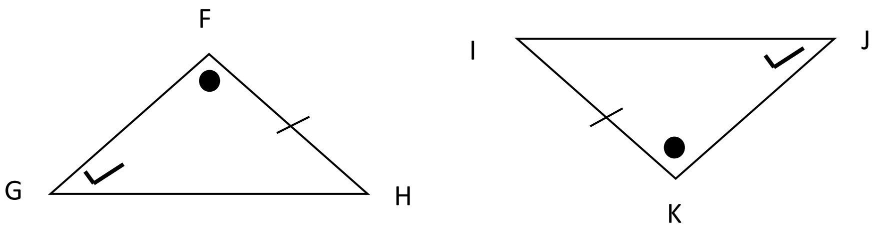
3



$\triangle$ **ABC** $\equiv$ $\triangle$ **DEF** **(ASA)**

i.e. If two angles and a side between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

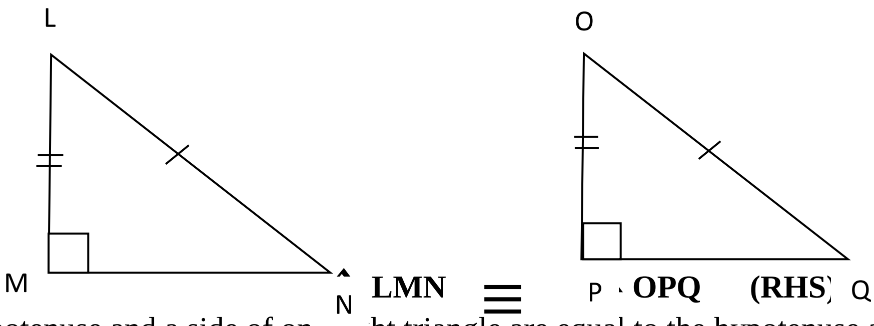
4



$\triangle$ **FGH** $\equiv$ $\triangle$ **KJI** **(AAS)**

i.e. If two angles and a side NOT between them in one triangle are equal to the corresponding parts in another triangle, then the triangles are congruent.

5



$\triangle$ **LMN** $\equiv$ $\triangle$ **OPQ** **(RHS)**

i.e. If the hypotenuse and a side of one right triangle are equal to the hypotenuse and side of another triangle, then the triangles are congruent.

SUMMARY

- For Grade 10, congruency is limited to triangles only.
- Conditions for triangles to be congruent:
  - → S,S,S
  - → S,A,S (Included Angle)
  - → A,A,S

→ A,S,A(Included Side)
→ R,H,S

QUADRILATERALS

A **quadrilateral** is a plane figure bounded by four sides.

Definitions of quadrilaterals

A **trapezium** is a quadrilateral with only one pair of sides parallel.

A **kite** is a quadrilateral with two pairs of adjacent sides equal but with no side common to both pairs.

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

A **rhombus** is a parallelogram with a pair of adjacent sides equal.

A **rectangle** is a parallelogram with a right angle.

A **square** is a rectangle with a pair of adjacent sides equal or a **square** is a rhombus with a right angle.

Five ways of proving that a quadrilateral is a parallelogram

Try to prove that:
Both pairs of opposite sides are parallel
Or both pairs of opposite sides are equal
Or both pairs of opposite angles are equal
Or the diagonals bisect each other
Or one pair of opposite sides is parallel **and** equal.

GRADE 11 EUCLIDEAN GEOMETRY

Note:

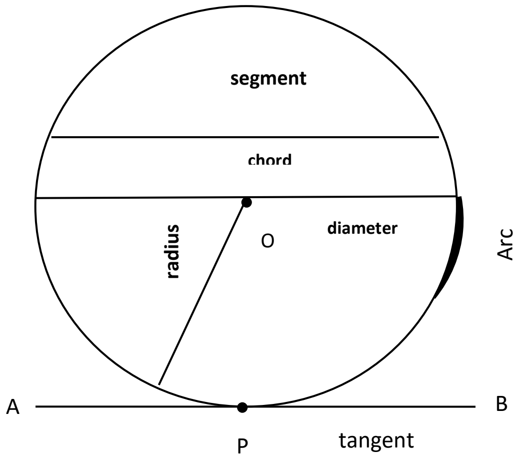
Proofs of theorems can be asked in examinations, but their converses (where they hold) cannot be asked.

CIRCLE GEOMETRY

Definitions

- A **circle** is a set of points that are equidistant from a fixed point called the **center.**
- The **circumference** of the circle is the distance around the edge of a circle.
- The **radius** is a line from the centre to any point on the circumference of the circle.
- A **chord** divides the circle into two **segments**.
- A **diameter** is a chord that passes through the centre. It is the longest chord and is equal to twice the radius.
- An **arc** is part of the circumference.
- A **semi-circle** is half the circle.
- A **tangent** is a line touching the circle at a point.
- **Cyclic quadrilaterals** have all their vertices on the circumference of a circle.

PARTS OF A CIRCLE



Ways Of Presenting A Proof Argument

1. **Classical method:** This is done by writing two columns (HOLY-CROSS), the first being a list of statements and the second column a matching list of legal justifications. i.e. theorems and axioms referred to as REASONS.

e.g.

Statement	Reason

TYPES OF QUESTIONS FOR EUCLIDEAN GEOMETRY

1. Calculations
2. Expressing an angle in terms of:
3. Proof type
1. Most of PROOF type questions would require that we first prove that angles are equal. How to Prove that angles are equal:

Suppose we are to prove that $\hat{A} = \hat{B}$

Scenario 1: (1) Write $\hat{A}$ and below it write $\hat{B}$

Statement	Reason
$\hat{A}$	
$\hat{B}$	

(2) Look for any angle equal to $\hat{A}$ (with reason)

Statement	Reason
$\hat{A} = \hat{P}$	Vertically opposite
$\hat{B}$	

(3) Compare $\hat{P}$ with $\hat{B}$. In most cases the two angles will be equal, then:

Statement	Reason
$\hat{A} = \hat{P}$	Vertically opposite
$\hat{B} = \hat{P}$	tan-chord theorem

(4) Then conclude:

Statement	Reason
$\hat{A} = \hat{P}$	vertically opposite
$\hat{B} = \hat{P}$	tan-chord theorem
$\therefore \hat{A} = \hat{B}$	transitivity of equality

2. **For any PROOF type questions, there is always 3 steps to follow.**

   Step 1: Ask a question of WHEN

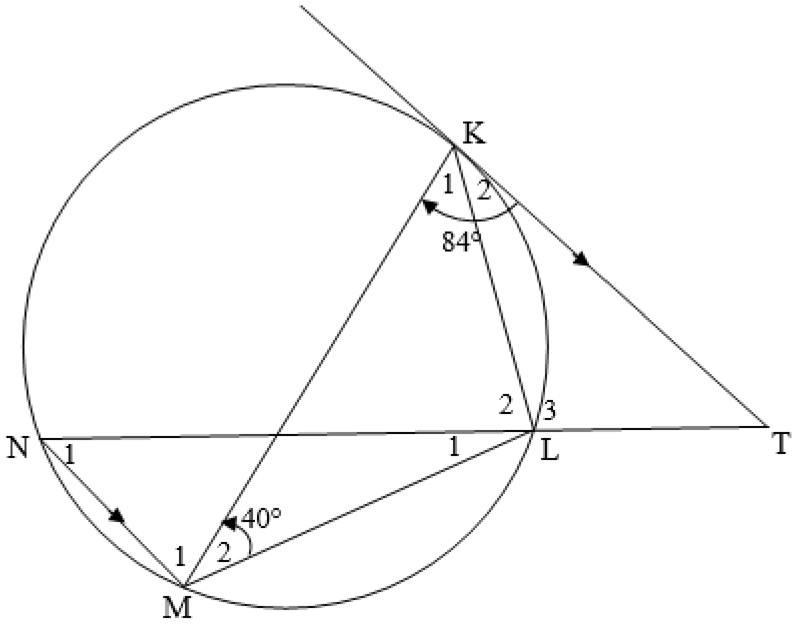
   Step 2: Give possible answers to the question above.

   Step 3: Prove one of the answers in Step 2 above.

EXAMPLES

Examples 1

In the diagram below, tangent KT to the circle at K is parallel to the chord NM. NT cuts the circle at L. ΔKML is drawn. $M_2 = 40^0$ and $M\hat{K}T = 84^0$



Determine, giving reasons, the size of:

1.1 $\hat{K}_2$

1.2 $\hat{N}_1$

1.3 $\hat{T}$

1.4 $\hat{L}_2$

1.5 $\hat{L}_1$

Solutions:

From the given information, key words are tangent parallel lines and chords. Therefore statements and reasons will be based on theorems which have these words. Use those theorems to determine the sizes of angles which their sizes are not given. When you put the size of an angle, write a short hand reason.

i.e. $M_1 = 84^0$ Alternate angles, NM // KT , $M_1 = L_2$ subtended by the same arc KN, $K_2 = 40^0$ tan chord theorem, $K_1 = N_1 = 44^0$ subtended by ML, $L_1 = 180^0 - (124 + 40^0) = 16^0$ sum of angles of a triangle etc.

1.1 $\hat{K}_2 = \hat{M}_2 = 40^0$ tan chord theorem

1.2 $\hat{N}_1 = \hat{K}_1 = 84^0 - 40^0 = 44^0$ subtended by chord ML

1.3 $\hat{T} = N_1 = 44^0$ Alternate angles NM // KT

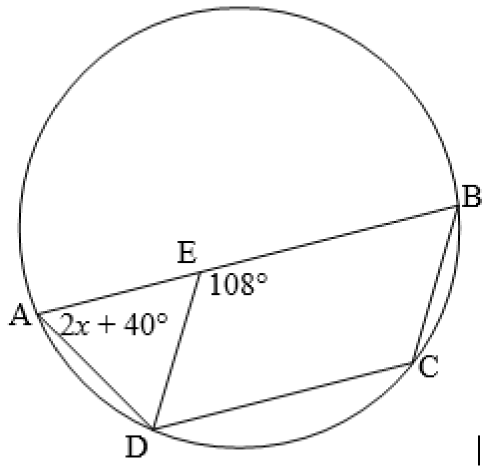
1.4 $\hat{M}_1 = 84^0$ Alternate angles, NM // KT

$L_2 = M_1 = 84^0$ subtended by the same arc KN

1.5 $\hat{L}_1 = 180^0 - (\hat{M} + \hat{N}_1) = 180^0 - (124 + 40^0) = 16^0$ sum of angles of a triangle

Example 2

In the diagram below, AB and DC are chords of a circle. E is a point on AB such that BCDE is a parallelogram. $D\hat{E}B = 108°$ and $D\hat{A}E = 2x + 40°$.



Calculate, giving reasons, the value of x.

Solution:

Properties of parallel gram and cyclic quadrilateral.

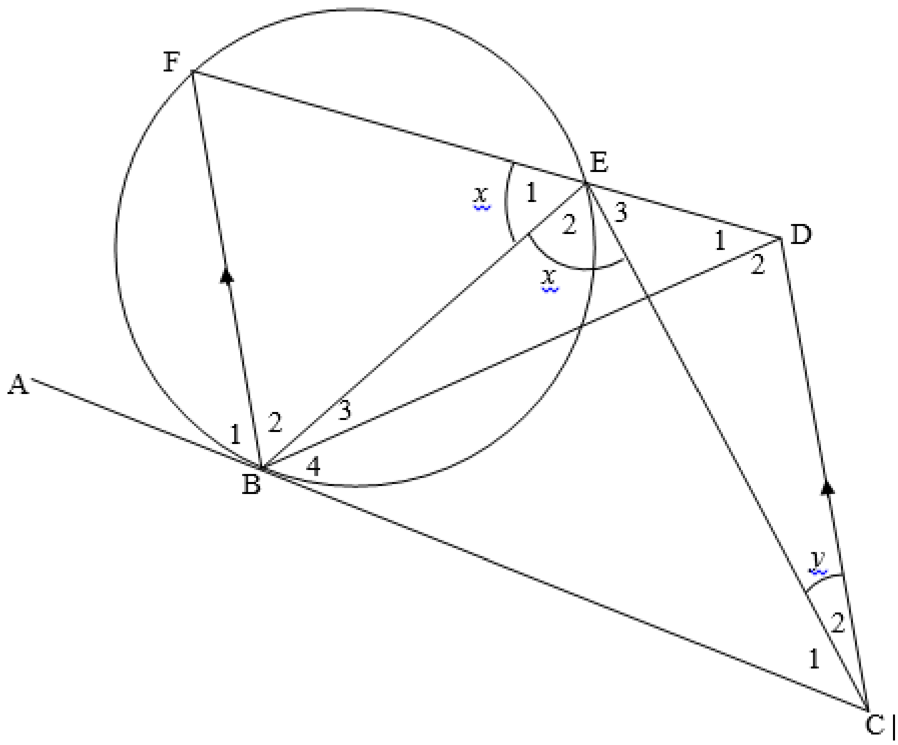
$\hat{C} = 108^0$ Opposite angles of a parallelogram are equal

$\hat{C} + D\hat{A}E = 180^0$;$108^0 + 2x + 40^0 = 180^0$ Opposite angles of cyclic quadrilateral ABCD

$\therefore 2x = 180^0 - 148^0 \Rightarrow 2x = 32^0$

$x = 16^0$

Example 3



3.1 Give reasons for the following statements

3.1.1 $\hat{B}_1 = x$

3.1.2 $B\hat{C}D = \hat{B}_1$

3.2 Prove that BCDE is a cyclic quadrilateral.

3.3 Which TWO other angles are each equal to x?

3.4 Prove that $\hat{B}_2 = \hat{C}_1$.

Solutions

3.1.1 tangent chord theorem

3.1.2 corresponding; FB || DC

3.2 $\hat{E}_1 = B\hat{C}D$

$\therefore$ BCDE = cyclic quad [converse ext $\angle$ cyc quad]

3.3 $\hat{D}_2 = \hat{E}_2$ [$\angle s$ in the same segment/$\angle e$ *in dies segment*]

$\hat{D}_2 = F\hat{B}D$ [alt $\angle s$, BF$\parallel$CD/*verwiss* $\angle e$,*BF*||*CD*]

3.4 $\hat{B}_3 = y$ *OR* $\hat{B}_3 = \hat{C}_2$ [$\angle$s in the same segment]

$\hat{B}_2 = x - y$ *OR* $\hat{B}_3 + \hat{B}_2 = x$

$\hat{C}_1 = x - y$

$\therefore \hat{B}_2 = \hat{C}_1$

OR/*OF*

In ΔBFE and ΔBEC

$\hat{E}_1 = \hat{E}_2$ [$= x$]

$\hat{F} = \hat{B}_3 + \hat{B}_4$ [tan - chord theorem]

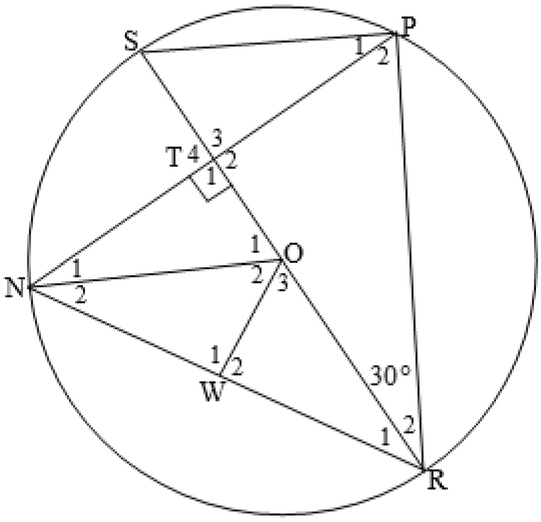
$\therefore \Delta BFE /// \Delta CBE$ [$\angle, \angle, \angle$]

$\therefore \hat{B}_2 = \hat{C}_1$

EXERCISE

QUESTION 1

In the diagram, the vertices of ΔPNR lie on the circle with centre O. Diameter SR and chord NP intersect at T. Point W lies on NR. OT $\perp$ NP. $\hat{R}_2 = 30°$.



1.1 $\hat{S}$

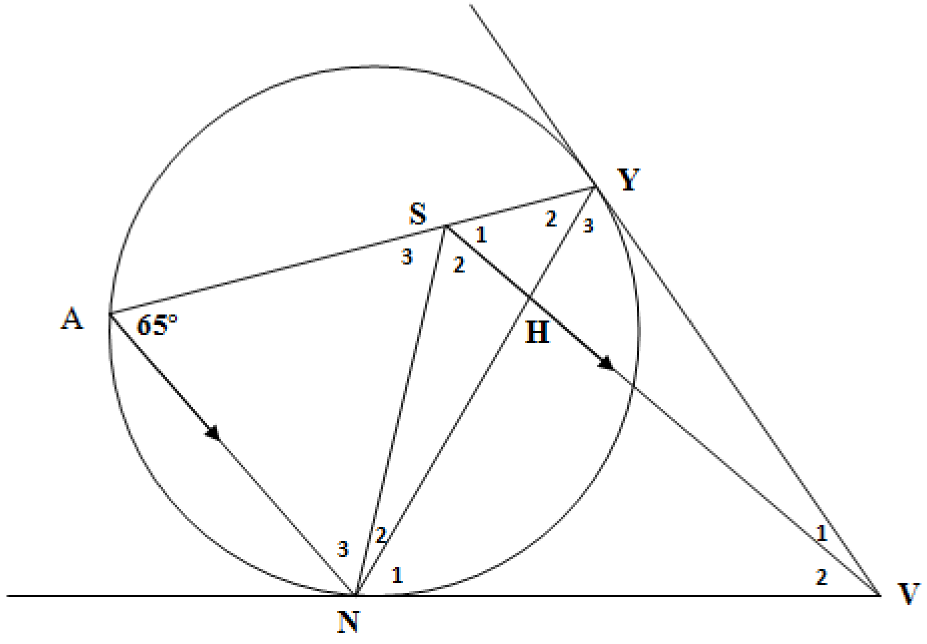
1.2 $\hat{R}_1$

1.3 $\hat{N}_1$

1.4 If it is further given that NW = WR, prove that TNWO is a cyclic quadrilateral.

QUESTION 2

VN and VY are tangents to the circle at N and Y. A is a point on the circle, and AN, AY and NY are chords so that $\hat{A} = 65°$. S is a point on AY so that AN || SV. S and N are joined



2.1 Write down, with reasons, THREE other angles each equal to 65°

2.2 Prove that VYSN is a cyclic quadrilateral.

2.3 Prove that ΔASN is isosceles.

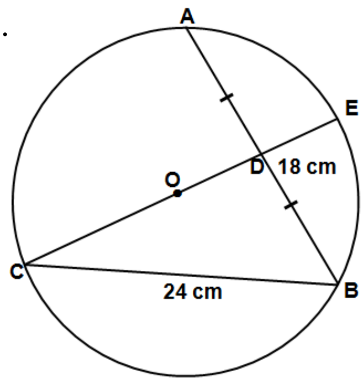
QUESTION 3

3.1 Complete the following so that the Euclidean Geometry statement is true:

A line drawn from the centre of a circle to the midpoint of the chord is ....to the chord (1)

3.2 In the circle with centre O, chord AB = 18cm and AD = DB.

Chord CB = 24 cm.



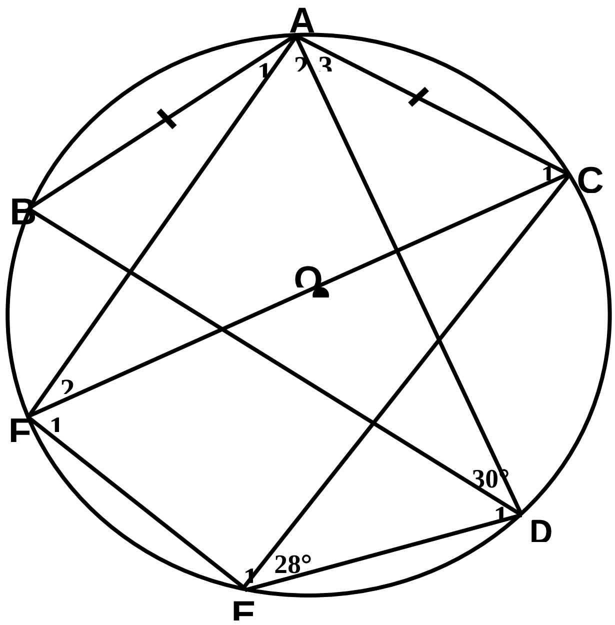
3.2.1 Calculate the length of CD. Leave the answer in simplest surd form. (4)

3.2.2 If $\frac{CD}{DE} = 3$, calculate the length of DE. (2)

[7]

QUESTION 4

In the diagram, O is the centre of the circle. Chords AB = AC. $C\hat{E}D = 28^\circ$ and $A\hat{D}B = 30^\circ$



Calculate, with reasons, the sizes of the following angles:

4.1 $\hat{E}_1$ (2)

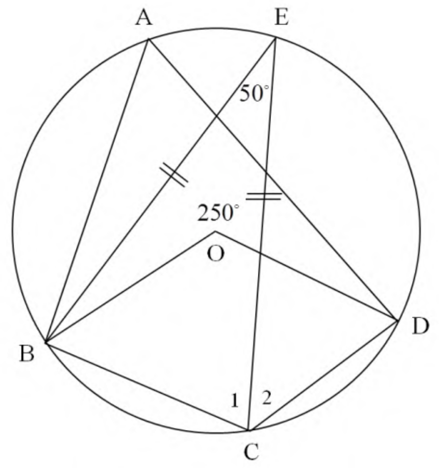
4.2 $\hat{A}_2$ (3)

4.3 $\hat{F}_2$ (2)

[7]

QUESTION 5

Refer to the figure below:



The circle, centred at O, has points A, B, C, D and E on the circumference of the circle. Reflex angle $B\hat{O}D = 250°$ and $B\hat{E}C = 50°$. Chord BE = EC. Determine the following, **stating all necessary reasons**:

5.1 $\hat{A}$ (2)

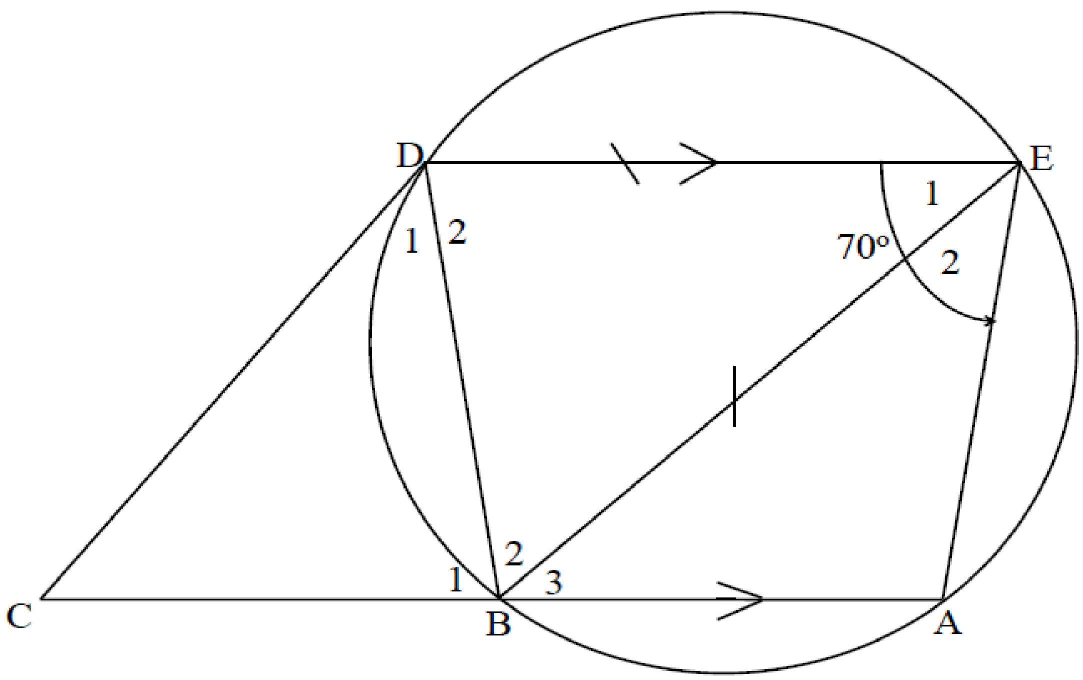
5.2 $B\hat{C}D$ (1)

5.3 $\hat{C}_2$ (2)

[5]

QUESTION 6

1.1 In the diagram below, BAED is a cyclic quadrilateral with BA || DE. BE = DE and $A\hat{E}D = 70°$. The tangent to the circle at D meets AB produced at C.



Calculate, with reasons the sizes of the following.

6.1.1 $\hat{A}$ (2)

6.1.2 $\hat{B}_1$ (2)

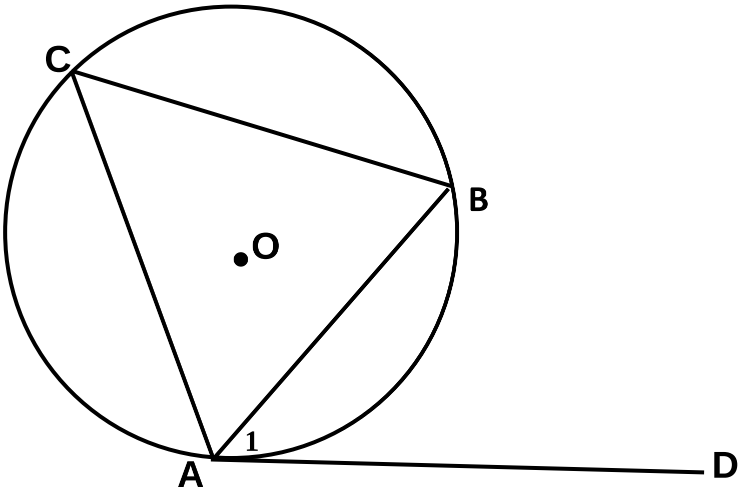
6.1.3 $\hat{D}_2$ (2)

6.1.4 $\hat{B}_2$ (2)

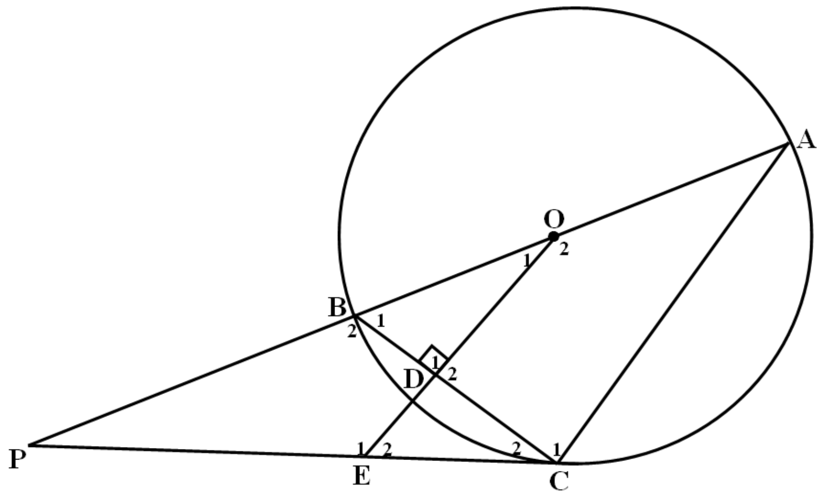
6.1.5 $\hat{D}_1$ (3)

QUESTION 7

7.1 Use the diagram to prove the theorem that states that $\hat{A}_1 = \hat{C}$ (5)



7.2 In the diagram, AB is a diameter of circle, centre O. AB is produced to P. PC is a tangent to the circle at C. OE ⊥ BC at D.



7.2.1 Prove, with reasons, that EO || CA. (4)

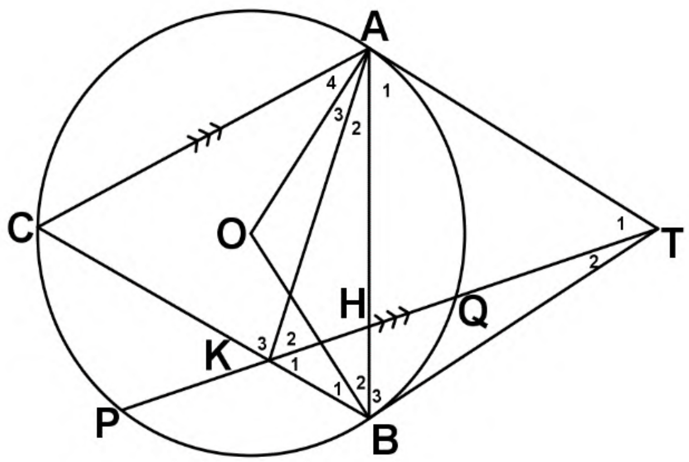
7.2.2 If $\hat{C}_2 = x$, name with reasons, two other angles each equal to x. (3)

7.2.3 Calculate the size of $\hat{P}$ in terms of x. (2)

[14]

QUESTION 8

In the diagram O is the centre of the circle passing through C,A and B.
TA and TB are two tangents to the circle at A and B.TQP cuts the circle at Q and P.
CA ║PT. QP cuts AB and BC at H and K respectively.



Prove that:

8.3.1 AOBT is a cyclic quadrilateral. (4)

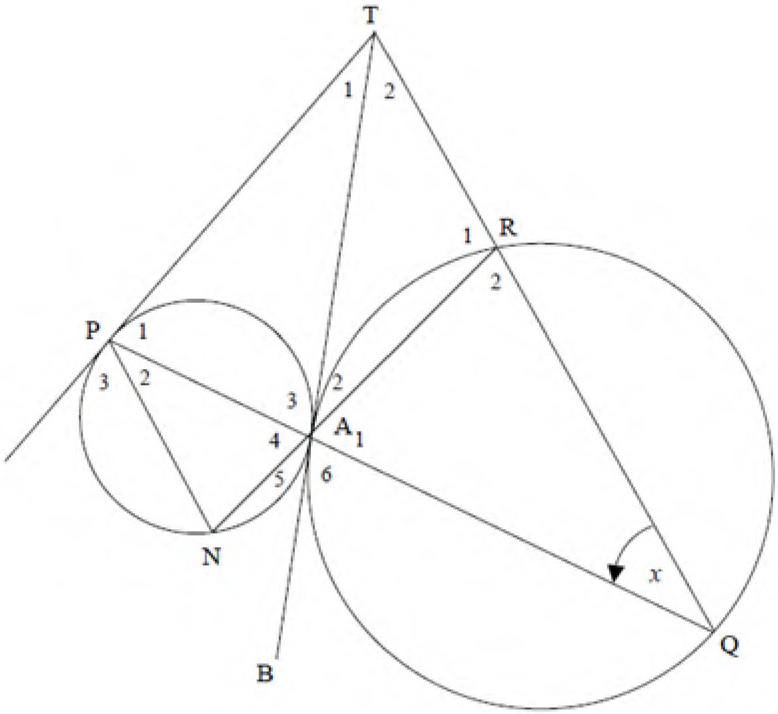
8.3.2 $H\hat{K}B = \hat{A}_1$. (4)

8.3.3 TA is a tangent to the circle through A, H and K. (4)

[20]

QUESTION 9

In the diagram below, two circles have a common tangent TAB. PT is a tangent to the smaller circle. PAQ, QRT and NAR are straight lines.



Let $\angle Q_1 = x$.

9.1.1 Name with reasons THREE other angles equal to *x*. (6)

9.1.2 Prove that APTR is a cyclic quadrilateral. (6)