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education

Department:
Education
PROVINCE OF KWAZULU-NATAL

CURRICULUM GRADE 10 -12 DIRECTORATE

NCS (CAPS)

TEACHER SUPPORT DOCUMENT

GRADE 12

MATHEMATICS

STEP AHEAD PROGRAMME

2021

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PREFACE

This support document serves to assist Mathematics learners on how to deal with curriculum gaps and learning losses as a result of the impact of COVID-19 in 2020. It also captures the challenging topics in the Grade 10 -12 work. Activities should serve as a guide on how various topics are assessed at different cognitive levels and also preparing learners for informal and formal tasks in Mathematics. It will cover the following topics:

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1. FUNCTIONS

TOPIC: Revision of grade 11 functions (Lesson 1)

RELATED CONCEPTS/ TERMS/VOCABULARY

- Function – A relationship for which each element of the domain (x) corresponds to exactly one element of the range(y). For every x- value there is only one possible y-value.
- Increasing function - A function that is going upwards when looking at it from left to right.
- Decreasing function – A function that is going 'downwards when looking at it from left to right. N.B if a function has a turning point, this is the point at which a function could change from decreasing to increasing
- Horizontal shift – A translation of the graph either to the left or the right
- Vertical shift – A translation of the graph either up or down
- Average gradient – The gradient between two points on a curved graph.
- Domain – All the possible x values that are covered by the function/ graph (to find the domain always study the graph from left to right)
- Range – All the possible y values that are covered by the function/ graph (to find the range always study the graph from bottom to top)
- x- intercept- Where a function cuts the x- axis
- y- intercept- where a function cuts the y- axis
- Axis of symmetry – A line that cuts the function exactly in half
    - A parabola has a vertical line of symmetry ($x =$…..)
    - A hyperbola has 2 axes of symmetry which form a cross. These are in the form $y = mx + c$
    - An exponential graph does not have an axis of symmetry
- Asymptote – A straight line that a curved graph gets closer to but never touches
- Turning point – the point at which a function (parabola, sin or cos graph) changes from increasing to decreasing or from decreasing
- Maximum – The highest the graph can be and is always the y-value of the turning point
- Minimum – The lowest the graph can be and is always the y-value of the turning point. N.B Min and Max only involves functions that have a turning point.

RESOURCES

- Grade 11 Textbooks

NOTES

In Grade 10 we discussed hyperbolic function of the form $y = \frac{a}{x} + q$. We will now focus on hyperbolic functions of the form $y = \frac{a}{x+p} + q$ where $x + p \neq 0$

Consider the example:

Given $f(x) = \frac{2}{x-2} + 1$

It is clear that from the revision of Grade 10 hyperbolas that the graph of $y = \frac{2}{x-2} + 1$ has a **horizontal asymptote** at $y = 1$.

It is also the case that $x - 2 \neq 0$, i.e. $x \neq 2$ because then the denominator will be zero and the expression $\frac{2}{x-2}$ would be undefined.

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In other words, the graph is not defined for $x = 2$. The graph there, therefore has a **vertical asymptote** at $x = 2$.

Horizontal shift – affects the *x* values in the equation.

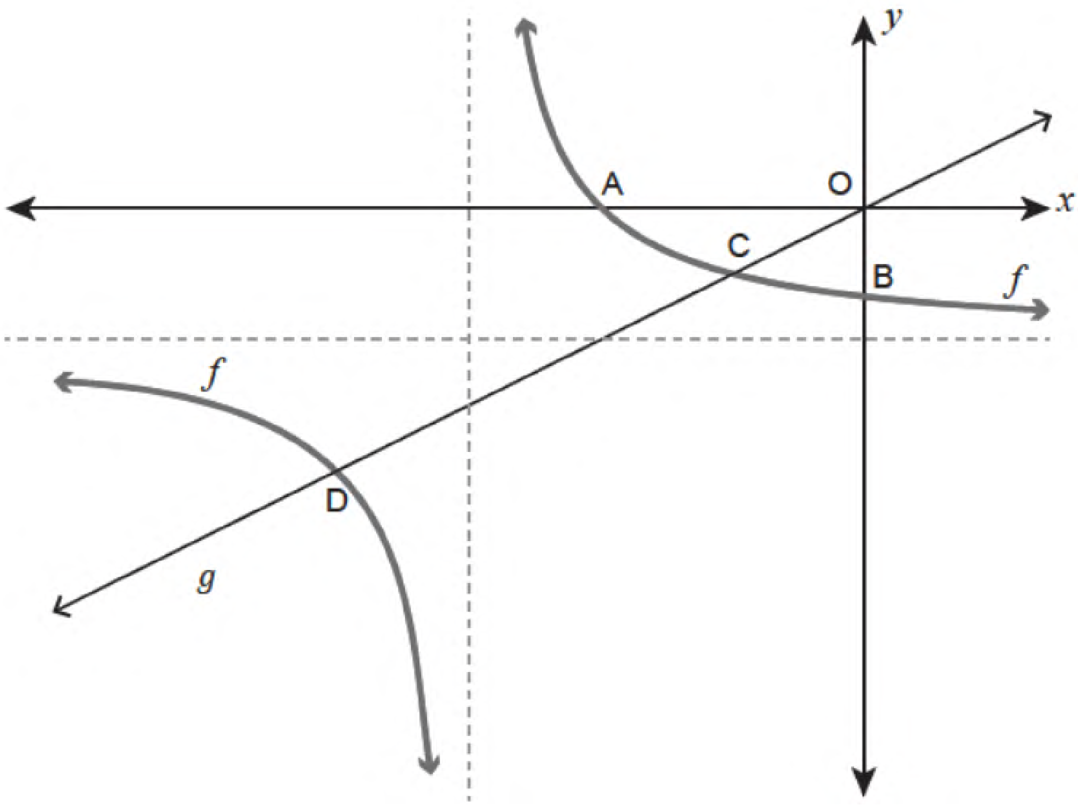
$f(x + 2)$ – shift of 2 units to the left

$f(x - 2)$ – shift of 2 units to the right

The diagram below shows the graph of $f(x) = \frac{1}{x+3} - 1$ and $g(x) = \frac{1}{2}x$

The graph of *f* intersects the *x*-axis at A and the *y*-axis at B.

The graph of *f* and g intersect at points C and D.



Questions

Write down the equations of the asymptotes of *f*

Determine the domain of *f*

a) Calculate the length of OB and OA

b) Determine the coordinates of C and D

c) Use the graphs to obtain the solution to $\frac{1}{x+3} \geq \frac{x+2}{2}$

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Solutions

a) $x = -3 \quad y = -1$	b) $x \in R\ ;\ x \neq -3$
c) $f(x) = \frac{1}{x+3} - 1$ x-int ; $y = 0$ $0 = \frac{1}{x+3} - 1$ $1 = \frac{1}{x+3}$ $x + 3 = 1$ $x = -2$ $(-2;0)$ $\therefore\ OA = 2$ units y-int ; $x = 0$ $y = \frac{1}{0+3} - 1$ $y = \frac{1}{3} - 1$ $y = \frac{-2}{3}$ $(0;\frac{-2}{3})$ $\therefore\ OB = \frac{2}{3}$ units	d) $\frac{1}{x+3} - 1 = \frac{1}{2}x$ LCD $2(x + 3)$ $2 - 2(x + 3) = x(x + 3)$ $2 - 2x - 6 = x^2 + 3x$ $0 = x^2 + 5x + 4$ $0 = (x + 1)(x + 4)$ $\therefore\ x = -1$ and $x = -4$ Substitute into $y = \frac{1}{2}x$ $y = \frac{1}{2}(-1)$ $y = -\frac{1}{2}$ $y = \frac{1}{2}(-4)$ $y = -2$ The two points are: $(-1\ ;\ \frac{-1}{2})$ $(-4\ ;\ -2)$ $\therefore\ C\,(-1\ ;\ \frac{-1}{2})$ $D\,(-4\ ;\ -2)$

e) $\frac{1}{x+3} \geq \frac{x+2}{2}$

$\frac{1}{x+3} \geq \frac{x}{2} + 1$

$\frac{1}{x+3} - 1 \geq \frac{x}{2}$

$f(x) \geq g(x)$

$\therefore\ x \leq -4$ or $-3 < x \leq -1$

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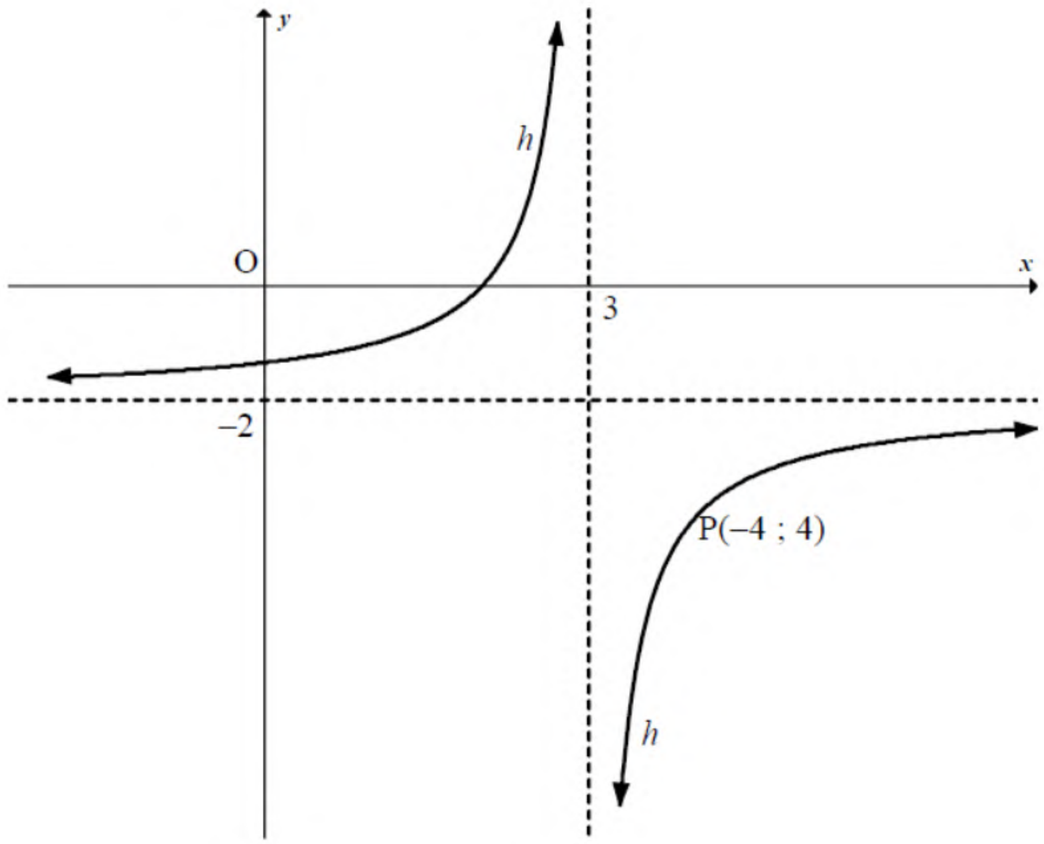
ACTIVITIES/ASSESSMENTS

1.1 The diagram below shows the graph of $h(x) = \frac{a}{x+p} + q$. The lines $x = 3$ and $y = -2$ are asymptotes of h. P(4; -4) is a point on h.



1.1.1 Write down the values of p and q. (2)

1.1.2 Calculate the value of a. (2)

1.1.3 Calculate the coordinates of the y – intercept of h. (2)

1.1.4 If $g(x) = h(x + 2)$, write down the equation of the vertical asymptote of g. (2)

1.1.5 If the graph of h is symmetrical about the line $y = -x + c$ (2)

[10]

Given: $f(x) = \frac{-3}{x-2} + 1$

1.1.6 Calculate the coordinates of the y-intercept of f. (2)

1.1.7 Calculate the coordinates of the x-intercept of f. (2)

1.1.8 Sketch the graph of f, clearly showing the asymptotes and the intercepts with the axes. (3)

1.1.9 Write down the range of f. (2)

1.1.10 Another function h, is formed by translating f 3 units to the right and 4 units down.

Write down the equation of h. (2)

1.1.11 For which value(s) of x $h(x) \leq -4$? (3)

1.1.12 Determine the equations of the asymptotes of $k(x) = \frac{3x-5}{x-1}$ (3)

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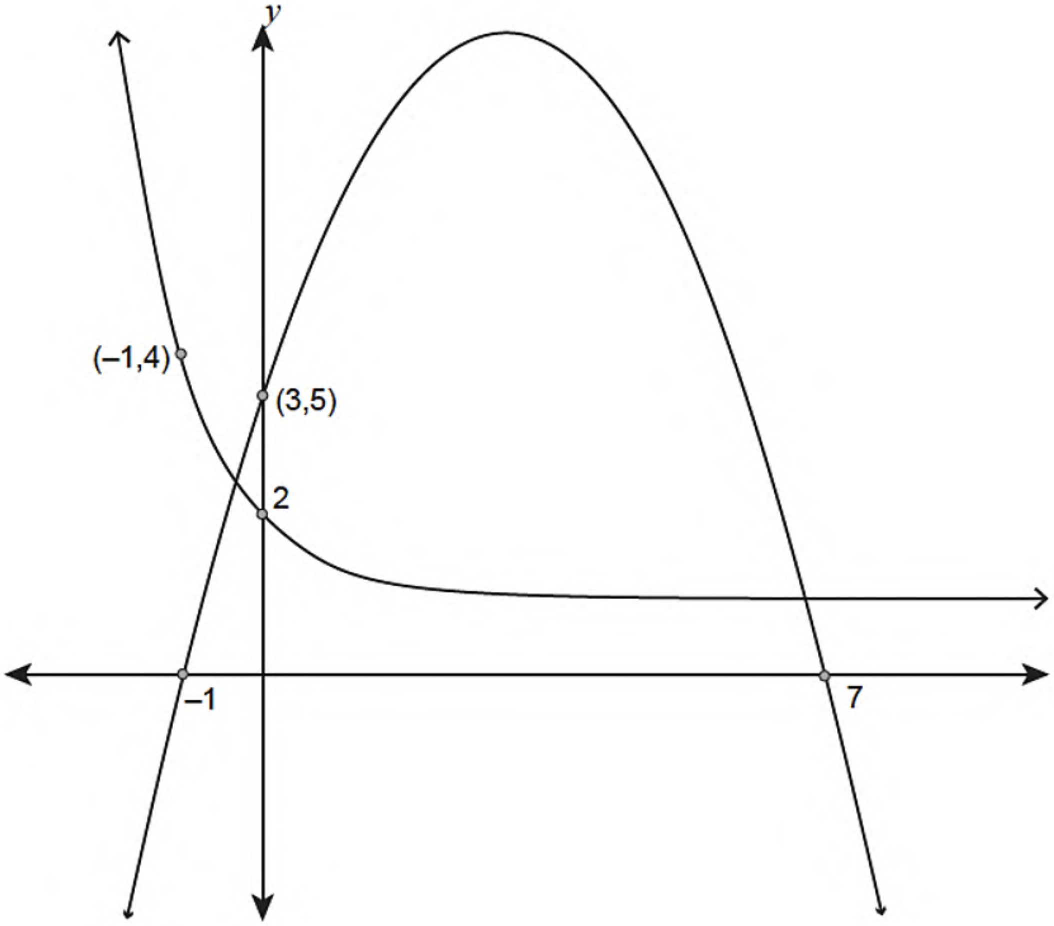
TOPIC: Revision on grade 11 functions **Lesson 2**

RESOURCES

- Grade 11 Textbook

NOTES

The graphs of the form $f(x) = ax^2 + bx + c$ and $g(x) = d^x + e$ are sketched below. The y-intercept of the parabola is (0 ; 3.5) and the x-intercepts are (–1 ; 0) and (7 ; 0).



Questions

(a) Find the values of a, b and c

(b) Find the values of d and e

(c) Determine the coordinate of the turning point of $f(x)$

(d) Find the range of $f(x)$

(e) Determine the equation of the axis of symmetry of $f(x)$

(f) Determine the equation of the asymptote of $g(x)$

(g) Determine the values of x for which the graph $f(x)$ increases

(h) Determine the equation of $h(x)$ which is the graph formed by moving $g(x)$ 2 units to the left and 5 units down.

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Solutions	
a) $y = a(x - x_1)(x - x_2)$ $y = a(x - (-1))(x - 7)$ $y = a(x + 1)(x - 7)$ Substitute (0 ; 3.5) $3.5 = a(0 + 1)(0 - 7)$ $3.5 = a(1)(-7)$ $3.5 = -7a$ $\frac{-1}{2} = a$ $y = a(x + 1)(x - 7)$ $y = \frac{-1}{2}(x + 1)(x - 7)$ $y = \frac{-1}{2}(x^2 - 6x - 7)$ $y = \frac{-1}{2}x^2 + 3x + \frac{7}{2}$ $\therefore\ a = \frac{-1}{2} \quad b = 3 \quad c = \frac{7}{2}$	b) $y = d^x + e$ Substitute (0 ; 2) $2 = d^0 + e$ $2 = 1 + e$ $1 = e$ Substitute (–1 ; 4) $4 = d^{-1} + 1$ $3 = d^{-1}$ $3 = \frac{1}{d}$ $3d = 1$ $d = \frac{1}{3}$
c) $x = \frac{-b}{2a}$ $x = \frac{-3}{2\left(\frac{-1}{2}\right)}$ $x = 3$ $y = \frac{-1}{2}(3)^2 + 3(3) + \frac{7}{2}$ $y = 8$ Turning point: (3 ;8)	d) $y \in (-\infty ; 8]$

e) $x = 3$	f) $y = 1$
g) $x \in (-\infty ;3)$	h) $h(x) = \left(\frac{1}{3}\right)^{x+2} + 1 - 5$ $h(x) = \left(\frac{1}{3}\right)^{x+2} - 4$

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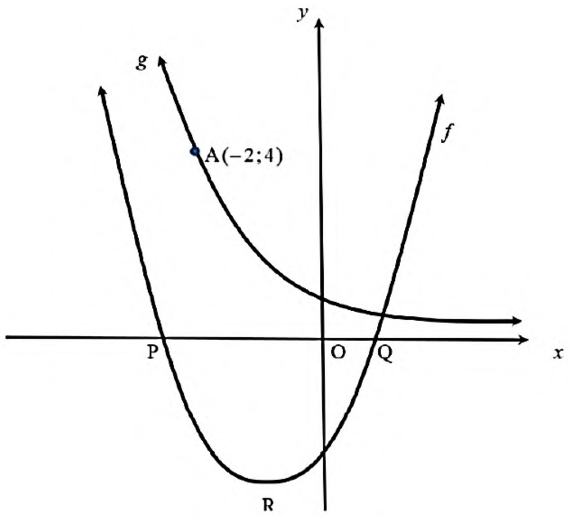
ACTIVITIES/ ASSESSMENT

1.2

The graphs of $f(x) = 2(x+1)^2 - 8$ and $g(x) = \left(\frac{1}{2}\right)^x$ are represented in the sketch alongside.

P and Q are the x-intercepts of f and R is the turning point of f.

Point A$(-2;4)$ is a point on the graph of g.



1.2.1 Write down the equation of the axis of symmetry of *f*. (1)

1.2.2 Write down the coordinates of the turning point of *f*. (1)

1.2.3 Determine the length of PQ. (4)

1.2.4 Write down the equation of *k*, if *k* is the reflection of *f* in the *y*-axis.

Give your answer in the form $y = ax^2 + bx + c$ (3)

1.2.5 For which value(s) of x will $x.f(x)<0$. (4)

1.2.6 Describe the transformation to *g(x)* if $h(x) = \frac{2^{-x}}{8}$. (3)

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TOPIC: Introduction to inverse functions	Lesson 3
RESOURCES	
• Grade 12 Textbook	

NOTES

Definition of a function

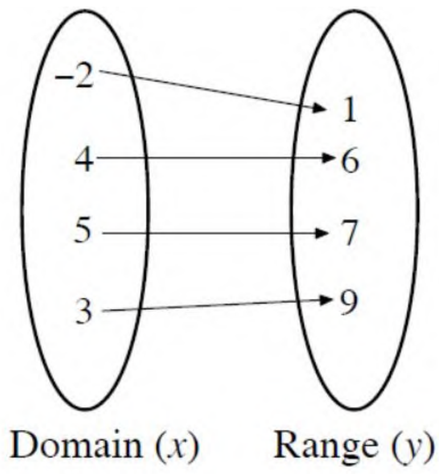
A relation is any relationship between two variables. A function is a special kind of relation in which: For every x-value, there is at most one y-value. Each element of the domain (x) is associated with only one element of the range (y). In other words, the x-values are never repeated in the set of ordered pairs of a function.

If each element of the domain is associated with only one element of the range. The relation is a **one-to-one function**.

Example 1:

(a) {(-2 ;1) ; (4 ; 6) ; (5 ; 7) ; (3 ; 9)}

- Here, each element of the domain is associated with only one element of the range. In other words, each x-value associates with only one y-value. In this case, the relation is said to be a **one-to-one function**.



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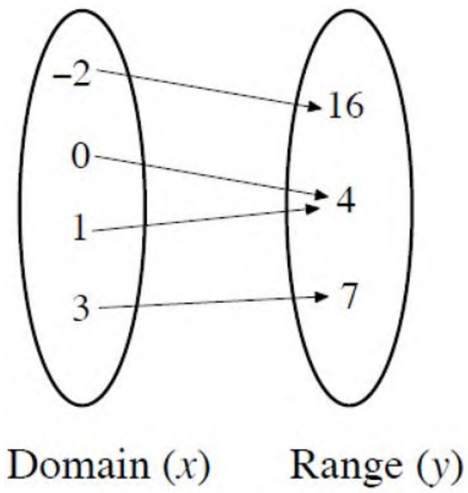
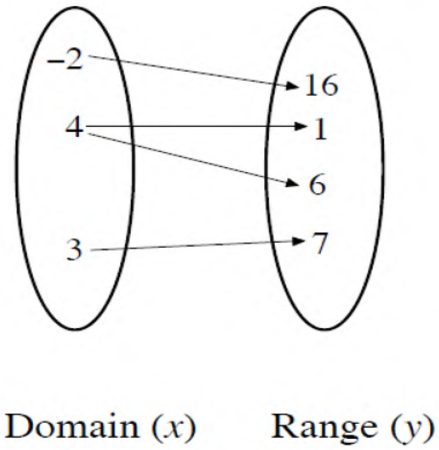
(b) {(-2 ;16) ; (0 ; 4) ; (1; 4) ; (3 ; 7)}

- Here, each element of the domain is associated with only one element of the range. However, the *x*-values 0 and 1 are associated with the same element of the range (namely 4).
  In this case, the relation is said to be a **many-to-one function**. Each *x*-value still associates with only one *y*-value.



(c) {(-2 ;16) ; (4 ;1) ; (4 ; 6) ; (3 ; 7)}

- Here, the *x*-value 4 in the domain is associated with more than one element of the range (1 and 6). In this case, the relation is **one-to-many** and is **not a function**.

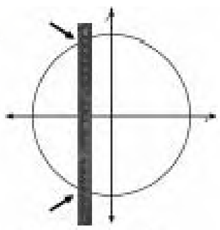


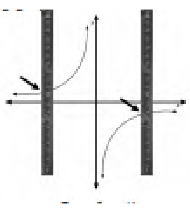
HOW TO DETERMINE WHETHER A GRAPH IS A FUNCTION:

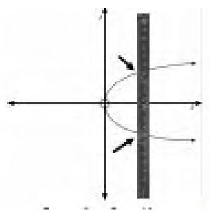
The Vertical and Horizontal Line Tests

You can use a ruler to perform the "**vertical line test**" on a graph to see whether it is a function or not. Hold a clear plastic ruler parallel to the y-axis, i.e. vertical. Move it from left to right over the axes. If the ruler only ever cuts the curve in **one** place, then the graph **is** a function. If the ruler at any stage cuts the graph in more than one place, then the graph is not a function. This is because the same x-value will be associated with more than one y-value.

The "**horizontal line test**" determines if the graph is a one-to-one or many-to-one function. If the ruler is positioned horizontally so that it is parallel to the x-axis, and the movement of the ruler is horizontally up or down, the following holds true: If the ruler only ever cuts the curve in one place, then the graph is a one-to-one function. If the ruler at any stage cuts the graph in more than one place, then the graph is a many-to-one function. See the graphs below.

1.

Not a function

2.

Function

3

Not a function

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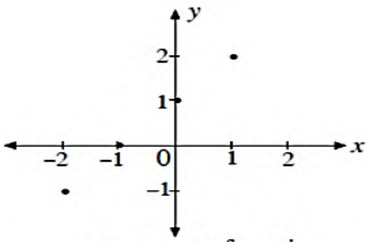
Example 2

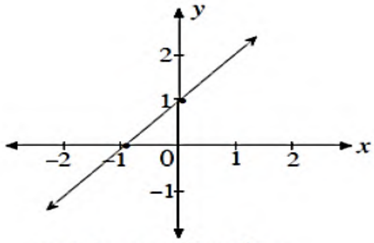
Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

(a)



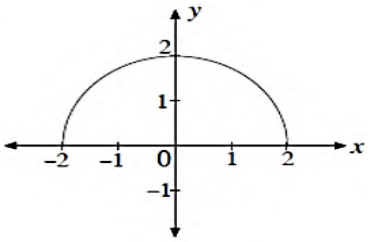
one-to-one function

(b)



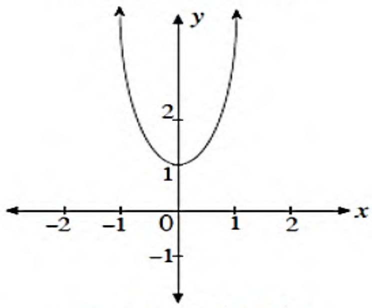
one-to-one function

(c)



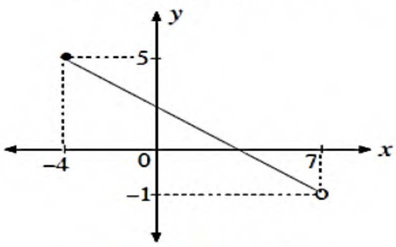
many-to-one function

(d)



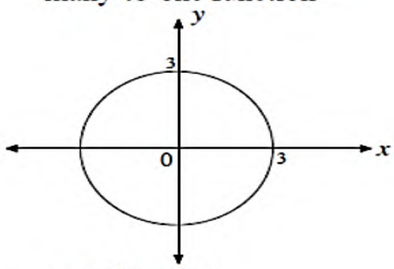
many-to-one function

(e)



one-to-one function

(f)



not a function

Functional notation:

Since functions are special relations, we reserve certain notation strictly for use when dealing with functions. Consider the function $f= \{(x{:}y) / y = 3x\}$ This function may be represented by means functional notation.

Functional notation

$f(x)$=3x

This is read as "f of x is equal to 3x". The symbol (x) is used to denote the element of the range to which x maps. In other words, the y-values corresponding to the x-values are given by $f(x)$,

i.e. y= (x).

For example, if x=4, then the corresponding y-value is obtained by substituting x=4 into 3x. For x=4 the y-value is y =3(4) =12.

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ACTIVITIES/ ASSESSMENT

1.3 The x and y values in the tables represent a relation between x and y. State whether each relation is one-to-one, one-to-many, many-to-one or many-to-many. Use your answers to say whether each relation in 1.1−1.5 is a function or not.

1.3.1

x	y
1	3
2	6
3	9
4	12
5	15

1.3.2

x	y
-2	5
-1	2
0	1
1	2
2	5

1.3.3

x	y
2	1
2	-1
3	-4
3	4
4	5

1.3.4

x	y
4	1
8	2
12	3
16	4
20	5

1.3.5

x	y
0	1
1	3
2	5
3	3
4	1

Given: $f(x) = -3x + 6$.

1.3.6 Calculate $f(-1)$, $f(0)$ and $f(2)$.

1.3.7 Use the answers in 1.3.6 to write down three elements of the domain of f and three elements of the range of f.

1.3.8 Draw the graph of $f(x)$ for the domain $x \epsilon$ R.

1.3.9 Use the graph to explain why $f(x)$ is a function.

1.3.10 What kind of function is $f(x)$?

Given $g(x) = -3x^2 + 3$

1.3.11 Calculate g(−1), $g(0)$ and $g(1)$.

1.3.12 Use the answers to 1.3.11 to write down three elements of the domain of g and two elements of the range of g.

1.3.13 Draw the graph of $g(x)$ for the domain $x \epsilon$ R.

1.3.14 Use the graph to explain why $g(x)$ is a function.

1.3.15 What kind of function is $g(x)$?

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TOPIC: Inverses **Lesson 4**

RESOURCES

- Grade 12 Textbook

NOTES

- Swopping of domain and range (x *to* y)
- Points of intersection (i.e. where the graphs are equal)
- Notation $f(0) = 2$ (0;2) y-intercept
  $f(2) = 0$ (2;0) x-intercept
  $f(3) = 4$ (3;4)
- Reflection of $f(x)$ about the line $y = x$ results to an inverse of $f(x)$

Method 1: Inverse function by swopping *x* and *y*

So if $y = 2x - 1$ (*f*)

Then $x = 2y - 1$ (Interchange x and y)

$-2y = -x - 1$

$2y = x + 1$

$y = \frac{x+1}{2}$

We then say, inverse function of f^{-1}, then

$f^{-1} = \frac{x+1}{2}$

If $x = 5$, is substituted in f^{-1}, then
$y = \frac{5+1}{2} = 3$

So rule *f*, maps 3 onto 5 and, the reverse (or inverse rule)
f^{-1} maps 5 back onto 3.

Method 2: Inverse function using flow diagrams

A flow diagram could also help you to understand the concept of inverse functions:

If $f(x)$=2x–1, complete a flow diagram the function
x→ multiply by 2 → 2x→ subtract 1 → 2x–1

So the inverse does the reverse, so you perform the operations back to front. That which was performed last is now performed first.

Inverse: x →Add 1 → x+1 → Divide by 2 → $\frac{x+1}{2}$

Can You:

Determine the inverse function of,

$f(x) = -3x + 4$

by:

1. Method 1 and
2. Method 2

Solution: $f^{-1}(x) = \frac{x-4}{-3}$

Given the function $f(x)$, we determine the inverse $f^{-1}(x)$ by:

- Interchanging x and y in an equation;
- Making y the subject of the formula
- Expression the new equation in the function notation

Inverses of one-to-one linear functions
An inverse function of a function f, is a function which does the "reverse" of a given function f.

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f^{-1}, is the notation used for the inverse of function f.
Since linear function is a $one\ x \rightarrow one\ y$ relation and if domain and range is swopped, the inverse will remain a $one\ x \rightarrow one\ y$.

Example 1

Consider the function $f(x) = 2x - 1$

f is the rule that maps values in the domain (*x*) to the values in the range (*y*). Note: $2x - 1 = y$.
If $x = 3$, then the function f maps this x- value to a corresponding *y*- value in the range as follows:

$f(x) = 2x - 1$
$\therefore f(3) = 2(3) - 1$
$\therefore f(3) = 5$
$\therefore y = 5$
So if $x = 3$, then $y = 5$

The rule that reverses this process and maps 5 back to 3 is called the inverse of the original function *f* and is denoted by f^{-1}.

Method 1: Inverse function by swopping *x* and *y*

So if $y = 2x - 1$ (*f*)

Then $x = 2y - 1$ (Interchange *x* and *y*)

$-2y = -x - 1$

$2y = x + 1$

$$y = \frac{x+1}{2}$$

We then say, inverse function of f^{-1}, then

$$f^{-1} = \frac{x+1}{2}$$

If $x = 5$, is substituted in f^{-1}, then

$$y = \frac{5+1}{2} = 3$$

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Method 2: Inverse function using flow diagrams

A flow diagram could also help you to understand the concept of inverse functions:

If $f(x)$=2x−1, complete a flow diagram the function
x→ multiply by 2 → 2x→ subtract 1 → 2x–1
So the inverse does the reverse, so you perform the operations back to front. That which was performed last is now performed first.

Inverse: x →Add 1 → x+1 → Divide by 2 → $\frac{x+1}{2}$

Can You:

Determine the inverse function of,

$f(x) = -3x + 4$

by:

3. Method 1 and
4. Method 2

Solution: $f^{-1}(x) = \frac{x-4}{-3}$

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Given the function $f(x)$, we determine the inverse $f^{-1}(x)$ by:

- Interchanging x and y in an equation;
- Making y the subject of the formula
- Expression the new equation in the function notation

If $f(x) = 2x - 4$

1. Determine f^{-1}, that is the inverse of f.
2. Sketch the graphs of f , f^{-1} and $y = x$ on the same system of axes.
3. Determine the coordinates of the point of intersection and indicate it on your sketch.
4. Write down the domain and range for f and f^{-1}.

Solution:

1.

So if, $f(x)$=2x−4

Then: y=2x−4 (f)

Then: x=2y−4 (interchange x and y)

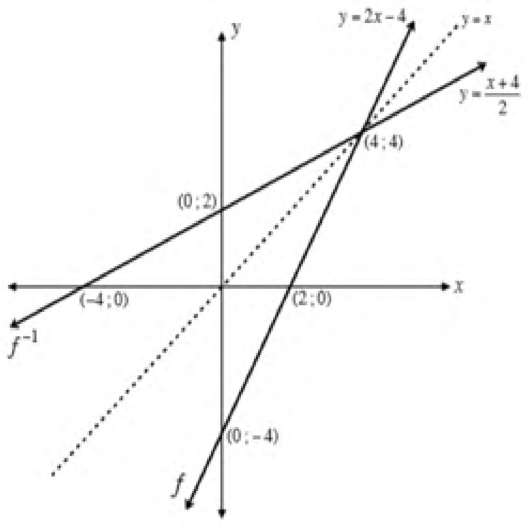
∴−2y=−x−4

∴2y=x+4

$\therefore y = \frac{x+4}{2}$

We then say, inverse function of f is:

$$f^{-1}(x) = \frac{x+4}{2}$$



2.

Sketching of these functions:

y=2x−4 (***f***)

***y* – intercept: let *x*=0**

y=2(0)−4= −4

***x* – intercept: let *y*=0**

0=2x−4 ∴ −2x= −4

∴x=2

$$y = \frac{x+4}{2} \ (\boldsymbol{f^{-1}})$$

***y* – intercept: let *x*=0**

$$y = \frac{0+4}{2} = 2$$

***x* intercept: let *y*=0**

$0 = \frac{x+4}{2}$

∴x=−4

3.

You can also find the point of intersection of these two graphs by solving the equation:

$f(x)$= $f^{-1}(x)$

∴2x−4= $\frac{x+4}{2}$

∴4x−8=x+4 (LCD = 2)

∴3x=12

∴***x*=4**

∴y=2(4)−4

∴***y*=4**

The coordinates of the point of intersection are (4; 4)

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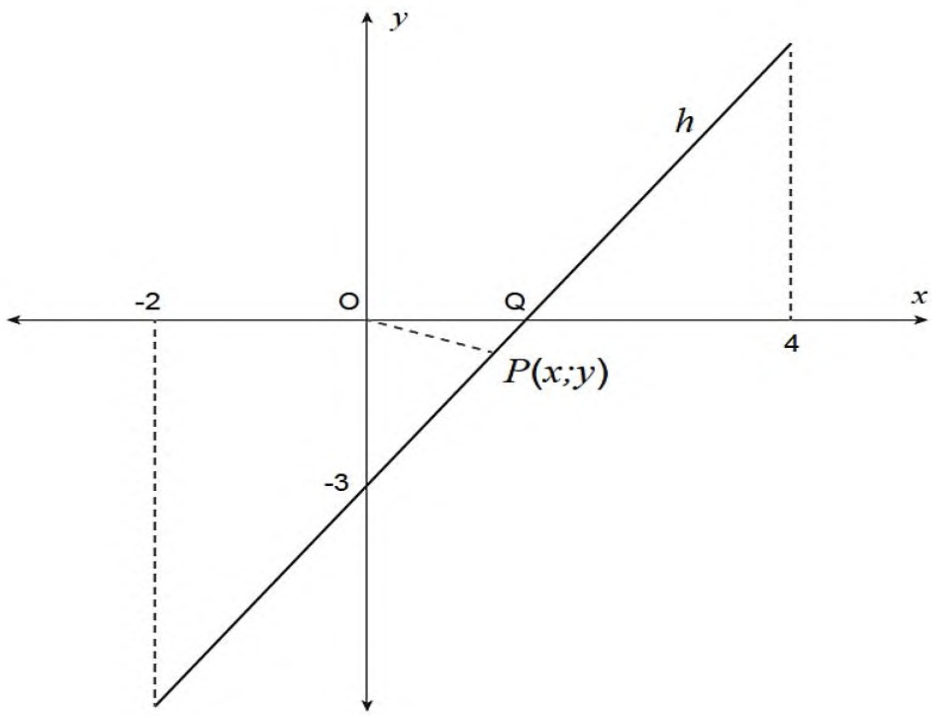
4.
Domain of f: $x \in R$
Range of f: $y \in R$
Domain of f^{-1} : $x \in R$
Range of f^{-1}: $y \in R$

Do you notice that:
The function (x)=$2x$–4 is a one-to-one linear function.
And it's inverse $f^{-1}(x) = \frac{x+4}{2}$ is also a one-to-one function.

ACTIVITIES/ ASSESSMENT

Given: $h(x) = 2x - 3$ for $-2 \leq x \leq 4$. The intercept of h is Q.



1.4.1 Determine the coordinates of Q.

1.4.2 Write down the domain of h^{-1}.

1.4.3 Sketch the graph of h^{-1}, clearly indicating the y-intercept and the end points.

1.4.4 For which values of x, will $h(x) = h^{-1}(x)$?

TOPIC: Inverses **Lesson 5**

RESOURCES

- Grade 12 Textbook

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Use of logs in the parabolic inverse
- Learners do not apply restrictions in the surd, so that the value is a real number (i.e. restrict the domain)
- Confusion between restrictions

NOTES

Consider the many–to-one function $f(x)$=x^2

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Let's sketch: $\boldsymbol{f}\,(\boldsymbol{x})= x^2$ **which can also be written as** $y= x^2$, by using the table with some values for x.

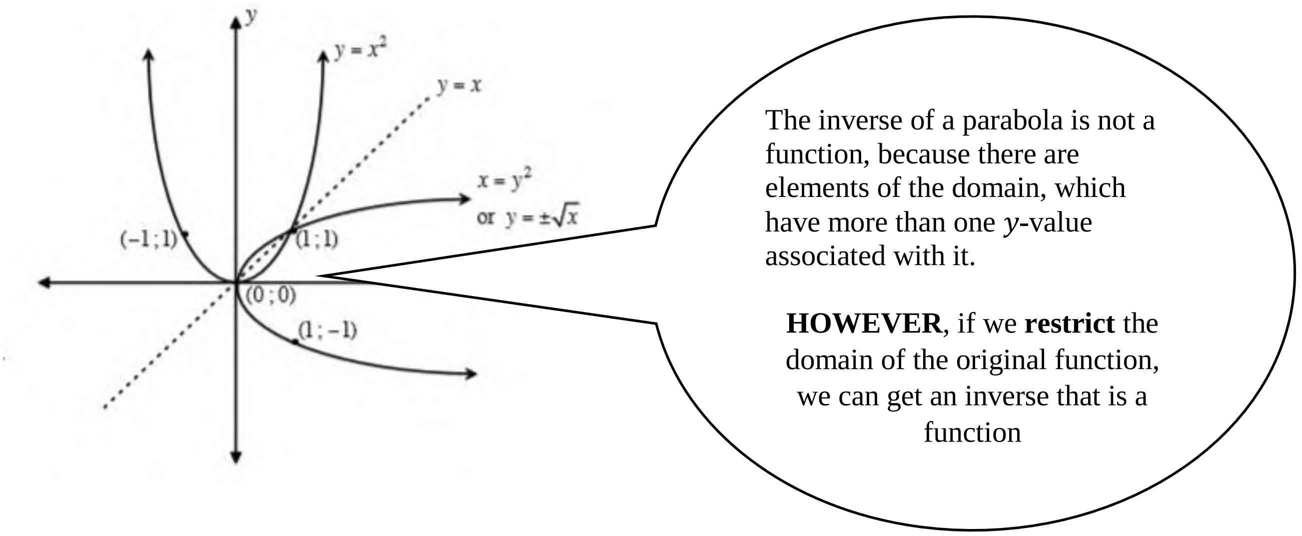
$\boldsymbol{x}$	-1	0	1
$\boldsymbol{y}$	1	0	1

Interchange x and y of, y=, then x=y^2

Use the table below to sketch, x=y^2

$\boldsymbol{y}$	-1	0	1
$\boldsymbol{x}$	1	0	1

It is possible to make y the subject of the formula for the inverse relation. $\therefore y^2$=x
$\therefore y= \pm\sqrt{x}$ **Provided** $x\geq 0$
The graph of $y= \pm\sqrt{x}$ is not a function because a vertical line will cut the graph in two points as it moves from left to right. So we will need to do something to the graph of $y= x^2$ so that when we determine the inverse, this inverse will also be a function.



Please note
If a function is not a one –to-one function, the inverse will not be a function.
However, below you will see that the domain of a many-to-one function can be restricted so that its inverse is a function.
Two different restrictions can be placed on the domain so that the inverse is a function.

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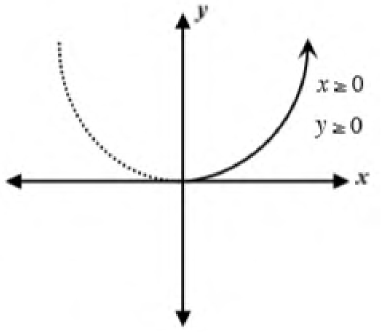
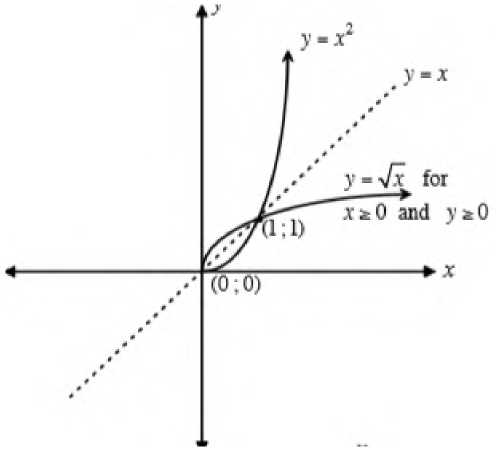
Situation 1

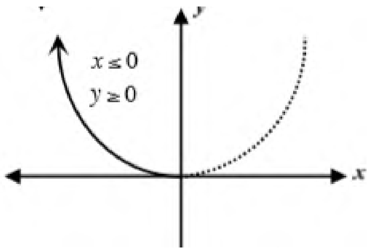
Restrict the domain of $f(x)= x^2$ as follows:
$f(x)= x^2$ where $x\geq 0$
Note that the graph of this parabola will be the one-half of the parabola, where the x-values are positive. The range of this function is the same as for the original function, $y \in [0; \infty)$.
The inverse of the graph of the function f, is the image when f is reflected about y=x. See the adjacent sketch.
The equation of the inverse function is then defined as,
$f^{-1}(x) = \sqrt{x}$ where $x\geq 0$ and $y\geq 0$.
Note, both f and f^{-1} are one-to-one functions.

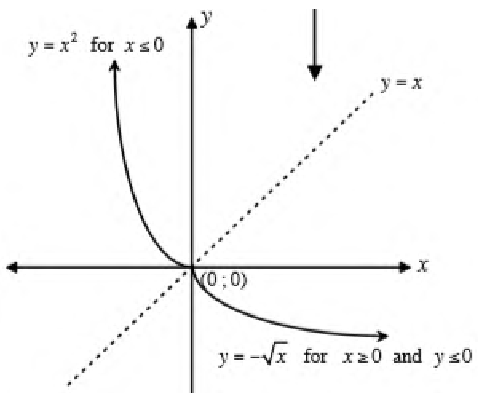




Situation 2

Restrict the domain of
$f(x)= x^2$as follows:
$f(x)= x^2$ where $x\leq 0$
Note that the graph of this parabola will be the half of the parabola, where the x-values are negative.
The range of this function is the same as for the original function, $y \in [0; \infty)$.
The inverse of the graph of the function f, is the image when f is reflected about y=x. See the adjacent sketch. It is clear that the inverse of the graph of the function
(x)=x^2 where $x\leq 0$ is also a function. The equation of the inverse function is then defined as $f^{-1}(x) = -\sqrt{x}$ where $x\geq 0$ and $y\leq 0$
Note, both f and f^{-1} are one-to-one functions.





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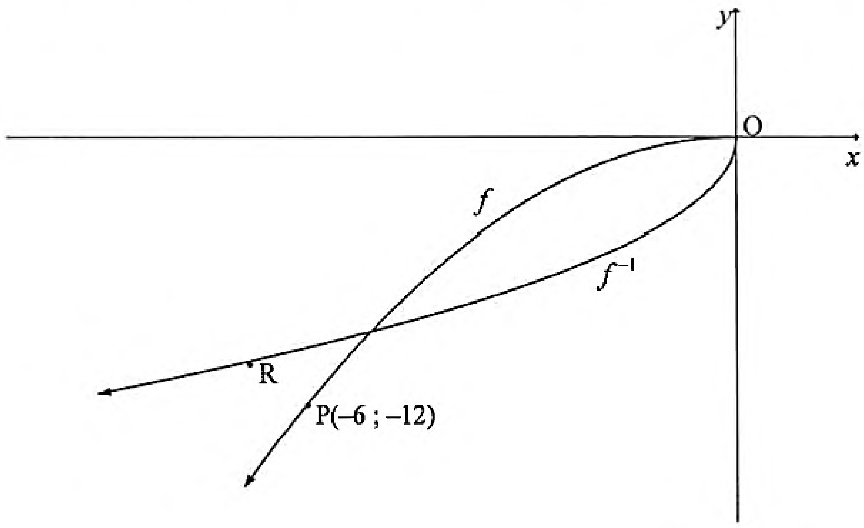
ACTIVITIES/ASSESSMENT

1.5

In the diagram below, the graph of $f(x) = ax^2$ is drawn in the interval $x \leq 0$.
The graph of f^{-1} is also drawn. P(–6 ; –12) is a point on f and R is a point on f^{-1}.



1.5.1 Is f^{-1} a function? Motivate your answer. (2)

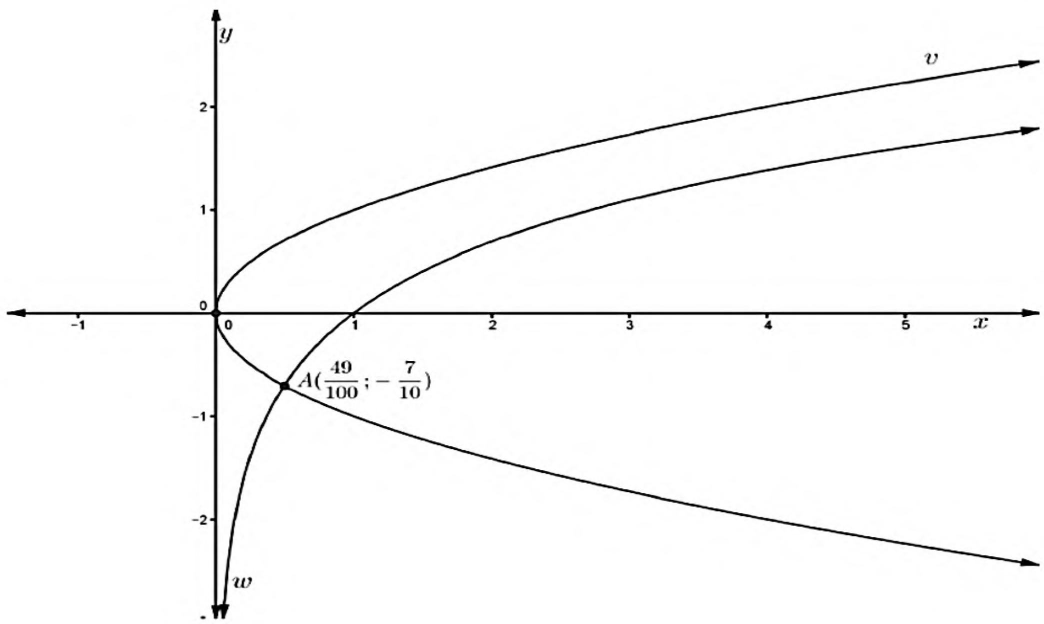
1.5.2 If R is the reflection of P in the line $y = x$, write down the coordinates of R. (1)

1.5.3 Calculate the value of a. (2)

1.5.4 Write down the equation of f^{-1} in the form $y = \cdots$ (3)

[8]

Study the diagram which shows the graphs of $v(x) = \pm\sqrt{x}$ and $w(x) = \log_a x$ then answer the questions that follow:



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1.5.5 State whether $v(x)$ is a function or not, and motivate your answer. (2)

1.5.6 Write down the conditions that will make v a function. (2)

1.5.7 Determine all values of:

1.5.7.1 y for which $w(x) < 0$ (1)

1.5.7.2 x for which $w(x) < -\frac{7}{10}$ (2)

1.5.8 If a function is as determined in 2.2, write down the equation(s) of $v^{-1}(x)$. (3)

1.5.9 If $h(x) = w(x) - \sqrt{x}$ where the range of $\sqrt{x}$ is $(0; \infty)$, calculate the range of $h(1)$. (1)

[11]

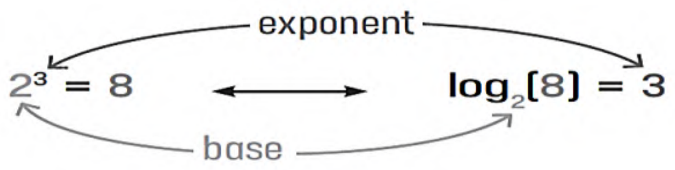
TOPIC: Inverses	**Lesson 6**
RESOURCES	

- Grade 12 Textbook

NOTES

In general, we can rewrite $\boldsymbol{product = base^{exponent}}$ in logarithmic form as follows:

$$\mathbf{log}_{\boldsymbol{base}}\,\boldsymbol{product} = \boldsymbol{exponent}$$



Examples that involves logarithms

1) Express the following in logarithmic form.

(a) $16 = 4^2$
$\therefore \log_4 16 = 2$

(b) $x = 2^y$
$\therefore y = \log_2 x$

2) Express the following in exponential form.

a) $\log_2 64 = 6$
$\therefore 64 = 2^6$

(b) $\log_3 y = x$
$\therefore y = 3^x$

1. What is an inverse function?
(A reflection of the function in the line $y = x$).

2. Draw a Cartesian plane and the line $y = x$ onto it.

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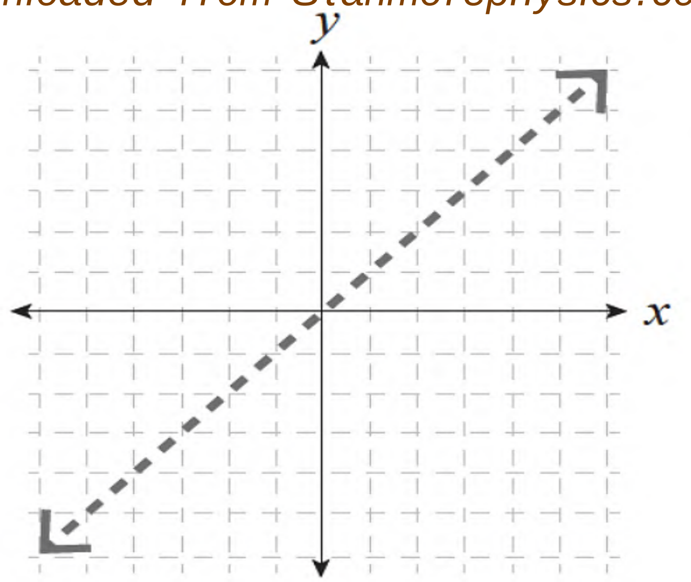
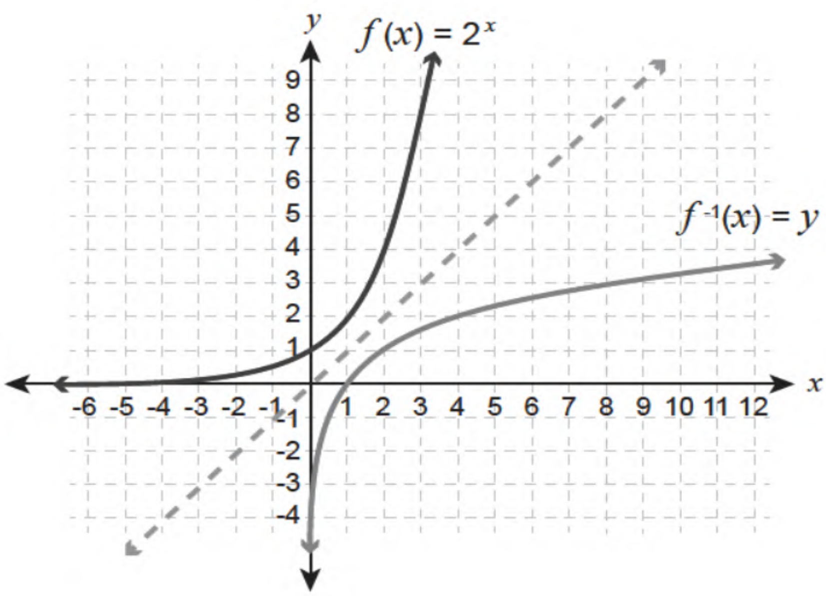


3. sketch the function $y = 2^x$ on the same Cartesian plane, marking the y-intercept clearly.
4. Draw in the inverse function. If you folded your paper along the $y = x$ line the two graphs should land on top of one another. All x-values become y-values and all y-values become x-values. Focus on the asymptote of the exponential function and think about where the asymptote will be of the inverse function.



5. The graph need not be accurate. It just needs to show the y-intercept of the exponential graph and the x-intercept of the inverse function. It should also be clear that the line $y = 0$ (the x-axis) is the asymptote of the exponential function and that the line $x = 0$ (the y-axis) is the asymptote of the inverse function. Discuss these points with learners.
6. *Is the exponential function increasing or decreasing?*
   (Increasing).
   Is the inverse function increasing or decreasing?
   (Increasing).
7. Find the equation of the inverse function:
   $x = 2^y$
   Use logarithms to make y the subject of the formula.
   $y = \log_2 x$
8. Write this equation next to the log graph you drew.
9. Draw another Cartesian plane with the line $y = x$ drawn in, then draw the function $y = \left(\frac{1}{2}\right)^x$.

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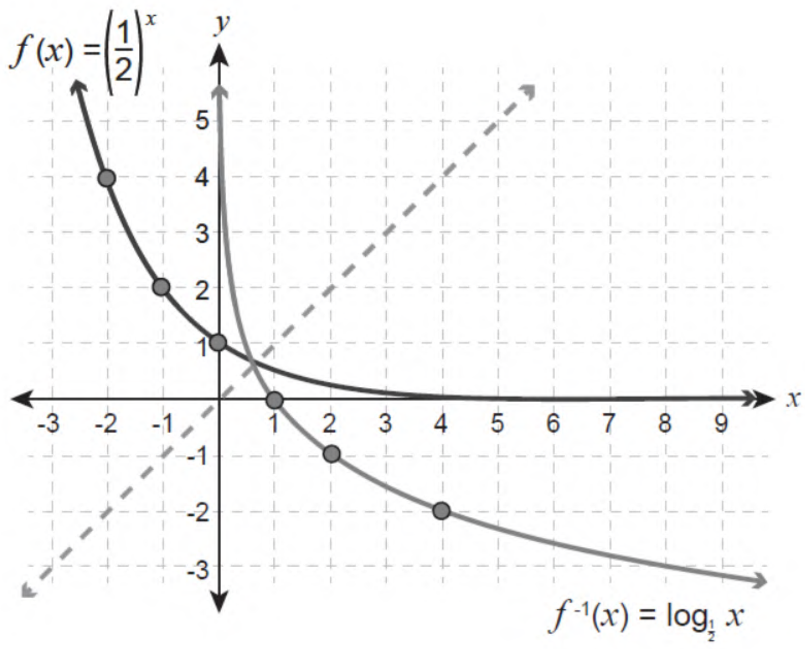
10. Draw the inverse of the function. Consider the fact that it must be reflected in the line $y = x$ and also consider the intercepts and the asymptotes.



NB: Be reminded that because the base lies between 0 and 1, the function will decrease.

Question	Solution
Write down the range of $f(x) = \left(\frac{1}{2}\right)^x$	$y \in (0;\infty)$ or $y > 0$
Write down the domain of $f^{-1}(x) = \log_{\frac{1}{2}} x$	$x \in (0;\infty)$ or $x > 0$
Write down the values of x for which $f(x) \leq 4$	$x \in (-2;\infty)$ or $x > -2$
Write down the values of x for which $f^{-1}(x) > -2$	$x \in (0;4)$ or $0 < x < 4$

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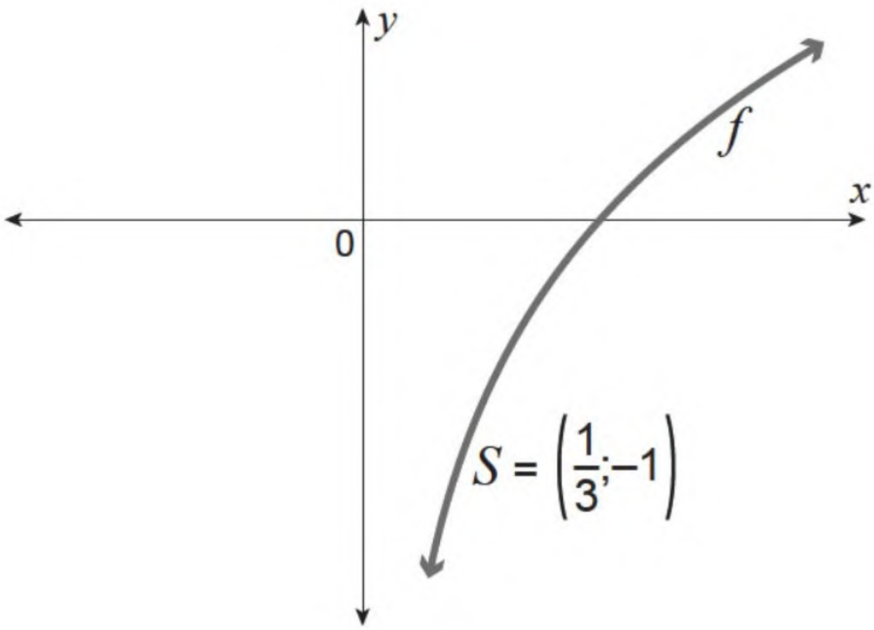
Example 1

Given: $f(x) = \log_a x$ where $a > 0$. S $\left(\frac{1}{3}; -1\right)$ is a point on the graph of f.



a) Prove that $a = 3$.
b) Write down the equation of h, the inverse of f, in the form $y = \ldots$
c) If $g(x) = -f(x)$, determine the equation of g.
d) Write down the domain of g.
e) Determine the values of x for which $f(x) \geq -3$

Solution:

a) $f(x) = \log_a x$

$-1 = \log_a \frac{1}{3}$

$\therefore a^{-1} = \frac{1}{3}$

$a = \left(\frac{1}{3}\right)^{-1}$

$a = 3$

b) $f(x) = \log_3 x$

$x = \log_3 y$

$y = 3^x$

c) $f(x) = \log_3 x$

$\therefore -f(x) = -\log_3 x$

$\therefore g(x) = -\log_3 x$ (other options: $g(x) = \log_3 \frac{1}{x}$; $g(x) = \log_{\frac{1}{3}} x$

d) $x > 0$ or $x \in (0;\infty)$

e) $\log_3 x = -3$

$x = 3^{-3}$

$x = \frac{1}{27}$

$\therefore x \geq \frac{1}{27}$

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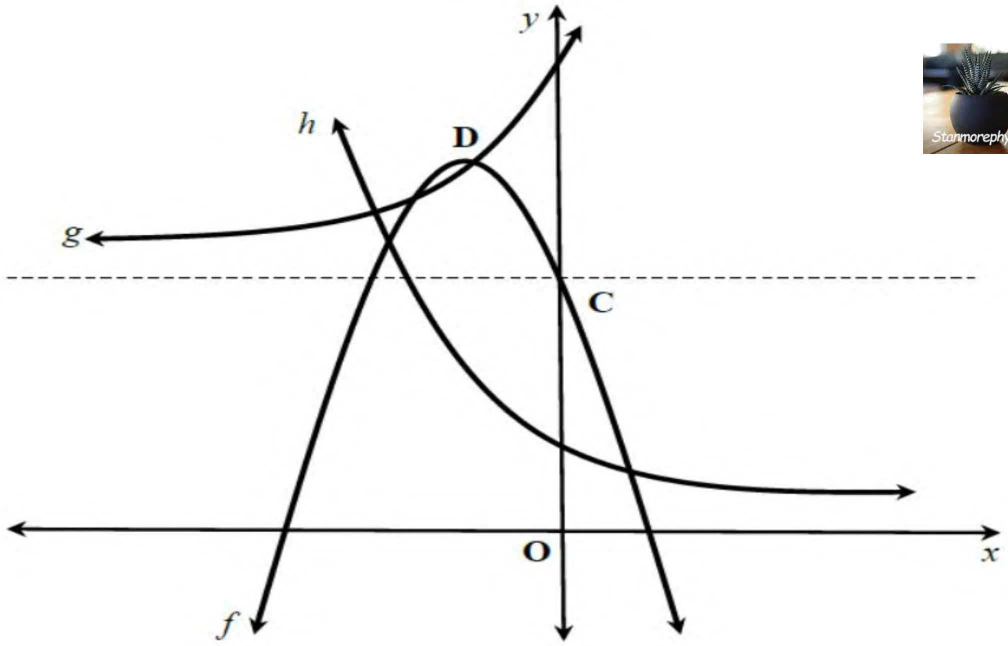
ACTIVITIES/ ASSESSMENT

1.6 The graphs of $h(x) = 3^{-x}$, $f(x) = -(x+1)^2 + 9$ and $g(x) = a.2^{x} + q$ are represented in the sketch below. D, the turning point of *f*, is also a point of intersection of *g* and *f*. The asymptote of *g* passes through C, the *y*-intercept of *f*.



1.6.1 Write down the coordinates of C. (2)

1.6.2 Calculate the values of a and q. (3)

1.6.3 Write down the range of g. (2)

1.6.4 Write down the coordinates of D', if D is reflected about the line $y = 8$ (1)

1.6.5 If $k(x) = (x+2)^2 + 9$, describe the transformation from f to k. (3)

1.6.6 Write down the equation of $h^{-1}(x)$ in the form $y = \cdots$ (1)

1.6.7 Determine the minimum value of $y = \left(\frac{1}{3}\right)^{f(x)-5}$ (2)

[14]

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TOPIC: FINANCE,GROWTH AND DECAY (LESSON 1)	Weighting	15 marks	Grade	10

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

1. Recap on the facts related to Simple and Compound Interest

Recap activity on Grade 10 (Simple and Compound Interest)

(a) At the end of 2006 Thando invested R5000 into a savings account. At the end of 2010 he had a total of R7400. Calculate the simple interest rate Thando received.
(b) Thando has R4500 in his savings account. The bank pays him a compound interest rate of 4,25%p.a. Calculate the amount Thando receives if he decided to withdraw the money after 30 months

Expect the Calculations of **A**, **P** and **i**

SIMPLE INTEREST – Interest is calculated on the original amount invested or borrowed. Simple Interest is used for short-term loans (Hire- Purchase accounts) and investments.

$$\boldsymbol{A = P(1 + i \times n)}$$

A→ Future value/ accumulated amount/ final amount

P → Principal amount/ initial amount/ starting amount

$\boldsymbol{i}$ → Interest rate **(written as decimal i÷ 100)**

$\boldsymbol{n}$ → Period **(usually years)**

COMPOUND INTEREST- Interest is calculated on the origanl sum plus interest already earned. Compound interest is used with long- term loans and investments.

$$\boldsymbol{A = P(1 + i)^n}$$

ACTIVITY /ASSESSMENT 2.1

(a) Lusanda wants to save R22 000 for an overseas holiday. If she can afford to invest R15 800 into a savings account that pays 13% simple interest per annum, how long will she have to wait to go on holiday? Give your answer in nearest year.

(b) Lonwabo guessed 4 correct numbers in the Lottery and invested her winnings at a rate of 7,5% per annum simple interest. After 2 years he had R17 250 in his saving account. How much did Lonwabo win?

(c) Nothile invested money into a saving account which promised an interest rate of 9% per annum compounded annually. After ten years she has R14 204,18 in her account. How much Nothile did initially invested.

(d) Lesedi invested R6 550 into a savings account. If the interest was compounded annually, calculate the yearly interest rate that Lesedi received if he had R11543,34 in his account after 5 years.

(e) If Yanelisa's salary is R25 000 per month and has increased at the same rate as inflation since she started working 20 years ago, Calculate her salary when she started working, if the average rate of inflation over the last twenty years has been 8,5% per annum.

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TOPIC FINANCE,GROWTH AND DECAY (LESSON 2)	**Weighting**	15 marks	**Grade**	**11**

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

Summary of the facts related to growth and decay

- Once you buy expensive items that lose value over a period of time. e.g. car or furniture loses its value as it becomes a scrap over a period of time. This is called **DEPRECIATION**
- Other items **gain value** over a period of time.
- e.g. Property (house), house itself gain value as the home owner performs exterior and interior renovations that add to the price tag of the house. This is **APPRECIATION.**
- That is why people are encouraged to invest on property rather than spend their money on buying expensive cars that will lose value over a period of time.

  APPRECIATION(+)

  - $\mathbf{A} = \boldsymbol{P}(\mathbf{1} + \boldsymbol{i})^{\boldsymbol{n}}$ COMPOUND APPRECIATION
  - $\mathbf{A} = \boldsymbol{P}(\mathbf{1} + \boldsymbol{in})$ SIMPLE APPRECIATION (derivation can be explained)

  DEPRECIATION(−)

- $\mathbf{A} = \boldsymbol{P}(\mathbf{1} - \boldsymbol{i})^{\boldsymbol{n}}$ COMPOUND DEPRECIATION
- $\mathbf{A} = \boldsymbol{P}(\mathbf{1} - \boldsymbol{in})$ SIMPLE DEPRECIATION

NB! 2 methods that might be asked
- Straight line method, use simple interest
- **Reducing balance** method use compound interest formula

- Calculate the REPLACEMENT/ EXPECTED/ NEW COST(**APPRECIATION**)
- Calculate the SCRAP/ BOOK/ TRADE-IN/ DECAY VALUES**(DEPRECIATION)**

EXAMPLE

A fridge costs R9999. Calculate what it will be worth in 5 years' time if it depreciates:

(a) On a Reducing Balance at 8% p.a.

Solution

$A = P(1-i)^n$

$A = 9999(1-0.08)^5$

$A = R6590.16$

(b) On a straight line basis at 10% p.a.

Solution

$A = P(1-i\times n)$

$A = 9999(1-0.10\times 5)$

$A = R5999.40$

ACTIVITY /ASSESSMENT 2.2

(a) Jane bought a laptop for R9 600. Calculate the book value of the laptop after 3 years, if it depreciates at 20% p.a. on the reducing balance method.

(b) Enzokuhle's car which was bought 5 years ago is now worth R80 434,88. What was the original purchase price of the car if it depreciated at 8% p.a. compounded daily?

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(c) A tractor bought for R120 000 depreciates to R11 090,41 after 12 years by using the reducing balance method. Calculate the rate of depreciation per annum. (The rate was fixed over the 12 years.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 3)	**Weighting**		**Grade**	**11**

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

1. COMPOUNDING PERIODS

In grade 10 we only calculated amount invested where interest earned once a year. NOW it is possible to earn interest anytime so if interest is not earned once a year, in the formula we need to adjust i and n to match with given compounding period

Calculation of Interest	i	n (number of years)
Annually/ Yearly	i	n
Semi- Annually/ Half yearly	$i \div 2$	$n \times 2$
Quarterly	$i \div 4$	$n \times 4$
Monthly	$i \div 12$	$n \times 12$

NB! Semi-annually/ half yearly, monthly, and quarterly must be known for exam purposes

EXAMPLE 1

Sam invested R100 for 5 years, Calculate the value of his investment if interest is calculated at 12% per annum compounded;

a) Monthly

$$A = P(1+i)^n$$

$$A = 100\left(1 + \frac{0.12}{12}\right)^{5\times 12}$$

$$A = R181.67$$

b) Quarterly

$$A = P(1+i)^n$$

$$A = 100\left(1 + \frac{0.12}{4}\right)^{5\times 4}$$

$$A = R180.61$$

c) Semi-annual/ half yearly

$$A = P(1+i)^n$$

$$A = 100\left(1 + \frac{0.12}{2}\right)^{5\times 2}$$

$$A = R179.08$$

EXAMPLE 2

Michael invests 5 000 into an account that offers an interest rate of 7.2% p.a. compounded monthly. How much will Michael have after 9 months.

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$A = P(1+i)^n$

$A = 5000\left(1 + \frac{0.072}{12}\right)^9$

$A = R5\ 276.57$

Note n is not multiplied because it is already given in months.

ACTIVITY /ASSESSMENT 2.3

(a) Lwezi invests R5 800 into an account that offers an interest rate of 12,7% p.a. compounded monthly. How much will Lwezi have after 12 months.

(b) Which one is the better investment over a year: 10,5% p.a. compounded daily and 10.55% p.a. compounded monthly?

(c) An investment grows from R6 700 to R8 510,59 in 3 years. Determine the interest rate received if interest was compounded monthly.

(d) Mlando invest R6000 into an account which offers an interest rate of 11% p.a. compounded semi – annually. How much will he have after 3,5 years?

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 4)	**Weighting**		**Grade**	**11**

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

You are encouraged to highlight or underline all the key figures (amounts, interest rates and years).

The different times mentioned over the entire period will be written as follows: T_1 or T_6 (Term 1 for year 1 and Term 6 for year 6). The beginning of the entire period is represented as T_0.

<u>Explanation of when do we use time line:</u>

- ✓ When there are changes (additional deposits or withdrawals) in amount invested.
- ✓ Calculations of more than one interest rate.

- Additional (deposit)- positive sign must be used
- Withdrawal – negative sign must be used

(Take note that: treat each amount separately and let it grow it up to the last period)

Examples

(a) Mrs Mkhize deposits R12 000 into a savings account. Two years later she adds an additional R6 000 to her savings. How much will she have after 5 years if she receives an interest rate of 7,5% per annum compounded monthly?

solution

- List all the key times (T_0, T_2, T_5)
- Fill in the invested amount $T_0(R12000)$ and T_2 $(+R6\ 000)$
- Consider the interest rate $(i = \frac{0{,}075}{12})$

$A = P(1+i)^n$

$A = 12000\left(1 + \frac{0.075}{12}\right)^{12\times5} + 6000\left(1 + \frac{0.075}{12}\right)^{12\times3}$

$A = R24\ 948{,}21$

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(b) At the beginning of 2007, Johannes deposited R9 000 into a Money Market Investment account at an interest rate of 12% compounded semi - annually. At the beginning of 2010 the interest rate dropped to 5,5% p.a. compounded monthly. How much will Johannes have in his account at the beginning of 2012?

Solution

$T_0 = R9000$,

$A = P(1+i)^n$

$$A = R9000\left(1+\frac{0,12}{2}\right)^{3\times 2}.\left(1+\frac{0,055}{12}\right)^{2\times 12}$$

$= R14247.57$

ACTIVITY /ASSESSMENT 2.4

(a) R 5000 is invested into an account which offers an interest rate of 13% p.a. compounded monthly. 3 years later an additional R2 000 is deposited into the account. 2 years after that R3000 is withdrawn. 4 years after the initial investment the interest rate decreases to 10% p.a. compounded quarterly. How much will be in the account after 7 years.

(b) R60 000 is invested in a account which offers interest at 7% p.a. compounded quarterly for the first 18 months. There after interest rate changes to 5% p.a. compounded monthly. Three years after the initial investment, R5000 is withdrawn from the account. How much will be in the account at the end of 5 years?

(c) R150 000 is deposited in an investment account for a period of 6 years at an interest rate of 12% p.a. compounded half-yearly for the first 4 years and then 8.5% p.a. compounded yearly for the rest of the period. A deposit of R8 000 is made into the account after the first year and then another deposit of R2000 is made 5 years later after the initial investment. Calculate the value of the investment at the end of the 6 year period.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 5)	**Weighting**		**Grade**	**11**

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

- For the conversion of rates, all calculations must be of **one year**.
- Importance of noting the compounding periods e.g. quarterly i_4
- Effective interest rate- where the stated period and compounding period are the same. It is where the compounding period is taken into consideration.
- Nominal interest rate - where the stated period and compounding period are not the same. The interest rate that is quoted in the question.

$$\boxed{(1+i_{eff}) = \left(1+\frac{i_{nom}}{m}\right)^m}$$

i_{eff} → effective interest rate

i_{nom} → Nominal interest rate

m → number of times interest receive a year **(compounding period)**

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Example 1

Determine the effective interest rate if an investment earns interest at a nominal interest rate of 11,5% p.a. compounded quarterly.

$$1 + i_{eff} = \left(1 + \frac{i_m}{m}\right)^m$$

$$1 + i_{eff} = \left(1 + \frac{i_4}{4}\right)^4$$

$$i_{eff} = \left(1 + \frac{0.115}{4}\right)^4 - 1$$

$$i_{eff} = 0{,}120055 \ldots \times 100$$

$$i_{eff} = 12{,}01\%$$

Example

Calculate the nominal interest rate compounded monthly if effective interest rate is 10,5% p.a..,

$$1 + i_{eff} = \left(1 + \frac{i_m}{m}\right)^m$$

$$1 + 0.105 = \left(1 + \frac{i_{12}}{12}\right)^{12}$$

$$\sqrt[12]{1 + 0.105} = \sqrt[12]{\left(1 + \frac{i_{12}}{12}\right)^{12}}$$

$$\sqrt[12]{1 + 0.105} - 1 = \frac{i_{12}}{12}$$

$$i_{12} = 12 \times (\sqrt[12]{1 + 0.105} - 1) \times 100$$

$$i_{12} = 10{,}03\%$$

Note: Effective rate **<u>is a little</u>** higher than the nominal rate.

ACTIVITY /ASSESSMENT 2.5

(a) Calculate the effective interest rate if interest is 9,8% p.a. compounded

(i)monthly (ii) quarterly (iii) semi- annually

(b) Calculate the nominal interest rate compounded quarterly if effective interest rate is 11,7% p.a.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 6)	**Weighting**		**Grade**	**12**
RESOURCES				
DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document				
NOTES				
Recap on solving equation using the Laws of Exponents and /or logarithms. <u>Recap activity</u>				

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(a) $2^{(n+1)} = 32$
(b) $3^n = 20$
(c) $75(1{,}025)^{x-1} = 300$

EXAMPLE

An investment of R25 000 grows to an amount of R55 267,04. If the account offers an interest rate of 12% per annum compounded annually, for **how long** was the money invested?

Solution

$A = R555267.04$

$P = R25000$

$i = 12\% = 0.12$

$n = ?$

$A = P(1+i)^n$

$R55267.04 = R25000(1+0.12)^n$

$\frac{R55267.04}{R25000} = (1+0.12)^n$

$n = \log_{(1+0.12)}(\frac{55267.04}{25000})$

$n = 7\,years$

The same skill will be required if the number of payments is required in calculations involving the present or future value because this is represented by an exponent.

ACTIVITY /ASSESSMENT 2.6

(a) Calculate **how many years** it will take for an investment, earning 7,5% p.a. compounded monthly to be triple in value

(b) How long must any amount of money be invested for , in order it to **double** at an interest rate of 8,5% p.a. compound interest. Give your answers in years and months.

(c) A motor cycle which costs R250 000 depreciates at a rate of 3,8% per annum compounded monthly. How long will it take for the motor cycle to be worth R206677,47? Give your answer to the nearest year.

(d) A photocopier valued at R24 000 depreciates at a rate of 18% p.a. on the reducing- balance method. After how many years will its value be R15 000?

(e) Convert an interest rate of 12% **compounded monthly** to an interest **compounded quarterly.**
use $\left(1+\frac{i_{new}}{n}\right)^n = \left(1+\frac{i_{nom}}{m}\right)^m$

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 7)	**Weighting**		**Grade**	**12**
RESOURCES				
DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document				
NOTES				
Notes on Future Value Annuity				

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- **Annuity**-is a number of equal payments made at regular intervals for a certain amount of time. An annuity is subject to a rate of interest
- **Future value** – is used in investments when you save money for future e.g. **Savings account**, **Retirement fund** and **Sinking fund**.
- Regular payment (**usually monthly payments**) - it is like a present value that will collect interest over a period of time.

$$\text{Fv} = \frac{x[\,(1+\ i)^n - 1\,]}{i}$$

Where F→ Future Value

$\boldsymbol{x}$→ Value of regular instalments

$\boldsymbol{i}$→ Interest rate

$\boldsymbol{n}$→ Number of payments (**usually months**)

HINTS ON FUTURE VALUE CALCULATIONS

- Calculate FUTURE VALUE – (The are 3 cases when calculating Fv)
- **F**, $\boldsymbol{x}$ and $\boldsymbol{n}$ → **Straight forward.** Won't be asked to calculate $\boldsymbol{i}$

3 cases

$$\left\{\begin{array}{l} \text{1. Payment made in } \textbf{one month's time} \rightarrow \textbf{use formula as it is} \\ \text{2. Payment starting at the } \textbf{end of first month} \rightarrow \textbf{use formula as it is.} \end{array}\right\} \textbf{Same little difference}$$

3. Payment start **immediately** and **end on the last day** → (**include n + 1** in the formula) $\text{FV} = \frac{x[\,(1+i)^n - 1\,]}{i}$

   - Since we have **immediately** and **end** at the same time it means one of the months it was **repeated**, meaning that payments were made twice. That is why we have +1 for that additional payment made. **Have more focus on this tricky case**

Examples

(a) Lusanda starts to save for his retirement. He opens an investment account and immediately deposits R800 into account, which earns 12.5% p.a. compounded monthly. Thereafter, he deposits R800 at the end of each month for 20 years. What is the value of his retirement savings at the end of 20 years period?

Solution

$$F_v = \frac{x[(1+i)^n - 1}{i}$$

$$F_v = \frac{800(1+\frac{0.125}{12})^{20\times 12} - 1}{\frac{0.125}{12}}$$

$$F_v = R856415.66$$

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(b) On her 25th birthday, Zoleka decided to accumulate R5 000 000 by her 50th birthday. She plans to make equal monthly payments into account that pays 10% interest p.a. compounded monthly. If Zoleka makes her first payment a month after her 25th birthday and her last payment on her 50th birthday, determine how much she will need to deposit monthly to accumulate R5000 000 on her 50th birthday.

Solution

$Given: F_v = R5000000$

$x = R?$

$i = 10\% = \frac{0.10}{12}$

$n = 25 \times 12$

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$$R5000000 = \frac{x[(1+\frac{0.1}{12})^{25\times12} - 1]}{\frac{0.1}{12}}$$

$$x = \frac{5000000(\frac{0.1}{12})}{[(1+\frac{0.1}{12})^{25\times12} - 1]}$$

$x = R3768.37$

Investment could stop early and **grow compounded** for the rest of the period now with (A =P$(1+i)^n$)

ACTIVITY /ASSESSMENT 2.7

(a) Nicky opened a savings account with a single deposit of R1 000 on 1 April 2020. She then makes 18 monthly deposits of R700 at the end of every month. Her first Payment is made on 30 April 2020 and her last payment on 30 September 2021. The account earns interest at 15% per annum compounded monthly.
   (i) Determine the amount that should be in her savings account immediately after her last deposit is made (that is on 30 September 2021).
   (ii) If she makes no further payments but leaves the money in the account, how much money will be in the account on 30 September 2022.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 8)	Weighting		Grade	12

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

Notes on Present Value Annuity

- Present value - used for loans e.g. Student loan (NSFAS), Vehicle loan to buy cars, loan to buy house (Bond or Mortgage or Home loan).
- First money received and paid later. Once pay deposit, reduce loan amount
- Present value (P) - it is always the outstanding balance with n payments to go.

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$$P = \frac{x[\,1-(1+\,i)^{-n}\,]}{i}$$

Where P→ Present Value
$\boldsymbol{x}$→ Value of regular instalment
$\boldsymbol{i}$→ Interest rate
$\boldsymbol{n}$→ Number of **payments left / remaining**

HINTS ON PRESENT VALUE CALCULATIONS

- Before calculation 1st know when the loan is due, usually when take loan its first payment is due after a month of granting the loan
- Calculate P, x and n. **Won't be asked to calculate $\boldsymbol{i}$**
- OUTSTANDING BALANCE (actual remaining amount owed) →**2 *options***
- Option 1→ Use future value formula but n→ number of payments which have been made. Outstanding balance = Money owed – money paid/FV

Outstanding balance $= L\,(1+i)^n - \frac{x[\,(1+i\,)^n-1]}{i}$ where L→***for loan*** amount***

- Option 2→Use present value formula but in n→ substitute number of payments left/ still to go.

3. FINAL PAYMENT→ (***you cannot calculate without outstanding balance***)
→ 1st find outstanding balance
→ 2nd use $P(1+i)^1$
Where P→***outstanding balance***
n =1 (since final payment will be made 1 month later)

- TOTAL INTEREST PAID = Monthly payment×(n×compounding period) - loan amount

Examples

1. Yandiswa takes a loan from a bank to start his own business. His monthly repayments are R30 428 a month for 15 years. The interest rate is 9% compounded monthly.
1.1 Determine how much Yandiswa initially borrowed from the bank.
Solution

1.1 1st, How many years to go? Thus outstanding Balance

$Given: P = ?$

$x = R30428$

$i = 9\% = 0.09 \div 12$

$n = 15 \times 12$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P_v = \frac{30428[1-(1+\frac{0.09}{12})^{-15\times12}]}{\frac{0.09}{12}}$$

$$P_v = R3000000.24$$

1.2 Determine the **balance** of the loan at the end of 5 years.

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At the end of 5 years, it was 15 years, it means 10 years to go

Given:

$P = ?$

$x = R30428$

$i = 9\% = 0.09/12$

$n = 10 \times 12$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P_v = \frac{R30428[1-(1+\frac{0.09}{12})^{-10\times12}]}{\frac{0.09}{12}}$$

$$P = R2402037.83$$

2. John buys a car and needs to take a loan for R115 000. The bank charges 15,5%p.a. compounded monthly and is told the loan period will be 4 years. Calculate Johns monthly payment.

Solution

$$P_v = R115000$$

$$i = \frac{0.155}{12}$$

$$n = 4 \times 12 = 48$$

$$x = ?$$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$115000 = \frac{x[1-(1+\frac{0,155}{12})^{-48}]}{\frac{0,155}{12}}$$

$$\frac{0,155}{12} \times 115000 = \frac{x[1-(1+\frac{0,155}{12})^{-48}]}{\frac{0,155}{12}} \times \frac{0,155}{12}$$

$$x = R3229,76$$

John's monthly payments will be R3 229,76

ACTIVITY /ASSESSMENT 2.8

1. Siphokazi bought a house. She paid a deposit of R102 000, which is equivalent to 12% of the selling price of the house. She obtained a loan from the bank to pay the balance of the selling price. The bank charges her interest of 9% per annum, compounded monthly.
   (a) Determine the **selling price** of the house.
   (b) The period of the loan is 20 years and she starts repaying the loan one month after it was granted calculate her **monthly instalments**.
   (c) How **much interest** will she pay over the period of 20 years? Round off your answer correct to the nearest rand.
   (d) Calculate the outstanding balance of her loan immediately after her 85th instalment.

2. Nolusizo took out a loan of R1 500 000 to buy a house He will repay the loan with monthly payments over 20 years. The interest rate is 8% p.a. compounded quarterly.
   (a) Showing ALL your calculations and formulae, prove that his monthly installment will be R12 499,96.
   (b) Calculate the outstanding balance after 12 years.

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3. Buhle decided to start saving before retirement. She makes payments of R10 000 monthly into an account yielding 7,72% p.a. compounded monthly, starting on 1 November 2016 with a final payment on 1 April 2026.
   (a) Calculate how much will be in the saving account immediately after the last deposit is made.
   (b) At the end of the investment period Buhle re-invested the full amount in order for her to be able to draw a monthly pension from the fund. She re-invested the money at an interest rate of 10% p.a. compounded monthly. If she draws an amount of R30 000 per month from this investment for how many full months she will be able to receive R30 000?
   (c) After withdrawing R30 000 for 20 months Buhle requires R1 500 000.Determine whether she can access this amount of the money from this annuity

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 9)	Weighting		Grade	12
RESOURCES				
DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document				
NOTES				

Notes and steps on how to calculate final payments and interest paid

FINAL PAYMENT

- Firstly, calculate n
- Secondly calculate Outstanding Balance
- Thirdly compound the Outstanding Balance for the last month using A=$P(1+i)^1$ (*note n=1and P=Outstanding Balance)

TOTAL INTEREST PAID = Monthly payment×(n×compounding period) - loan amount

Examples

1. Michael borrows R5000 from a MMS lender at an interest rate of 28% p.a. compounded monthly. He repays the loan by means of equal monthly payments of R800 and a final payment of less than R800.

1.1 Determine the **number of payments** at R800?

$Solution$
$Given : P = R5000$
$x = R800$
$i = 28\% = 0.28/12$
$n = ?$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

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$$R5000 = \frac{R800[1-(1+\frac{0.28}{12})^{-n}]}{\frac{0.28}{12}}$$

$$\frac{R5000(\frac{0.28}{12})}{R800} = \frac{R800[1-(1+\frac{0.28}{12})^{-n}]}{R800}$$

$$-[\frac{500(\frac{0.28}{12})}{800} - 1] = (1+\frac{0.28}{12})^{-n}$$

$$-n = \log_{(1+\frac{0.28}{12})}[-\frac{5000(\frac{0.28}{12})}{800} + 1]$$

$$-n = 6.834037675$$

$$n = 6.834037675$$

$$n \approx 7\ payments$$

But only 6 payments will be R800. The last payment will be less than R800

1.2 What will be the value of **final payment** be?

Solution

For Final Payment, we will need to know the Balance outstanding first then use the appreciation formula to calculate the final payment from the outstanding balance.

Given:

$P = ?$
$x = R800$
$i = 28\% = 0.28/12$
$n = 0.834037675$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P_v = \frac{800[1-(1+\frac{0.28}{12})^{-0.834037675}}{\frac{0.28}{12}}$$

$$P_v = R653.2611593$$

$$A = P(1+i)^n$$

$$A = R653.2611593(1+\frac{0.28}{12})^1$$

$$A = R668.50$$

Thus His Final payment will be **R668.50**

ACTIVITY /ASSESSMENT 2.9

(DBE NOVEMBER 2016)

2.9.1. On 1 June 2016 a bank granted Thabiso a loan of R250 000 at an interest rate of 15% p.a. compounded monthly, to buy a car. Thabiso agreed to repay the loan in monthly instalments commencing on 1 July 2016 and ending 4 years later on 1 June 2020. However, Thabiso was unable to make the first two instalments and only commenced with the monthly instalments on 1 September 2016.

(a) Calculate the amount Thabiso owed the bank on 1 August 2016, a month before he paid his first monthly instalment.

(b) Having paid the first monthly instalment on 1 September 2016, Thabiso will still pay his last monthly instalment on 1 June 2020. Calculate his monthly instalment.

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(c) If Thabiso paid R9 000 as his monthly instalment starting on 1 September 2016, how many months sooner will he repay the loan?

(d) If Thabiso paid R9 000 as a monthly instalment starting on 1 September 2016, calculate the final instalment to repay the loan.

2.9.2. Nine years ago, a bank granted Mandy a home loan of R525 000. This loan was to be repaid over 20 years at an interest rate of 10% p.a.., compounded monthly. Mandy's monthly repayments commenced exactly one month after the loan was granted.

(a) Mandy decided to make monthly repayments of R6 000 instead of the required R5 066,36. How many payments will she make to settle the loan?

(b) After making monthly repayments of R6 000 for nine years, Mandy required money to fund her daughter's university fees. She approached the bank for another loan. Instead, the bank advised Mandy that the extra amount repaid every month could be regarded as an investment and that she could withdraw this full amount to fund her daughter's studies. Calculate the maximum amount that Mandy may withdraw from the loan account.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 10)	**Weighting**		**Grade**	**12**
RESOURCES				
DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document				
NOTES				

<u>Notes on Delayed payments or Missed Payments</u>

- **<u>Deferred payments</u>** – also referred to as **delayed** payments and generate compound interest with $A = P(1+i)^n$.
- **<u>Missed payments/difficulties to pay</u>** – e.g. 13rd ,14th , 15th payments missed ,generate profit outside also $A = P(1+i)^n$, n = 3 since 3 payments missed. **First calculate outstanding balance** after the 12th payment. Thereafter, the loan continues to accumulate interest for the period equal to the number of missed payments.

<u>Example</u>

1. A loan of R10 000, taken on 1 February 2016, is to be repaid in regular fixed instalments of R450 on the first day of each month. Interest is charged on the loan at 9,5 % p.a. compounded monthly. The first instalment is paid on 1 August 2016.Calculate:

1.1 The total amount payable on 1 July 2016.

$A = P(1+i)^n$

$A = 10000(1+\frac{0,095}{12})^5$

$A = R10402,15$

1.2 The number of payments that will be needed to settle the loan.

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Solution

$$P=\frac{x[1-(1+i)^{-n}]}{i}$$

$$10\ 402{,}15=\frac{450\left[1-\left(1+\frac{0{,}095}{12}\right)^{-n}\right]}{\frac{0{,}095}{12}}$$

$$\frac{10\ 402{,}15\times\frac{0{,}095}{12}}{450}=1-\left(1+\frac{0{,}095}{12}\right)^{-n}$$

$$\left(1+\frac{0{,}095}{12}\right)^{-n}=0{,}816999213$$

$$-n=\frac{\log 0{,}816999213}{\log\left(1+\frac{0{,}095}{12}\right)}$$

$$n=25{,}6315128$$

$$\therefore n=26 \text{ payments}$$

1.3 The balance outstanding on the loan after the 25th payment has been made.

Solution

Balance on Loan after the 25th payment

$$=10\ 402{,}15\left(1+\frac{0{,}095}{12}\right)^{25}-\frac{450\left[\left(1+\frac{0{,}095}{12}\right)^{25}-1\right]}{\frac{0{,}095}{12}}$$

$$=R12668{,}90-R12386{,}54$$

$$=R282{,}36$$

or

$$P=\frac{x[1-(1+i)^{-n}]}{i}$$

$$=\frac{450\left[1-\left(1+\frac{0{,}095}{12}\right)^{-0{,}6315128}\right]}{\frac{0{,}095}{12}}$$

$$=R282.36$$

ACTIVITY /ASSESSMENT 2.10

2.10.1. Melissa takes a loan of R950 000 to buy a house. The interest is 14,25% p.a. compounded monthly. His first instalment will commence one month after taking out the loan.

(a) Calculate the monthly repayments over a period of 20 years.

(b) Determine the balance on the loan after the 100th instalment.

(c) If Melissa failed to pay the 101st, 102nd, 103rd and 104th instalments, calculate the value of the new instalment that will settle the loan in the same time period.

2.10.2. Exactly 10 months ago, a bank granted Jane a loan of R800 000 at an interest rate of 10,25% p.a.., compounded monthly.

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- Must be repaid over 20 years
- Must be repaid by means of monthly repayments of R7 853,15 starting one month after the loan was granted.

(a) How much did Jane owe immediately after making her 6th repayment?

(b) Due to financial difficulties, Jane missed the 7th ,8th and 9th payments she was able to make payments from the end of the 10th month onwards. Calculate Jane's increased monthly payment in order to settle the loan in the original 20 years.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 11)	Weighting		Grade	12
RESOURCES				
DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document				
NOTES				

<u>Notes on Sinking Fund</u>

- **SINKING FUND**- It is a **saving account** which is set up in order to save money to **replace an old** item in future. It is used as a saving account that will allow investor to purchase/buy expensive items or to fund expensive capital outlays in few years' time.
- **Calculate** the REPLACEMENT/EXPECTED/NEW COST →(**APPRECIATION**)
- **Calculate** the SCRAP VALUE/BOOK/TRADE-IN/DECAY VALUE →(**DEPRECIATION**)

SINKING FUND = APPRECIATION - DEPRECIATION

To calculate **monthly instalment** in the **sinking fund** →Use the **<u>future value</u>** annuity formula

<u>EXAMPLES (SINKING FUND)</u>

A company purchases a new vehicle for R200 000. The vehicle is expected to depreciate at a rate of 24% per annum on a reducing balance. It is also expected that the vehicle will have to be replaced after 5 years. A sinking fund is set up for this purpose. If the replacement cost of the vehicle increases by 18% per annum compounded annually calculate

(a) The value of the current vehicle in 5 years' time.

Solution

$A = ?$

$P = R200000$

$i = 24\%$

$n = 5$

$A = R200000(1-0.24)^5$

$A = R50710.51$

(b)The cost of the new vehicle in 5 years' time

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Solution

$$A = R200000(1+0.18)^5$$
$$A = R457551.55$$

(c) The total required value of the sinking fund, if the old vehicle is sold and proceeds contribute towards the purchase of the new vehicle.

Solution

$$Sinking\ fund = Appreciation - Depreciation$$
$$Sinking\ fund = R457551.55 - R50710.51$$
$$Sinking\ fund = R406841.04$$

(d)The **monthly instalments** paid into the sinking fund If the interest rate is 15% p.a. compounded monthly and start when the vehicle is initially purchased.

Solution

$$F_v = R406841.04$$
$$x = ?$$
$$i = 15\% = \tfrac{0.15}{12}$$
$$n = 5$$
$$406841.04 = \frac{x[(1+\frac{0.15}{12})^{5\times 12} - 1]}{\frac{0.15}{12}}$$
$$\tfrac{0.15}{12} \times 406841.04 = x[(1+\tfrac{0.15}{12})^{5\times 12} - 1]$$
$$x = R4593.21$$

ACTIVITY /ASSESSMENT 2.11

2.11.1. A company purchased a photocopying machine for R270 000. The company expects to replace the machine in 5 years' time. They anticipated the cost of the machine to escalate at 16% p.a. compound interest. They expect their present machine to have a scrap value of R100 000 in 5 years' time when they sell it. The company set up a sinking fund to save for a new photocopying machine. They will use the amount they obtain from the scrap value of the old machine and the money in the sinking fund after 5 years, to purchase a new machine. The company will pay a fixed monthly amount into the sinking fund, starting in one month's time and will make the final payment at the end of the 5 year period. The interest earned on the sinking fund is 10% p.a. compounded monthly.

(a) Determine the cost of the new machine in 5 years' time
(b) Calculate the rate of depreciation of the old machine on the reducing balance method.
(c) Determine the value of the fixed monthly payments the company must pay into the sinking fund.

<u>QUESTION 1 (DBE NOVEMBER 2012)</u>

2.11.2.A business buys a machine that costs R120 000. The value of the machine depreciates at 9% per annum according to the diminishing-balance method.

(a) Determine the scrap value of the machine at the end of 5 years.

(b) After five years the machine needs to be replaced. During this time, inflation remained constant at 7% per annum. Determine the cost of the new machine at the end of 5 years.

(c) The business estimates that it will need R90 000 by the end of five years. A sinking fund for R90 000, into which equal monthly instalments must be paid, is set up. Interest on this fund is

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8,5% per annum, compounded monthly. The first payment will be made immediately and the last payment will be made at the end of the 5-year period. Calculate the value of the monthly payment into the sinking fund.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 12)	Weighting		Grade	12

RESOURCES

DBE Past exam papers, Mind Action Series (Textbook), Exam Guidelines, CAPS Document, Learner Support Document

NOTES

ACTIVITY /ASSESSMENT 2.12

(a) Two friends, Kuda and Thabo, each want to invest R5 000 for four years. Kuda invests his money in an account that pays simple interest at 8,3% per annum. At the end of four years, he will receive a bonus of exactly 4% of the accumulated amount. Thabo invests his money in an account that pays interest at 8,1% p.a.., compounded monthly

Whose investment will yield a better return at the end of four years? Justify your answer with appropriate calculations.

(b) A car that costs R130 000 is advertised in the following way: 'No deposit necessary and first payment due three months after date of purchase.' The interest rate quoted is 18% p.a. compounded monthly. Calculate the amount owing two months after the purchase date, which is one month before the first monthly payment is due.

(b) Herschel bought this car on 1 March 2009 and made his first payment on 1 June 2009.Thereafter he made another 53 equal payments on the first day of each month.

(1) Calculate his monthly repayments.

(2) Calculate the total of all Herschel's repayments.

TOPIC: FINANCE,GROWTH AND DECAY (LESSON 13)	Weighting		Grade	12

RESOURCES

DBE Past Exam Papers

REVISION ACTIVITIES

QUESTION 1 (FEB/MARCH 2016)

1.1 Diane invests a lump sum of R5 000 in a savings account for exactly 2 years.The investment earns interest at 10% p.a. compounded quarterly.

1.1.1 What is the quarterly interest rate for Diane’s investment? (1)

1.1.2 Calculate the amount in Diane’s savings account at the end of the 2 years. (3)

1.2 Motloi inherits R800 000. He invests all of his inheritance in a fund which earns interest at a rate of 14% p.a., compounded monthly. At the end of each month he withdraws R10 000

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from the fund. His first withdrawal is exactly one month after his initial investment.

1.2.1 How many withdrawals of R10 000 will Motloi be able to make from this fund? (5)

1.2.2 Exactly four years after his initial investment Motloi decides to withdraw all the remaining money in his account and to use it as a deposit towards a house.

(a) What is the value of Motloi's deposit, to the nearest rand? (4)

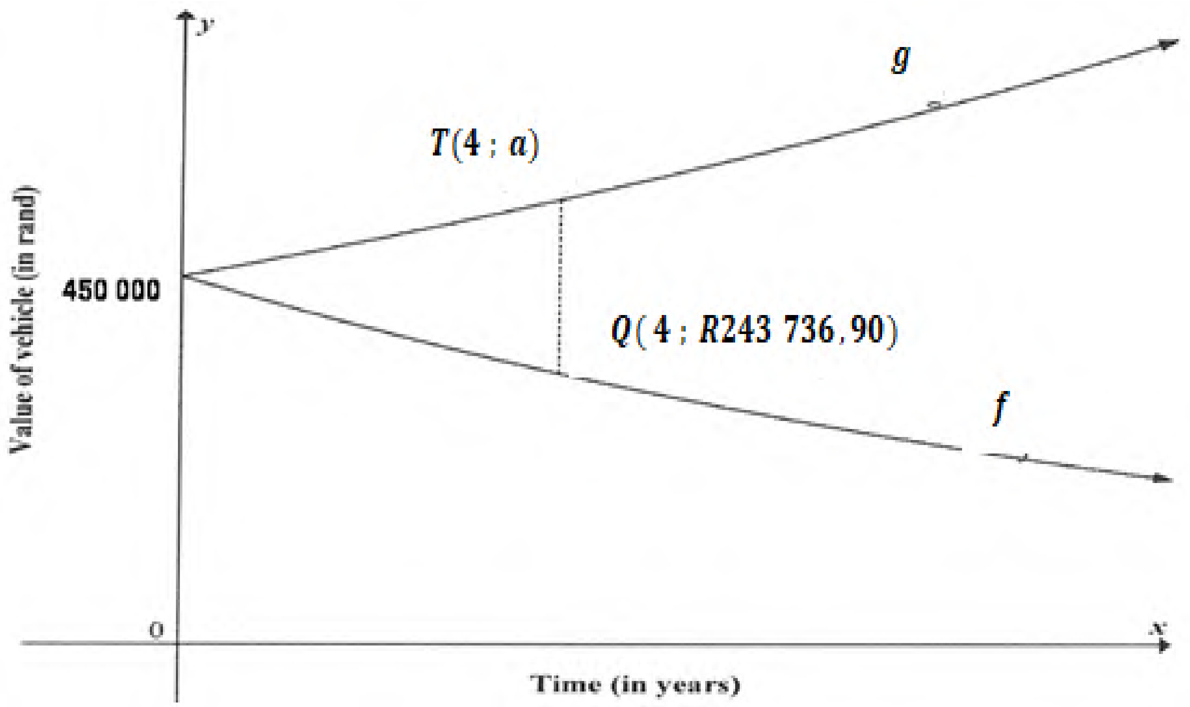
(b) Motloi's deposit is exactly 30% of the purchase price of the house. What is the purchase price of the house, to the nearest rand? (1)

[14]

QUESTION 2 (NOVEMBER 2015)

The graph of f shows the book value of a vehicle x years after the time Joe bought it.

The graph of g shows the cost price of a similar new vehicle x years later.



2.1 How much did Joe pay for the vehicle? (1)

2.2 Use the reducing – balance method to calculate the percentage annual rate of depreciation of the vehicle that Joe bought. (4)

2.3 If the average rate of the price increase of the vehicle is 8,1% p.a., calculate the value of a (3)

2.4 A vehicle that costs R450 000 now, is to be replaced at the end of 4 years. The old vehicle will be used as a trade-in. A sinking fund is created to cover the replacement cost of this vehicle. Payments will be made at the end of each month. The first payment will be made at the end of the 13th month and the last payment will be made at the end of the 48th month. The sinking fund earns interest at a rate of 6,2 % p.a., compounded monthly. Calculate the monthly payment to the fund. (5) **[13]**

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QUESTION 3 (SEPTEMBER 2016)

3.1 If a car valued at R255 000 depreciates on a reducing balance method at an interest rate of 12,5 % p.a.., calculate the book value of the car after 7 years. (3)

3.2 A loan of R10 000, taken on 1 February 2016, is to be repaid in regular fixed instalments of R450 on the first day of each month. Interest is charged on the loan at 9,5 % p.a. compounded monthly. The first instalment is paid on 1 August 2016.

Calculate:

3.2.1 The total amount payable on 1 July 2016. (2)

3.2.2 The number of payments that will be needed to settle the loan. (5)

3.2.3 The balance outstanding on the loan after the 25th payment has been made. (4)

QUESTION 5 (FEB/MARCH 2017)

5.1 On the 2nd day of January 2015 a company bought a new printer for R150 000.

- The value of the printer decreases by 20% annually on the reducing-balance method.
- When the book value of the printer is R49 152, the company will replace the printer.

5.1.1 Calculate the book value of the printer on the 2nd day of January 2017. (3)

5.1.2 At the beginning of which year will the company have to replace the printer? Show ALL calculations. (4)

5.1.3 The cost of a similar printer will be R280 000 at the beginning of 2020.04.07 the company will use the R49 152 that it will receive from the sale of the old printer to cover some of the costs of replacing the printer. The company sets up a sinking fund to cover the balance. The fund pays interest at 8,5% per annum, compounded quarterly. The first deposit was made on 2 April 2015 and every three months thereafter until 2 January 2020.

Calculate the amount that should be deposited every three months to have enough money to replace the printer on 2 January 2020. (4)

5.2 Lerato wishes to apply for a home loan. The bank charges interest at 11% per annum, compounded monthly. She can afford a monthly instalment of R9 000 and wants to repay the loan over a period of 15 years. She will make the first monthly repayment one month after the loan is granted. Calculate, to the nearest thousand rand, the maximum amount that Lerato can borrow from the bank. (5)

[16]

QUESTION 6 (NOVEMBER 2017)

6.1 Mbali invested R10 000 for 3 years at an interest rate of r% p.a., compounded monthly. At the end of this period, she received R12 146,72. Calculate r, correct to ONE decimal place. (5)

6.2 Piet takes a loan from a bank to buy a car for R235 000. He agrees to repay the loan over a period of 54 months. The first instalment will be paid one month after the loan is granted. The bank charges interest at 11% p.a., compounded monthly.

6.2.1 Calculate Piet's monthly instalment. (4)

6.2.2 Calculate the total amount of interest that Piet will pay during the

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first year of the repayment of the loan. (6)

[15]

QUESTION 7 (FEB/MARCH 2018)

7.1 On 30 June 2013 and at the end of each month thereafter, Asif deposited R2 500 into a bank account that pays interest at 6% per annum, compounded monthly.

He wants to continue to deposit this until 31 May 2018. Calculate how much money Asif will have in this account immediately after depositing R2 500 on 31 May 2018 (3)

7.2 On 1 February 2018, Genevieve took a loan of R82 000 from the bank to pay for her studies. She will make her first repayment of R3 200 on 1 February 2019 and continue to make payments of R3 200 on the first of each month thereafter until she settles the loan. The bank charges interest at 15% per annum, compounded monthly.

7.2.1 Calculate how much Genevieve will owe the bank on 1 January 2019. (3)

7.2.2 How many instalments of R3 200 must she pay? (5)

7.2.3 Calculate the final payment, to the nearest rand, Genevieve has to pay to settle the loan. (5)

[16]

QUESTION 8 (SEPTEMBER 2018)

8.1 A tractor costing R180 000 depreciates on the reducing balance method to R65 000 at the end of 8 years. Determine the rate at which the tractor is depreciating per annum. (3)

8.2 Tebogo buys a flat at the beach front for R850 000. She takes out a loan from the bank at an interest rate of 14,25 % per annum compounded monthly. Her first instalment will commence in one month after she has taken out the loan.

8.2.1 Calculate the monthly repayments over a period of 20 years. (4)

8.2.2 If the monthly repayment is increased by 20 % before the first payment is being made towards the loan, determine the number of payments that will now be made to settle the loan. (4)

8.2.3 Calculate the final payment to settle the loan in QUESTION 23.2.2. (4)

[15]

QUESTION 9 (NOVEMBER 2018)

9.1 Selby decided today that he will save R15 000 per quarter over the next four years. He will make the first deposit into a saving account in three months' time and he will make his last deposit at the end of four years from now.

9.1.1 How much will Selby have at the end of four years if interest is earned at 8,8% per annum, compounded quarterly? (3)

9.1.2 If Selby decided to withdraw R100 000 from the account at the end of three years from now, how much will he have in the account at the end of four years from now? (3)

9.2 Tshepo takes out a home loan over 20 years to buy a house that costs R1 500 000.

9.2.1 Calculate the monthly instalment if interest is charged at 10,5% p.a., compounded monthly. (4)

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9.2.2 Calculate the outstanding balance immediately after the 144^{th} payment was made. (5)

[15]

QUESTION 10 (MAY/JUNE 2019)

10.1 Sandile bought a car for R180 000. The value of the car depreciated at 15% per annum according to the reducing-balance method. The book value of Sandile's car is currently R79 866,96.

10.1.1 How many years ago did Sandile buy the car? (3)

10.1.2 At exactly the same time that Sandile bought the car, Anil deposited R49 000 into a savings account at an interest rate of 10% p.a. compounded quarterly. Has Anil accumulated enough money in his savings account to buy Sandile's car now? (3)

10.2 Exactly 10 months ago, a bank granted Jane a loan of R800 000 at an interest rate of 10,25% p.a.., compounded monthly.

- Must be repaid over 20 years
- Must be repaid by means of monthly repayments of R7 853,15 starting one month after the loan was granted.

10.2.1 How much did Jane owe immediately after making her 6^{th} repayment? (4)

10.2.2 Due to financial difficulties, Jane missed the 7^{th} ,8^{th} and 9^{th} payments She was able to make payments from the end of the 10^{th} month onwards. Calculate Jane's increased monthly payment in order to settle the loan in the original 20 years. (5)

[15]

QUESTION 11 (SEPTEMBER 2019)

11.1 A car depreciated at the rate of 13,5% p.a. to R250 000 over 5 years according to the reducing balance method. Determine the original price of the car, to the nearest rand. (3)

11.2 Melissa takes a loan of R950 000 to buy a house. The interest is 14,25% p.a. compounded monthly. His first instalment will commence one month after taking out the loan.

11.2.1 Calculate the monthly repayments over a period of 20 years. (4)

11.2.2 Determine the balance on the loan after the 100^{th} instalment. (4)

11.2.3 If Melissa failed to pay the 101^{st}, 102^{nd}, 103^{rd} and 104^{th} instalments, calculate the value of the new instalment that will settle the loan in the same time period. (4)

[15]

QUESTION 12 (NOVEMBER 2019)

12.2 Nine years ago, a bank granted Mandy a home loan of R525 000. This loan was to be repaid over 20 years at an interest rate of 10% p.a.., compounded monthly. Mandy's monthly repayments commenced exactly one month after the loan was granted. Mandy decided to make monthly repayments of R6 000 instead of the required R5 066,36.

12.2.1 How many payments will she make to settle the loan? (5)

12.2.2 After making monthly repayments of R6 000 for nine years, Mandy required money to fund her daughter's university fees. She approached the bank for another loan. Instead, the bank advised. Mandy that the extra amount repaid every month could be regarded as an investment and that she could withdraw this full amount to fund her daughter's studies. Calculate the maximum amount that Mandy may withdraw from the loan account. (4)

[14]

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3. STATISTICS

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 1	Weighting	20/150 in Paper 2	Grade	10

RESOURCES

KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books

NOTES

- ***Mode*** *is the most common value in a data set.*
- ***Median*** *is the middle value in an ordered data set*
- ***Mean*** *is the average of the data set*
- ***Minimum*** *value is the lowest value in a data set*
- ***Maximum*** *value is the highest value in a data set*
- ***Lower Quartile*** *is the median of the lower half of an ordered data set*
- ***Upper Quartile*** *is the median of the upper half of an ordered data set*
- ***Range*** *is the difference between the highest and the lowest value in a data set*
- ***Inter-quartile range*** *is the difference between the upper and lower quartile*
- ***Skewness*** *a measure of symmetry in a distribution*

Example 1

In an experiment, a group of 23 girls were presented with a page containing 30 rectangles. They were asked to name the colours of rectangles correctly as quickly as possible. The time, in seconds, taken by each of the girls is given in the table below.

12	**13**	**13**	**14**	**14**	**16**	**17**	**18**	**18**	**18**	**19**	**20**
21	**21**	**22**	**22**	**23**	**24**	**25**	**27**	**29**	**30**	**36**	

a) Calculate the mean of the data

b) Calculate the inter-quartile range of the data

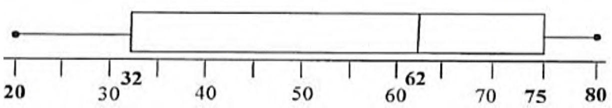
c) Draw a box and whisker diagram to represent the data and comment on the skewness of the data.

d) The five number summary of the times taken by a group of 23 boys in naming the colours of the rectangles is (15; 21: 23,5; 26; 38). Which of the two groups, girls or boys, had the lower median to correctly name the colours of the rectangles?

e) The first three learners who named the colours of all 30 rectangles correctly in the shortest time will receive a prize. How many boys will be among **X** these three prize winners? Motivate your answer

Example 2

The box and whisker diagram below shows the marks (out of 80) obtained in a History test by a class of nine learners.



a) Comment on the skewness of the data

b) Write down the range of the marks

c) If the leaners had to obtain 32 marks to pass the test, estimate the percentage of the class that failed

ACTIVITIES /ASSESSMENT

3.1.1. A group of 30 learners each randomly rolled two dice once and the sum of the values on the upper most faces of the dice was recorded. The data is shown in the frequency table below.

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Sum of the upper most values on faces	**Frequency**
2	0
3	3
4	2
5	4
6	4
7	8
8	3
9	2
10	2
11	1
12	1

a) Calculate the mean of the data.
b) Determine the median of the data.
c) Determine the standard deviation of the data.
d) Determine the number of times that the sum of the recorded values of the dice is within one standard deviation from the mean. Show your calculations.

3.1.2. An organization decided that it would set up donor clinics at various colleges. Students would donate blood over a period of 10 days. The number of units donated per day by students of college is shown in the table below.

Days	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Units	**45**	**59**	**65**	**73**	**79**	**82**	**91**	**99**	**101**	**106**

(a) Calculate the mean of the unit of blood donated per day over the period of 10 days
(b) It was discovered that there was an error in counting the number of units of blood donated by college **X** each day. The correct mean of the data is 95 units of blood. How many units of blood were NOT counted over the 10 days?
(c) Referring to the mean and median of the data, comment on the skewness of data

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 2	**Weighting**	20/150 in Paper 2	**Grade**	**10**
RESOURCES				
KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books				

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NOTES

Example 1

The arm spans (in cm) of the eleven players in each of two different soccer teams A and B are recorded.

a) The arm spans for **TEAM A** are:
203, 214, 187, 188, 196, 199, 205, 203, 199, 194 and 206

i) Calculate the mean of the arm spans using the formula: $\bar{x} = \frac{\sum x}{n}$.

ii) Copy and complete the table given.

iii) Calculate the standard deviation of the arm spans using the formula:

$\sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$.

b) For **TEAM B**, the variance is 875 cm^2. Calculate the standard deviation of the arm spans of **TEAM B.**

c) Make a comment about the dispersion of the arm spans of the players in both teams.

x	$x - \bar{x}$	$(x - \bar{x})^2$
203		
214		
187		
188		
196		
199		
205		
203		
199		
194		
206		
$n =$		$\sum(x-\bar{x})^2 =$

SOLUTION

a)

i) $\bar{x} = \frac{\sum x}{n} = \frac{203+214+187+188+196+199+205+203+199+194+206}{11} = 199{,}4545\ldots \approx 199{,}5$ cm

ii)

x	$x - \bar{x}$	$(x - \bar{x})^2$
203	203 – 199,5 = 3,5	$(3{,}5)^2 = 12{,}25$
214	214 – 199,5 = 14,5	$(14{,}5)^2 = 201{,}25$
187	187 – 199,5 = – 12,5	$(-12{,}5)^2 = 156{,}25$
188	188 – 199,5 = – 11,5	$(-11{,}5)^2 = 132{,}25$
196	196 – 199,5 = – 3,5	$(-3{,}5)^2 = 12{,}25$
199	199 – 199,5 = – 0,5	$(-0{,}5)^2 = 0{,}25$
205	205 – 199,5 = 5,5	$(5{,}5)^2 = 30{,}25$
203	203 – 199,5 = 3,5	$(3{,}5)^2 = 12{,}25$
199	199 – 199,5 = – 0,5	$(-0{,}5)^2 = 0{,}25$
194	194 – 199,5 = – 5,5	$(-5{,}5)^2 = 30{,}25$
206	206 – 199,5 = 6,5	$(6{,}5)^2 = 42{,}25$
$n = 11$		$\sum(x-\bar{x})^2 = 629{,}75$

iii) Standard deviation $= \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{629{,}75}{11}} = \sqrt{57{,}25} = 7{,}5663\ldots \approx 7{,}6$ cm.

b) For Team B, $\sigma = \sqrt{\text{variance}} = \sqrt{875} = 29{,}5803\ldots \approx 29{,}6$ cm.

c) The standard deviation of Team B is greater than that of Team A. This shows that the lengths of the arm spans of team B are more variable and spread out than those from Team A.

Example 2

The time (in minutes) taken by a group of athletes from Lesiba High School to run a 3km cross country race is: 18; 21; 16; 24; 28; 20; 22; 29; 19; 23

Use your calculator to determine:

a) The mean time taken to complete the race.

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b) The standard deviation of the time taken to complete the race.

ACTIVITIES /ASSESSMENT

3.2.1. Given the following data set:

15 23 45 28 32 35 52 25 70

a) Determine the variance and standard deviation for the above data.

b) If 15 were added to each value in the data set, what would be the new mean?

c) If 15 were added to each value in the data set, what would be the new standard deviation?

3.2.2. An organization decided that it would set up donor clinics at various colleges. Students would donate blood over a period of 10 days. The number of units donated per day by students of college is shown in the table below.

Days	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Units	**45**	**59**	**65**	**73**	**79**	**82**	**91**	**99**	**101**	**106**

a) Calculate the standard deviation of the data.

b) How many days is the number of units of blood donated at college **X** outside one standard deviation from the mean?

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 3	**Weighting**	20/150 in Paper 2	**Grade**	**10**

RESOURCES

KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books

NOTES

Example 1

Look again at the choir teacher's summary of attendance.

Number of learners at choir practice (x)	**Frequency (f)**
$0 < x \leq 10$	1
$10 < x \leq 20$	2
$20 < x \leq 30$	11
$30 < x \leq 40$	9
$40 < x \leq 50$	14
$50 < x \leq 60$	3
	$n = 40$

Find the approximate value of the mean number of learners who attended choir practice.

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SOLUTION:

First add in another column and work out the ***midpoint of each interval***. Then, add another column and calculate ***frequency*** $\times$ ***mid-point value***.

Number of learners at choir practice (x)	Frequency f	Midpoint of the interval X	$f \times X$
$0 < x \leq 10$	1	$\frac{0+10}{2} = 5$	$1 \times 5 = 5$
$10 < x \leq 20$	2	$\frac{10+20}{2} = 15$	$2 \times 15 = 30$
$20 < x \leq 30$	11	$\frac{20+30}{2} = 25$	$11 \times 25 = 275$
$30 < x \leq 40$	9	$\frac{30+40}{2} = 35$	$9 \times 35 = 315$
$40 < x \leq 50$	14	$\frac{40+50}{2} = 45$	$14 \times 45 = 630$
$50 < x \leq 60$	3	$\frac{50+60}{2} = 55$	$3 \times 55 = 165$
	$n = 40$		$\sum f.X = 1\,420$

Example 2

In a particular primary school in Pietermaritzburg, it was found that ninety of their Foundation Phase learners (Grades 1, 2 and 3) were accompanied to school by someone. The ages of the person accompanying the child were recorded, as shown in the table below.

Age (in years) (x)	Frequency (f)
$0 < x \leq 10$	12
$10 < x \leq 20$	30
$20 < x \leq 30$	18
$30 < x \leq 40$	12
$40 < x \leq 50$	9
$50 < x \leq 60$	6
$60 < x \leq 70$	3

Use the information given in the table to

a) Determine the modal interval.

b) Estimate the mean age of the person accompanying a learner from the Foundation Phase.

c) Estimate the median age of the person accompanying a learner from the Foundation Phase.

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SOLUTION:

a) The modal interval is $10 < x \leq 20$.
This means that more people in this age group accompanied the learners to school than any other age group.

b) To find the mean we have to take the midpoint of each class interval and then calculate *frequency* $\times$ *midpoint* for each class interval.

Age (in years) x	**Midpoint** X	**Frequency** f	$f.X$
$0 < x \leq 10$	$\frac{0+10}{2} = 5$	12	$12 \times 5 = 60$
$10 < x \leq 20$	$\frac{10+20}{2} = 15$	30	$30 \times 15 = 450$
$20 < x \leq 30$	$\frac{20+30}{2} = 25$	18	$18 \times 25 = 450$
$30 < x \leq 40$	$\frac{30+40}{2} = 35$	12	$12 \times 35 = 420$
$40 < x \leq 50$	$\frac{40+50}{2} = 45$	9	$9 \times 45 = 405$
$50 < x \leq 60$	$\frac{50+60}{2} = 55$	6	$6 \times 55 = 330$
$60 < x \leq 70$	$\frac{60+70}{2} = 65$	3	$3 \times 65 = 195$
		$n = 90$	$\sum f.X = 2\,310$

Mean $= \bar{X} = \frac{\sum f.X}{n} \approx \frac{2\,310 \text{ years}}{90} = 25{,}7$ years old

The mean tells us that if all the ages were added together, and then shared out equally amongst the 90 people, then each one would be 25,7 years old.

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ACTIVITIES /ASSESSMENT

3.3.1. The table below represents the ages of the 90 people accompanying Foundation Phase learners to another primary school in Pietermaritzburg:

Age (in years) (x)	Frequency (f)
$0 \leq x < 10$	4
$10 \leq x < 20$	12
$20 \leq x < 30$	25
$30 \leq x < 40$	14
$40 \leq x < 50$	10
$50 \leq x < 60$	20
$60 \leq x < 70$	5
	$n = 90$

a) Use the information given in the table to
   i) Determine the modal interval.
   ii) Estimate the mean age of the people accompanying a learner from the Foundation Phase.
   iii) Estimate the median age of the people accompanying a learner from the Foundation Phase.

b) What do the modal class, the mean and the median tell you about the ages of the people who accompany the children to school?

3.3.2. The table below has been adapted from Census 2011. It lists the property values of 262 properties in a part of the Ikwezi municipality.

Property Value in Rand (x)	Frequency (number of households) (f)
$0 \leq x < 50\,000$	172
$50\,000 \leq x < 100\,000$	51
$100\,000 \leq x < 150\,000$	18
$150\,000 \leq x < 200\,000$	12
$200\,000 \leq x < 250\,000$	9
	$n = 262$

a) Use the information given in the table to
   i) Determine the modal interval.
   ii) Estimate the mean property value of the properties.
   iii) Estimate the median property value of the properties.

b) What does the modal class, the mean and the median tell you about these property values?

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TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 4	**Weighting**	20/150 in Paper 2	**Grade**	**10**

RESOURCES

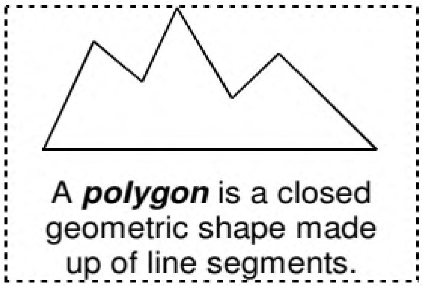
KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books

NOTES

- ***Continuous data*** *can be measured and broken in to smaller pieces. e.g. height, mass etc.*
- ***Frequency*** *tells us how many of each items there are in a data set/ group*.
- ***Interval class*** *subsets in which data grouped* ***(e.g.*** $\mathbf{5 < x \leq 10}$)
- ***Midpoint*** *the mid value of the data set* $\frac{5+10}{2}$
- ***Modal class*** *interval class with the highest frequency*
- ***Mode (estimate)*** *Midpoint of modal class*
- ***Histogram*** *a visual representation of grouped data showing frequency distribution among interval classes. It also shows the skewness of the data*

HISTOGRAM	BAR GRAPH
• It is a representation of grouped data • There is no gap between the bars ***For example****, you draw a HISTOGRAM to show the number of people whose heights (h) lie in the following intervals (measured in cm): 150 ≤ h < 160; 160 ≤ h < 170; etc*	• It is a representation of ungrouped data that does not have to be numerical • There is generally a gap between the bars ***For example*** *you draw a BAR GRAPH to show the number of learners in a class who wear glasses and the number who do not wear glasses.*

- ***Frequency polygon*** *a visual representation of grouped data in which frequency is plotted against* ***midpoints*** *of interval classes.*



Example 1

The following table lists the marks (given as percentage) obtained by the Grade 11 learners of Musi High School in their mathematics test:

24 70 50 22 63 45 48 52 56 38 65 68 65 17 32 60 62 53 63 45 49 44 56 12 55
83 54 22 67 54 34 77 46 50 58 80 81 39 84 75 55 76 73 80 66 71 62 40 23 76

a) Organise the data using a grouped frequency table.
b) Draw a histogram to illustrate the data.
c) Calculate the modal interval. What does this measure of central tendency tell you about the learners' marks?
d) Estimate the median. What does this measure of central tendency tell you about the learners' marks?

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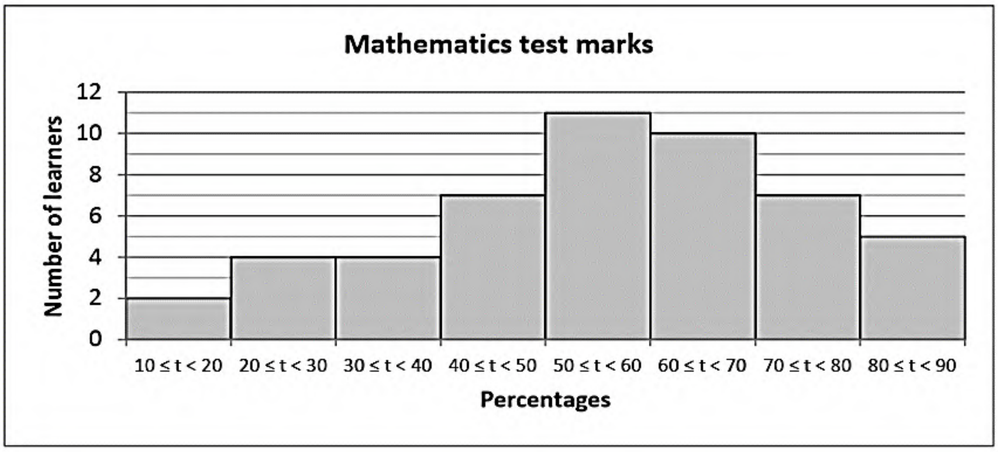
SOLUTIONS

a) The lowest mark was 12% and the highest mark was 84%
It is often easiest to use multiples of 10 as the class intervals, so start the first interval at 10% and end the last interval at 90%

Percentages (t)	Frequency (Number of learners)
$10 \leq t < 20$	2
$20 \leq t < 30$	4
$30 \leq t < 40$	4
$40 \leq t < 50$	7
$50 \leq t < 60$	11
$60 \leq t < 70$	10
$70 \leq t < 80$	7
$80 \leq t < 90$	5
TOTAL	50

b) Draw the histogram as follows:
STEP 1: Draw and label the horizontal and vertical axes.
STEP 2: Represent the frequency on the vertical axis and the classes on the horizontal axis.
STEP 3: Using the frequencies (or number of learners) as the heights, draw vertical bars for each class.



c) The modal interval is the interval with the largest frequency or largest number of learners. So the modal interval is $50 \leq t < 60$.
This tells us that more learners got marks in the interval $50 \leq t < 60$ than in any of the other intervals.

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d) There are 50 data items (marks/percentages).
The median lies between the 25th and the 26th marks.
Add up the frequencies until you reach 25 (or more than 25):

$$2 + 4 + 4 + 7 + 11 = 28$$

The 28th mark lies in the interval $50 \leq t < 60$
So the median lies in the interval $50 \leq t < 60$
The median ≈ 55% (the midpoint of the interval)
This tells us that 50% of the learners got marks that were less ***than 55%*** and 50% of the learners got marks that were ***more than 55%***

Example 2

Eighty of the learners at Alexandra High School were surveyed to find out how many minutes each week they spent collecting waste material for recycling. The grouped frequency table shows the results of the survey.

Number of minutes (t)	Number of learners (f)
$9 < t \leq 13$	8
$13 < t \leq 17$	28
$17 < t \leq 21$	27
$21 < t \leq 25$	12
$25 < t \leq 29$	4
$29 < t \leq 33$	1

I) Find the midpoint of the intervals

II) Use the table to draw a frequency polygon on a separate set of axes.

Solutions

i) Calculate the midpoint of each interval using the formula:
Midpoint = $\frac{lower\ limit\ of\ interval + upper\ limit\ of\ interval}{2}$

Number of minutes (t)	Mid points	Frequency (f)	Ordered pairs
$5 < t \leq 9$	$\frac{5+9}{2} = \frac{14}{2} = 7$	0	(7; 0)
$9 < t \leq 13$	$\frac{9+13}{2} = \frac{22}{2} = 11$	8	(11; 8)
$13 < t \leq 17$	$\frac{13+17}{2} = \frac{30}{2} = 15$	28	(15; 28)
$17 < t \leq 21$	$\frac{17+21}{2} = \frac{38}{2} = 19$	27	(19; 27)
$21 < t \leq 25$	$\frac{21+25}{2} = \frac{46}{2} = 23$	12	(23; 12)
$25 < t \leq 29$	$\frac{25+29}{2} = \frac{54}{2} = 27$	4	(27; 4)
$29 < t \leq 33$	$\frac{29+33}{2} = \frac{62}{2} = 31$	1	(31; 1)
$33 < t \leq 37$	$\frac{33+37}{2} = \frac{70}{2} = 35$	0	(35; 0)

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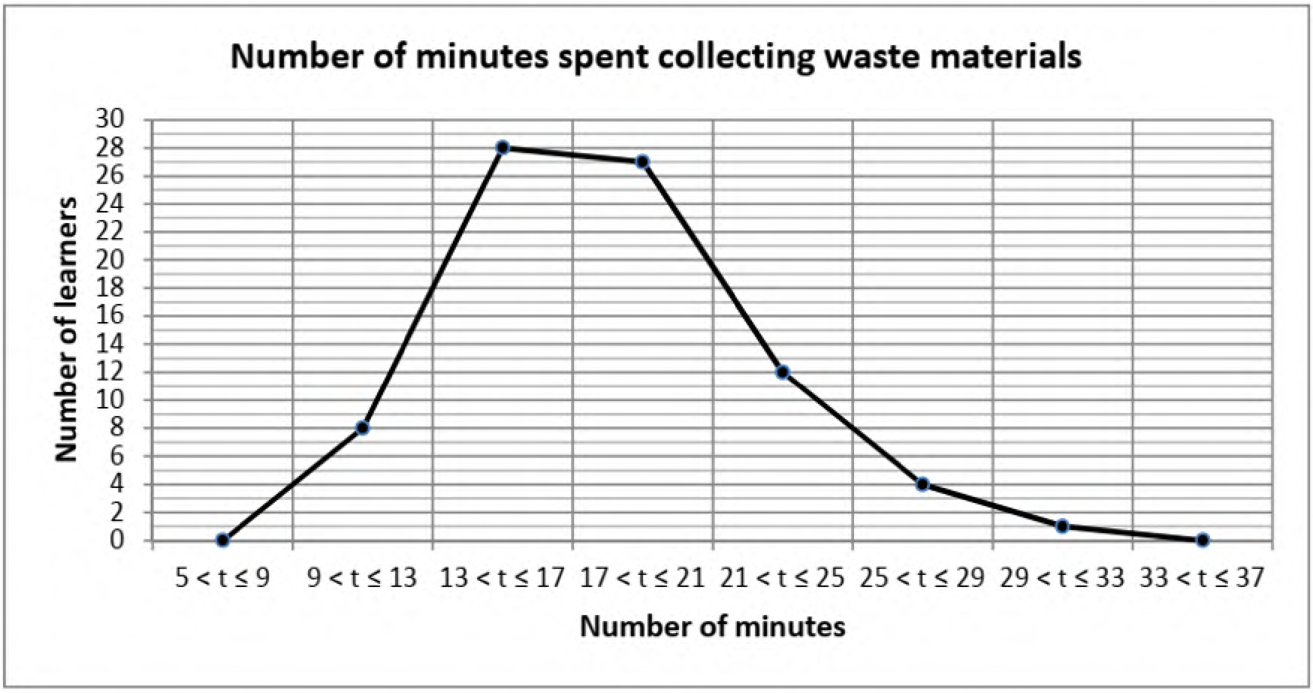
ii) Plot the ordered pairs (midpoint; frequency) and join them with straight lines. Make sure that the graph touches the horizontal axis on both sides.



ACTIVITIES /ASSESSMENT

3.4.1 The frequency table below represent the distribution of the amount of time (in hours) that 80 high school learners spent in one week watching their favourite sport.

Time in hours	Frequency
$10 < t \leq 15$	8
$15 < t \leq 20$	28
$20 < t \leq 25$	27
$25 < t \leq 30$	12
$30 < t \leq 35$	4
$35 < t \leq 40$	1

a) Draw a histogram to represent the data
b) Calculate
   i) the modal interval
   ii) an estimate of the median
c) What do these two measures of central tendency tell you about the amount of time the learners devote to watching their favourite sport?

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3.4.2. Some of the learners took part in the javelin competition. The best distances (in metres) thrown by each competitor in 2011 and 2012 are shown.

Distance thrown in metres (m)	Number of competitors 2011	Number of competitors 2012
$10 < m \leq 20$	0	1
$20 < m \leq 30$	3	4
$30 < m \leq 40$	14	19
$40 < m \leq 50$	21	13
$50 < m \leq 60$	7	11
$60 < m \leq 70$	0	2
TOTAL	**45**	**50**

a) On the same set of axes, draw frequency polygons to illustrate the 2011 and 2012 results.

b) By referring to the table and the frequency polygons, comment on the performance of the competitors in 2011 and 2012.

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 5	**Weighting**	20/150 in Paper 2	**Grade**	11

RESOURCES

KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books

NOTES

- ***Frequency*** tells us *how many of each item there are in a data set*.
- ***Cumulative frequency*** shows the number of results that are *less than* (<) or *less than or equal to* (≤) a stated value in a set of data.
- To find the ***cumulative frequency***,
    1. Add up the frequencies as you go down the frequency table.
    2. Write each *running total* or *cumulative frequency* in your table.
- An ***ogive*** or ***cumulative frequency curve*** is a graph that shows the information in a cumulative frequency table. The graph is useful for estimating the median and inter-quartile range of the grouped data.
- You can draw an ***ogive*** of ungrouped discrete data, grouped discrete data or grouped continuous data. It can be drawn from a grouped frequency table or an ungrouped frequency table.

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Example

The following frequency table shows the time (in minutes) taken by learners to travel to school.

Time taken to travel to school	Frequency	Cumulative Frequency	Ordered Pairs
$0 < t \leq 10$	4		
$10 < t \leq 20$	12		
$20 < t \leq 30$	28		
$30 < t \leq 40$	32		
$40 < t \leq 50$	29		
$50 < t \leq 60$	15		

a) Complete the table.
b) Draw an ogive to illustrate the information.

SOLUTION:

a) Steps to follow when completing the table:

- Add in an interval with a frequency of 0 before the first interval.
- Find the cumulative frequency by adding the frequencies.
- List the ordered pairs where *the first coordinate = upper limit of the interval* and *the second coordinate = cumulative frequency.*

Note: A cumulative frequency of 105 means that 105 learners or less spent 50 minutes or less to walk to school.

Time taken to travel to school	Frequency	Cumulative Frequency	Ordered Pairs
$-10 < t \leq 0$	0	**0**	**(0;0)**
$0 < t \leq 10$	4	**4**	**(10;4)**
$10 < t \leq 20$	12	**4 + 12 = 16**	**(20;16)**
$20 < t \leq 30$	28	**16 + 28 = 44**	**(30;44)**
$30 < t \leq 40$	32	**44 + 32 = 76**	**(40;76)**
$40 < t \leq 50$	29	**76 + 29 = 105**	**(50;105)**
$50 < t \leq 60$	15	**105 + 15 = 120**	**(60;120)**

b) Draw the ogive as follows:
   i) Draw the axes and label the variable on the x-axis and the cumulative frequency on the y-axis.
   ii) Plot the ordered pairs.
   iii) Join the points to form a smooth curve.

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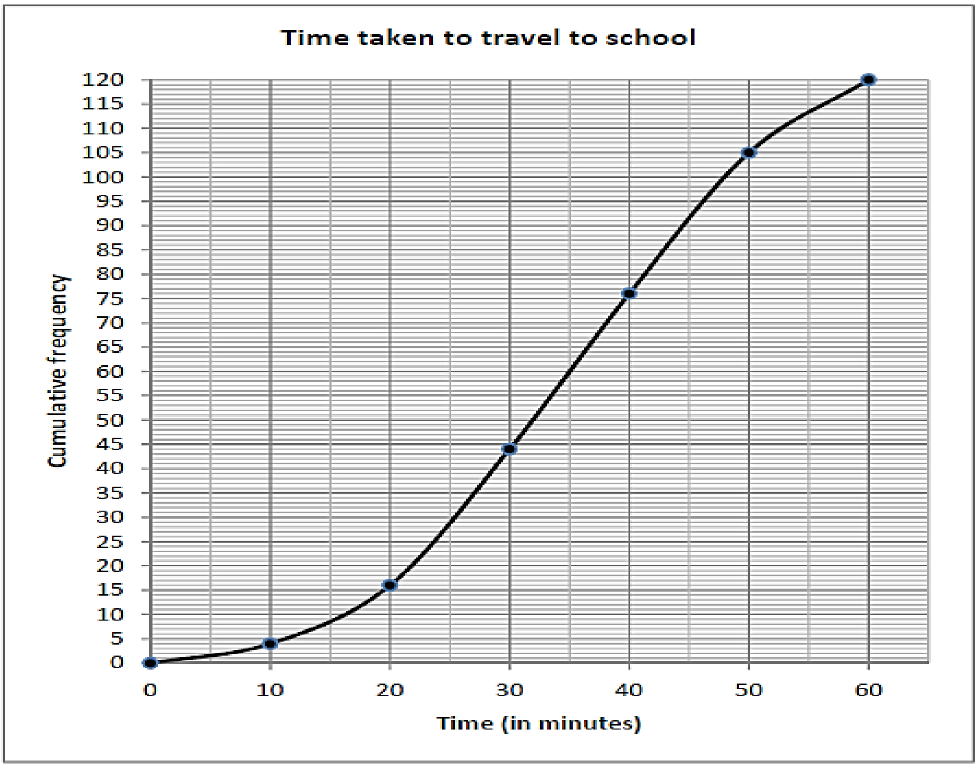
The ogive:



✓ Always remember when drawing cumulative frequency curve from a table of grouped data, the ***cumulative frequencies*** are plotted at the ***upper limit*** of the interval.

ACTIVITIES /ASSESSMENT (Homework)

3.5.1

1) In the 2009 Census@School learners were asked what their arm span was, correct to the nearest centimetre. The results of two hundred of the Grade 10, 11 and 12 learners who took part were recorded as follows:

Arm span in cm	Frequency	Cumulative Frequency
$130 < h \leq 135$	16	
$135 < h \leq 140$	26	
$140 < h \leq 145$	42	
$145 < h \leq 150$	54	
$150 < h \leq 155$	26	
$155 < h \leq 160$	22	
$160 < h \leq 165$	14	

To find your arm span: Open arms wide, measure the distance across your back from the tip of your right hand middle finger to the tip of your left hand middle finger.

a) Copy and complete the table.
b) Draw an ogive to illustrate the data.
c) Use your ogive to determine approximately how many learners have arm spans that are less than or equal to 152 cm.
d) Use your graph to determine approximately how many learners have arm spans of between 138 cm and 158 cm.

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TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 6	**Weighting**	20/150 in Paper 2	**Grade**	11
RESOURCES				
KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books				
NOTES				

Example 1

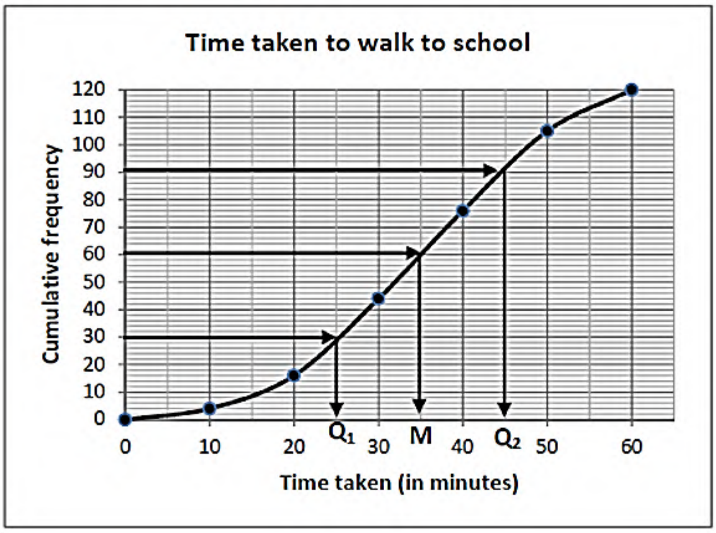
Use the Ogive drawn in the previous day's classwork to answer the questions below:

a) Determine the approximate values of
   i) the median
   ii) the lower quartile
   iii) the upper quartile of the set of data.

b) What does each of these values tell you about the time taken by the learners?

SOLUTION:

a) This is the ogive drawn in Example 4:



i) To find the approximate value of the *median (M)*, find the midpoint of the values plotted on the *cumulative frequency axis.*
- The maximum value is 120, so the median lies between the 60th and 61st term.
- Draw a horizontal line from just above 60 until it touches the ogive.
- From that point draw a vertical line down to the *horizontal* axis.

So the *median* $\approx$ *35 minutes*.

ii) To find the approximate value of the *lower quartile* (Q_1), find the midpoint of the lower half of the values plotted on the *cumulative frequency axis.*
- There are 60 terms in the lower half of the data, so the lower quartile lies between the 30th and the 31st term.
- Draw a horizontal line from just above 30 until it touches the ogive.
- From that point draw a vertical line down to the *horizontal* axis.

So the *lower quartile* $\approx$ *25 minutes*.

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iii) To find the approximate value of the *upper quartile* (Q_3), find the midpoint of the upper half of the values plotted on the *cumulative frequency axis.*

- There are 60 terms in the upper half of the data, so the upper quartile lies between 60 + 30 = 90th and the 91st term.
- Draw a horizontal line from just above 90 until it touches the ogive.
- From that point draw a vertical line down to the *horizontal* axis.

So the *upper quartile ≈ 45 minutes*.

b)

i) The *median* tells us that 50% of the learners took 35 minutes or less or to walk to school.

ii) The *lower quartile* tells us that 25% of the learners took 25 minutes or less to walk to school.

iii) The *upper quartile* tells us that 75% of the learners took 45 minutes or less to walk to school.

ACTIVITIES /ASSESSMENT

3.6.1 Fifty learners who travel by car to school were asked to record the number of kilometres travelled to and from school in one week. The following table shows the results:

Number of kilometres	Number of learners	Cumulative frequency
$10 < x \leq 20$	2	2
$20 < x \leq 30$		9
$30 < x \leq 40$		13
$40 < x \leq 50$		26
$50 < x \leq 60$		42
$60 < x \leq 70$		50
	TOTAL = 50	

a) Copy the table and then fill in the second column of the table.

b) Draw an ogive to illustrate the data.

c) Use your graph to estimate the median number of kilometres travelled per week.

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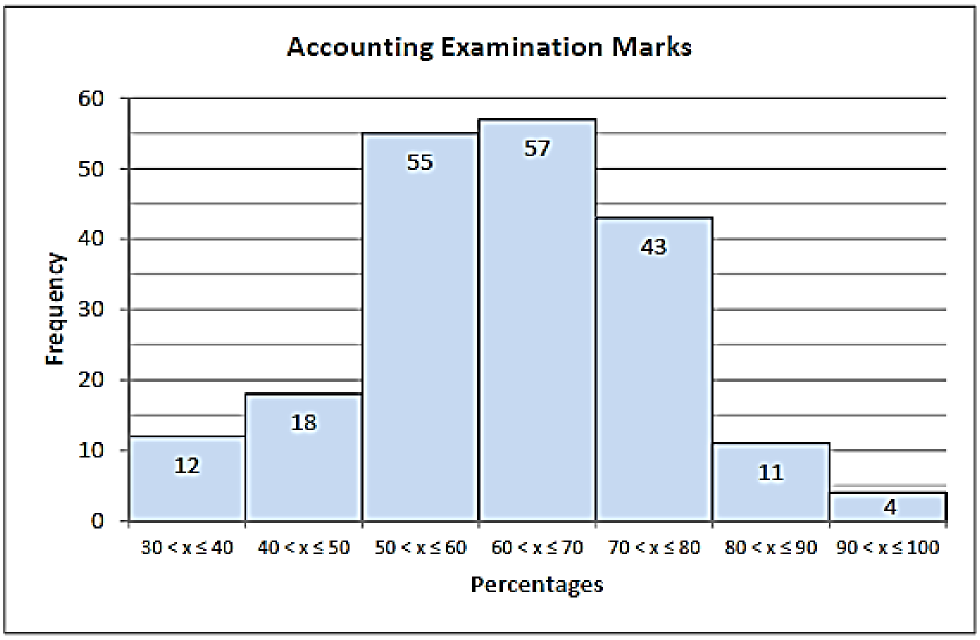
3.6.2.

The histogram below shows the distribution of the Accounting examination marks for 200 learners.



a) Draw a grouped frequency table to record the data shown on the histogram.

b) Draw an ogive to illustrate the data in the frequency table.

c) Use the ogive to estimate how many learners scored 72% or more for the examination.

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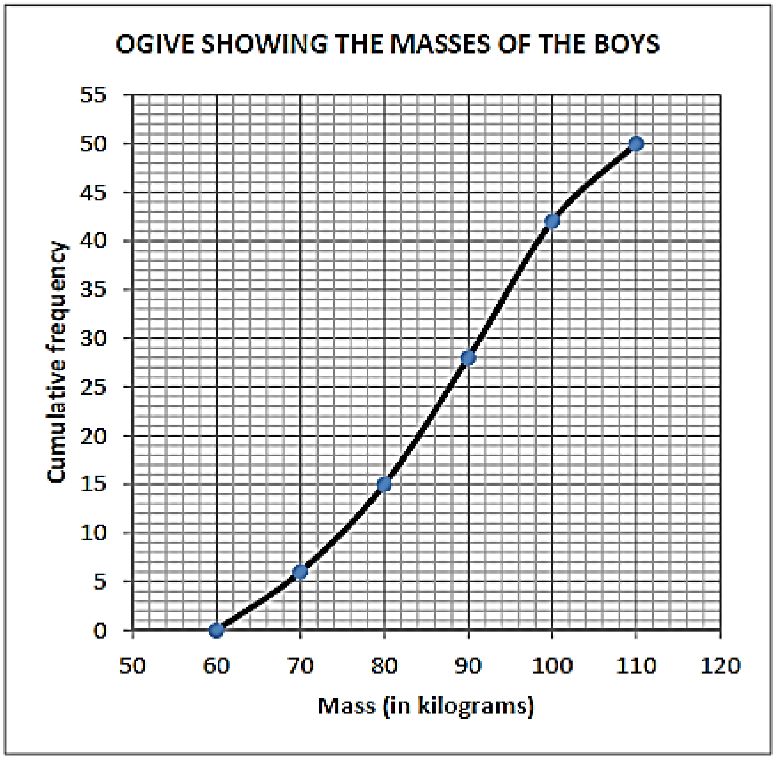
3.6.3. The masses of a random sample of 50 boys in Grade 11 were recorded. This cumulative frequency graph (ogive) represents the recorded masses.



a) How many of the boys had a mass between 90 and 100 kilograms?
b) Estimate the median mass of the boys.
c) Estimate how many of boys had mass less than 80 kilograms.

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 7	Weighting	20/150 in Paper 2	Grade	12

RESOURCES

KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books, trial papers. Any Mathematics textbook.

NOTES

- Scatter plot is a plot of bivariate data (data that has two variables) which shows a relationship between the two sets of data.
- Regression line (line of best fit) is used to show the general trend which a set of data follows.
- Interpolation is the term used to predict a value which lies in the domain and range of the given data set.
- Extrapolation is the term used to predict a value which lies outside the domain and range of the given data set
- Using a calculator to determine equation of regression line, mean point and correlation coefficient.
- Least squares regression line is a straight line in the form of $y = A + Bx$
    - This line will always pass through the mean point of the data $(\hat{X}; \hat{Y})$
    - A is y-intercept and B is gradient of a line.

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- To draw the regression line, plot the mean point and y-intercept the value of A

Step	Button to press/method
1	Mode
2	2(stat)
3	2($A + BX$)
4	Input each data value for x and y one at a time, pressing the = button after each entry.
5	Once all the data is entered press AC
6	Shift1 (stat) then 5(Reg) 1: A .2: B, 3:r
7	Pressing 1 will display the value of A, pressing 2 will display the value of B, pressing 3 will display the correlation coefficient.
8	To determine the mean point press 4(var) then select 2:$\hat{X}$ and 5:$\hat{Y}$

Example 1

The data below shows the marks obtained by ten Grade 12 learners from two different Mathematics classes sitting in the first row of each class.

Class/ Klas A	16	36	20	38	40	30	35	22	40	24
Class/ Klas B	45	70	44	56	60	48	75	60	63	38

1.1 Make use of the grid provided on the **DIAGRAM SHEET 1** to draw a scatter plot for the data.

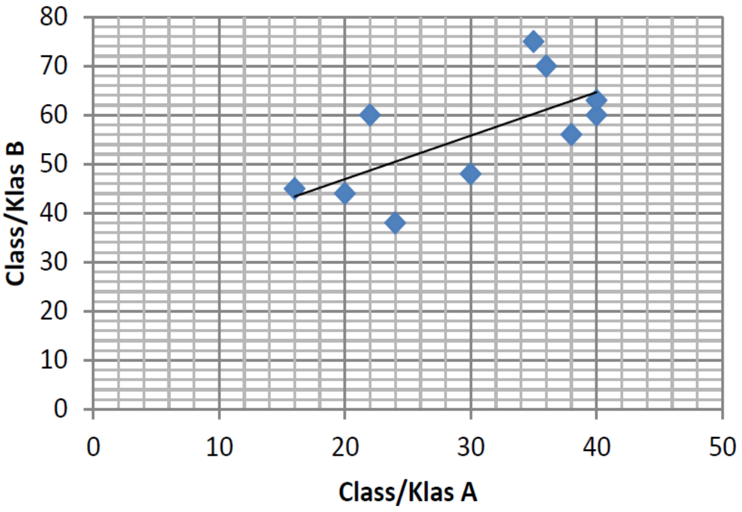
1.2 Calculate the equation of the least squares' regression line for this data.

1.3 Draw the least squares regression line for the data on the scatter plot diagram drawn in **QUESTION 1.2**

1.4 Learner scored 5 marks in class A. predict the mark he should get in Class B

Solution

1.1.



1.2. $A = 29,22$ and $B = 0,89$

$\therefore y = 29,22 + 0,89x$

1.3. See diagram

1.4 $y = 29,22 + 0,89(5) = 33,67$

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ACTIVITIES /ASSESSMENT

3.7.1. A recording company investigates the relationship between the number of times a CD is played by a national radio station and the national sales of the same CD in the following week. The data below was collected for a random sample of 10 CDs. The sales figures are rounded to the nearest 50.

Number of times CD is played	47	34	40	34	33	50	28	53	25	46
Weekly sales of the CD	3 950	2 500	3 700	2 800	2 900	3 750	2 300	4 400	2 200	3 400

a) Identify the independent variable.
b) Draw a scatter plot of this data on the grid provided on the class work book.
c) Determine the equation of the least squares' regression line.
d) How many times was the CD played if the weekly sales are 2700?

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 8	**Weighting**	20/150 in Paper 2	**Grade**	**12**

RESOURCES

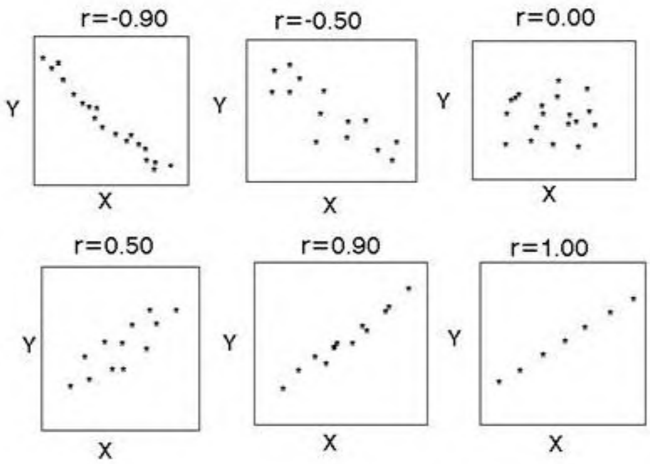
KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books, trial papers. Any Mathematics textbook.

NOTES

- Correlation coefficient
  - The correlation coefficient tells us about the strength of the relationship between the variables.
  - Correlation coefficient (r) always lies between -1and 1 $-1 < r < 1$

r	Comment
1	Perfect positive correlation
0,9	Strong positive correlation
0,5	Moderate positive correlation
0.2	Weak positive correlation
0	No correlation
-0,2	Weak negative correlation
-0,5	Moderate negative correlation
-0.9	Strong negative correlation
-1	Perfect negative correlation

- Ensure you refer above to the step-by-step use of a calculator
- Learners to fully understand *r* by graphical representation.



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Example 1
The data below shows the marks obtained by ten Grade 12 learners from two different Mathematics classes sitting in the first row of each class.

Class/ Klas A	16	36	20	38	40	30	35	22	40	24
Class/ Klas B	45	70	44	56	60	48	75	60	63	38

1.1. Calculate the correlation coefficient for the above data.

1.2. Comment on the strength of the relationship between Class A and Class B.

1.3. Calculate the mean and the standard deviation for Class B.

Solution

1.1. $\mathrm{r} = 0.66$
1.2. Moderate positive correlation
1.3. $\bar{x} = 55,90$ and $\sigma = 11,36$

Example 2
The goal-scorers in a netball game practice scoring at training during the week. In the tournament during the weekend, the number of goals they score from the total number of attempts they made is recorded as a percentage. The statistic is referred to as the successful goal- shooting average. The table below shows the number of goal shots practiced during the week and the successful goal-shoot average during the tournaments for 8 goal-scorers.

Number of goal shots practiced	280	400	540	595	375	430	500	650
Successful goal shoot average (%)	73	75	83	89	80	76	82	91

2.1. Determine the equation of the least squares regression line.
2.2. Calculate the correlation coefficient for the data
2.3. Comment on the correlation between the number of goal shots practised and the successful goal-shoot average.
2.4. A player practised 465 goal shots. What is their expected successful goal-shoot average for the next tournament?

Solution

2.1. $\mathrm{A} = 58.36$, $\mathrm{B} = 0.05$ $y = 58,36 + 0,05x$
2.2. $\mathrm{r} = 0.87$
2.3. Strong positive correlation
2.4. $y = 58,36 + 0,05(465) = 82$ goal-shoot average

ACTIVITIES /ASSESSMENT

3.8.1. A group of students attended a course in Statistics on Saturdays over a period of 10 months.

The number of Saturdays on which a student was absent was recorded against the final mark the Student obtained. The information is shown in the table below and the scatter plot is drawn for

Number of Saturday absent	0	1	2	2	3	3	5	6	7
Final mark (%)	96	91	78	83	75	62	70	68	56

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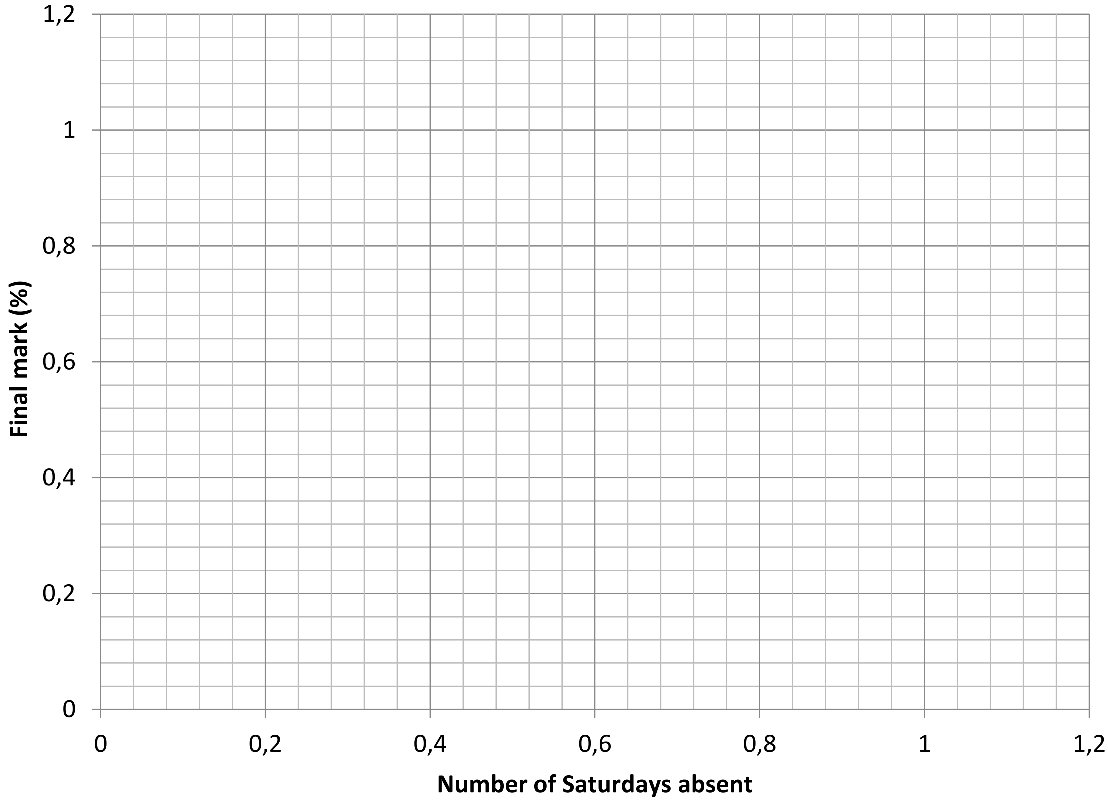

a) Calculate the equation of the least squares regression line
b) Draw the least squares regression line on the grid provided on DIAGRAM SHEET.
c) Calculate the correlation coefficient.
d) Comment on the trend of the data.
e) Predict the final mark of a student who was absent for four Saturdays.

TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 9	**Weighting**	20/150 in Paper 2	**Grade**	**12**
RESOURCES				
KZN Learner Assistance Revision Book Booklet Gr 12, 2020, Data Handling Study Guide from Stats SA, Graph Paper or books, trial papers. Any Mathematics textbook.				
NOTES				
This is a remedial to the lessons already done.				
ACTIVITIES /ASSESSMENT				

3.9.1. The table below gives the average exchange rate and the average monthly oil price for the year 2008.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Exchange rate in R/s	7.5	7.7	7.2	7.4	7.7	7.7	7.6	7.3	7.1	7.0	6.9	6.8
Oil Price in $	69.9	68.0	72.9	70.3	66.3	67.1	67.9	68.3	71.3	73.6	76.0	81.0

a) Draw a scatter plot to represent the exchange rate (in R/s) versus oil price (in $)
b) Determine the equation of the least square regression line.
c) Calculate the value of the correlation coefficient.
d) Comment on the strength of the relationship between exchange rate and the oil price
e) Determine the mean of the oil price

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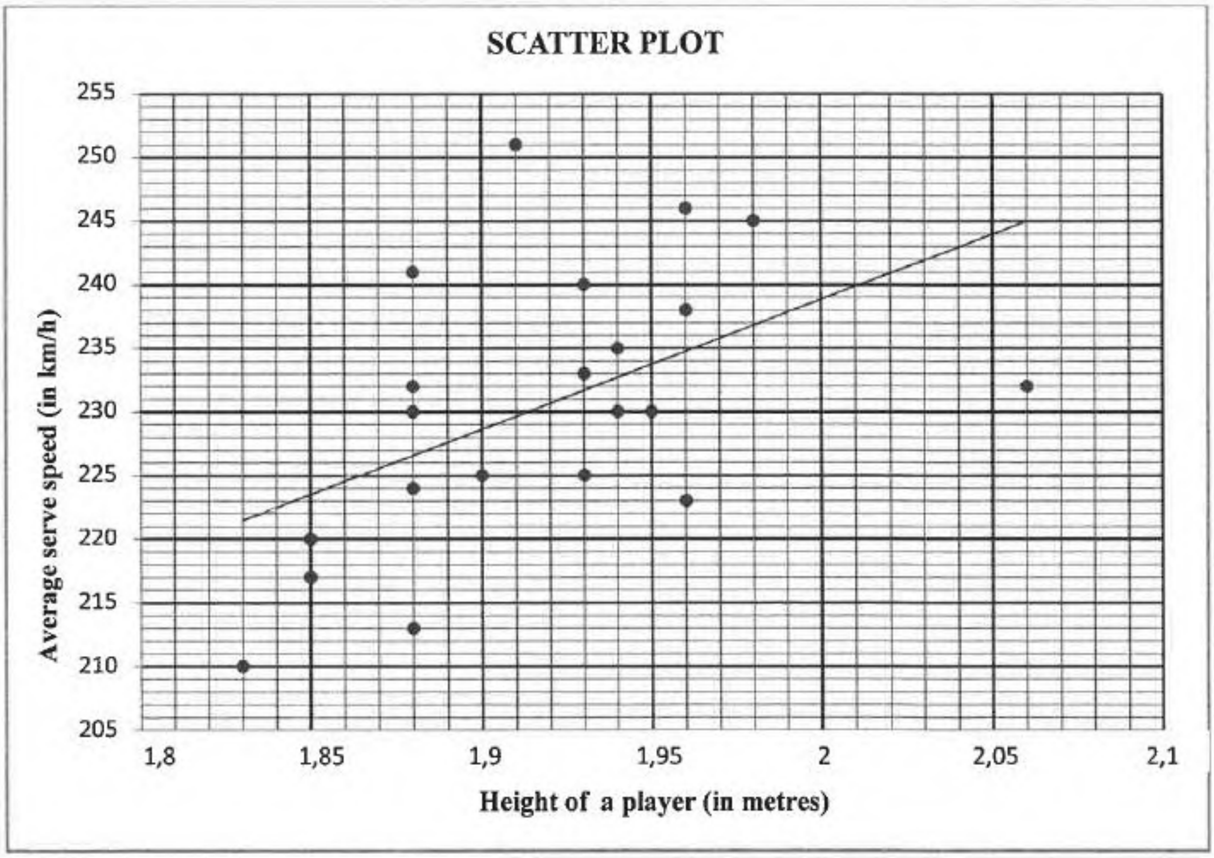
f) Determine the standard deviation of the exchange rate

g) Generally, there is a concern from public when oil price is higher than TWO standard deviations from the mean. In which months would the public have been concerned?

3.9.2

A familiar question among professional tennis players is whether the speed of a tennis serve (in km/h) depends on the height of a player (in metres). The heights of 21 tennis players and the average speed of their serves were recorded during a tournament. The data is represented in the scatter plot below. The least squares regression line is also drawn.



a) Write down the fastest average serve speed (in km/h) achieved in this tournament. (1)

b) Consider the following correlation coefficients:

A. $r = 0{,}93$ B. $r = -0{,}42$ C. $r = 0{,}52$

(i) Which ONE of the given correlation coefficients best fits the plotted data? (1)

(ii) Use the scatter plot and least squares regression line to motivate your answer to QUESTION 2.2.1. (1)

c) What does the data suggest about the speed of a tennis serve (in km/h) and the height of a player (in metres)? (1)

d) The equation of the regression line is given as $\hat{y} = 27{,}07 + bx$.
Explain why, in this context, the least squares regression line CANNOT intersect the y-axis at $(0\,;27{,}07)$. (1)

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TOPIC: Gr. 12 Statistics/ Data Handling: Lesson 10	Weighting	20/150 in Paper 2	Grade	12

RESOURCES

Short Test, Free State 2020 Trial Paper 2

ACTIVITIES /ASSESSMENT (Short Test)

SHORT TEST – SCATTER PLOTS AND REGRESSION

Time : 20 minutes **Marks: 13**

QUESTION 1

The table below gives the average exchange rate and the average monthly oil price for the year 2010.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Exchange rate in R/S	7.5	7.7	7.2	7.4	7.7	7.7	7.6	7.3	7.1	7.0	6.9	6.8
Oil price in $	69.9	68.0	72.9	70.3	66.3	67.1	67.9	68.3	71.3	73.6	76.0	81.0

1.1 Draw a scatterplot to represent the exchange rate (in R/S) versus the oil price (in $). (3)

1.2 Determine the equation of the least square regression line. (3)

1.3 Calculate the value of the correlation coefficient. (1)

1.4 Comment on the strength of the relationship between the exchange rate (in R/S) and the oil price (in $). (2)

1.5 Determine the mean oil price. (1)

1.6 Determine the standard deviation of the oil price. (1)

1.7 Generally there is a concern from the public when the oil price is higher than two standard deviations from the mean.

In which months would the public have been concerned? (2)

[13]

TOPIC: Probability Lesson 1	Weighting	15/150 in Paper 1	Grade	11

RESOURCES

Gr. 10 Textbooks, e.g. Mind Action Series Gr. 10.
A Revision exercise on Gr. 10 Probability, e.g. Mind Action Series Gr. 11 page 265 – 266.

NOTES

- $P(E)=\dfrac{n(E)}{n(S)}=\dfrac{\text{no. of favourable outcomes for this event}}{\text{total number of possible outcomes in the sample space}}$
- $P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$
- Mutually exclusive events
- Exhaustive events
- Complementary events

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- The use of Venn-diagrams to solve probability problems (limited to two events in a sample space in Gr. 10).
- For further practice (assessment), do. e.g. Nov. 2017 DBE, Question 7.

ACTIVITIES /ASSESSMENT

Activity 4.1.1:

1. In a recent survey, it was found that 90 people supported Kaiser Chiefs only, 80 people supported Orlando Pirates only and 5 people supported both teams. There were 10 people who did not support either team.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a person selected at random will support Kaiser Chiefs only.
   (c) Determine the probability that a person selected at random will support both teams.
   (d) Determine the probability that a person selected at random will support none of the teams.
   (e) Determine whether the events involved are inclusive or mutually exclusive?
   (f) Are these events complementary? Give a reason.

2. In the recent municipal elections in a certain town, there were 5014 votes for Candidate A, 3702 for Candidate B and 1215 for Candidate C.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a voter selected at random voted for Candidate A.
   (c) Determine whether the events in this situation are inclusive or mutually exclusive. Give reasons.
   (d) Are these events complementary? Give a reason.

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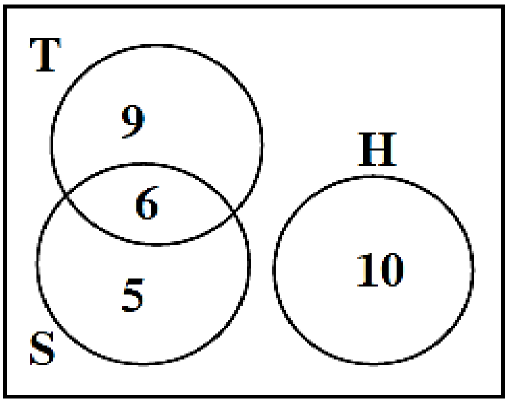
3. In a survey conducted by a local franchise selling pies, it was found that of the 220 customers, 170 bought chicken pies, 70 bought meat pies and 20 bought both.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a customer bought a chicken pie only?
   (c) Determine the probability that a customer bought a meat pie only?
   (d) Determine whether the events involved are mutually exclusive or inclusive. Give reasons.
   (e) Are these events complementary? Give a reason.

4. Thirty learners were asked to state the sports they enjoyed from swimming (S), tennis (T) and hockey (H). The numbers in each set are shown in the venn diagram. One student is then randomly selected.

   (a) Which events are inclusive? Give reasons.
   (b) Which events are mutually exclusive?
   (c) Which events are complementary?
   (d) What is the probability of selecting a learner who enjoyed either hockey or tennis?



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Activity 4.1.2:

November 2017 DBE Gr. 10:

QUESTION 7

7.1 Two events, A and B, are complementary and make up the entire sample space. Also, $P(A') = 0{,}35$.

7.1.1 Complete the statement: $P(A) + P(B) = \ldots$ (1)

7.1.2 Write down the value of $P(A \text{ and } B)$. (1)

7.1.3 Write down the value of $P(B)$. (1)

7.2 A survey was conducted among 150 learners in Grade 10 at a certain school to establish how many of them owned the following devices: smartphone (S) or tablet (T).

The results were as follows:

- 8 learners did not own either a smartphone or a tablet.
- 20 learners owned both a smartphone and a tablet.
- 48 learners owned a tablet.
- x learners owned a smartphone.

7.2.1 Represent the information above in a Venn diagram. (4)

7.2.2 How many learners owned only a smartphone? (3)

7.2.3 Calculate the probability that a learner selected at random from this group:

(a) Owned only a smartphone (1)

(b) Owned at most one type of device (2)

[13]

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TOPIC : PROBABILITY Lesson 2	**Weighting**	15/150 in Paper 1	**Grade**	11

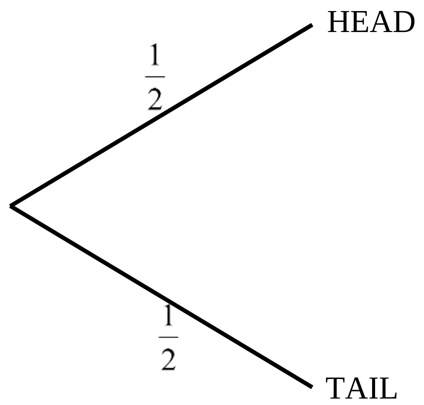
RESOURCES

Grade 11 textbooks.
Past question papers.

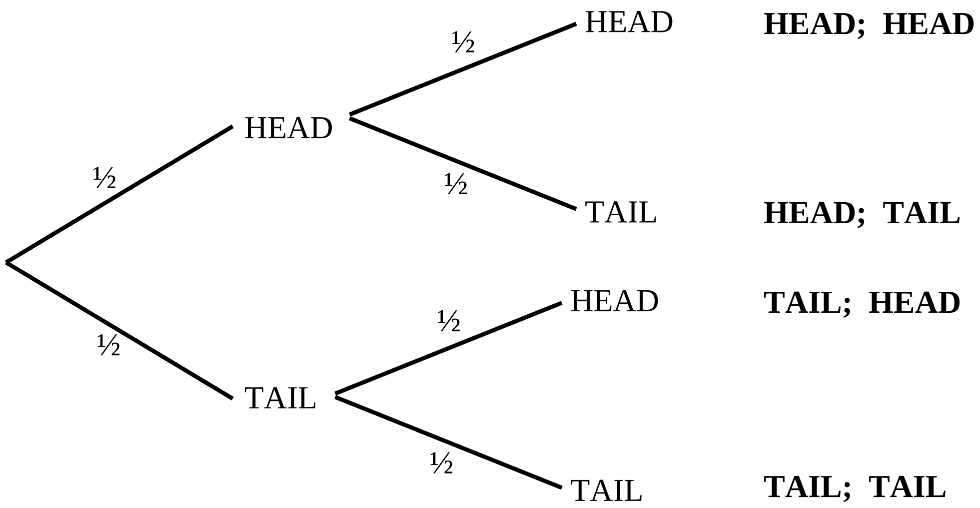
NOTES

Example 1

A coin is tossed twice, draw a tree diagram to represent all the possible outcomes and all the probabilities Start at a point and draw 2 branches representing the two possible outcomes. Write the outcomes at the end of the branches. Write the theoretical probability ON each branch.



- Extend the tree diagram to also represent the second toss. Even though there will only be one more toss, it needs to be represented twice – as if heads were tossed on the first throw and as if tails were tossed on the first throw – to cover ALL possibilities.



Write all the possible outcomes at the end of the branches. Run your finger along the different branches of the tree diagram to show all the possible combined outcomes.

- ***Are these events dependent or independent?***

  (Independent – how the coin landed the first time does not affect what will happen the second time)

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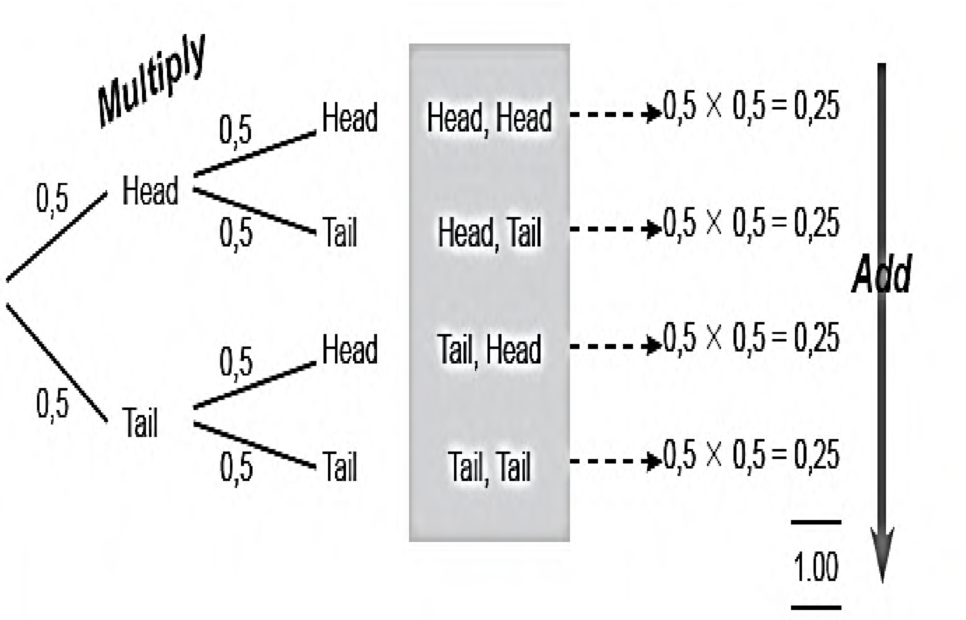
- Use the following diagram (add onto your final one) to show how probabilities are calculated using a tree diagram.



Consider the probability of getting <u>at least one head</u>. Confirm the meaning of this statement. In the two tosses, one head will be acceptable but so will two heads as this also covers the statement 'at least one'.

How many of these outcomes have at least one head in them? (Three of them do - only the last outcome has no heads at all).

To calculate the probability of at least one head being tossed, we first multiply along the branches that lead to each of these three outcomes, then add all of those answers together

$$\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$$

- To find the probability of something happening <u>AND</u> something else happening, multiply the probabilities together.
- To find the probability of something happening <u>OR</u> something else happening, <u>add up</u> the probabilities.

Example 2.

I have a bag with 7 red balls and 3 green balls in it. Without looking into the bag, I am going to take out one ball, and then another ball. The first ball will not be put back in the bag before the second one is drawn.

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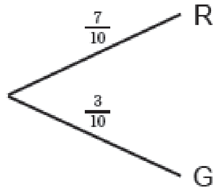
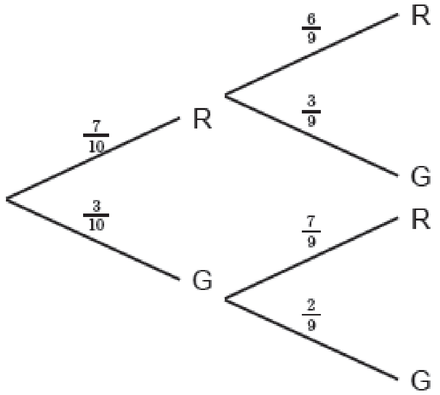
	Start at a point and draw 2 branches representing the two possible outcomes. Write the outcomes at the end of the branches. Write the theoretical probability ON the branch.
	Extend the tree diagram to represent the second draw. *Even though there will only be one more draw, it needs to be represented twice – as if a red ball was drawn on the first draw and as if a green ball was drawn on the first throw - to cover ALL possibilities.*

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	Teaching notes	Answer
How many possible outcomes are there? Name them.	The outcomes were listed at the end of the tree diagram. Remind learners that the end of each branch represents an outcome.	4 outcomes: RR RG GR GG
What is the probability of getting a red ball and then a green ball?	Say: *Look at the outcomes on the tree diagram – choose the outcome that is red first and green second.* *Now look at the probabilities along the branch that leads to this outcome. Multiply.*	$P(RG)=\frac{7}{10}\times\frac{3}{9}$ $=\frac{21}{90}$ $=\frac{7}{30}$
What is the probability of two colours the same?	Say: *Look at the outcomes on the tree diagram– choose the outcome that represent two colours the same.* *Now look at the probabilities along the branch that leads to these outcomes.* *Multiply along the branches and then add the possibilities.*	$P(RR)+P(GG)$ $=\left(\frac{7}{10}\right)\left(\frac{6}{9}\right)+\left(\frac{3}{10}\right)\left(\frac{2}{9}\right)$ $=\left(\frac{42}{90}\right)+\left(\frac{6}{90}\right)$ $=\frac{48}{90}$ $=\frac{8}{15}$ Point out why it was useful not to simplify until later – the fractions already had the same denominator when addition was required.

ACTIVITIES/ASSESSMENT

Activity 4.2.1:

1. A packet of sweets contains 3 pink, 2 green and 5 blue sweets. Two sweets are removed in succession from the packet without replacing them.
   a) Draw a tree diagram to determine all possible outcomes.
   b) Determine the probability that:
      (i) both sweets are blue
      (ii) a green and a pink sweet are selected. (round to 3 decimal places).

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2. The probability that the first answer in a maths quiz competition will be correct is 0,4. If the first answer is correct, the probability of getting the next answer correct rises to 0,5. However, if the answer is wrong, the probability of getting the next answer correct is only 0,3.

a) Represent the information on a tree diagram. Show the probabilities associated with each branch as well as the possible outcomes.

b) a) Calculate the probability of getting the second answer correct.

NSC NOV 2016.

3. Figures obtained from a city's police department indicate that of all the vehicles stolen, 70% were stolen by syndicates (gangs) to be sold off, and 30% were stolen by individual persons for their own use.

Of the vehicles stolen by syndicates:
- 10% were recovered (found back) within 24 hours;
- 30% were recovered after 24 hours; and
- 60% were never recovered.

Of the vehicles stolen by individual persons:
- 30% were recovered within 24 hours;
- 40% were recovered after 24 hours; and
- 30% were never recovered.

3.1 Draw a tree diagram to represent the above information. (3)

3.2 Calculate the probability that if a vehicle was stolen in this city, it would be stolen by a syndicate and recovered within 24 hours. (2)

3.3 Calculate the probability that a vehicle stolen in this city will not be recovered. (3)

KZN Common Test September 2016

TOPIC: Probability Lesson 3	**Weighting**	15/150 in Paper 1	**Grade**	11
RESOURCES				
Grade 11 textbooks. Past question papers.				

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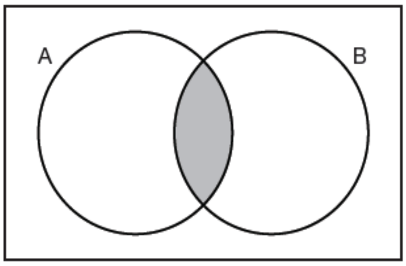
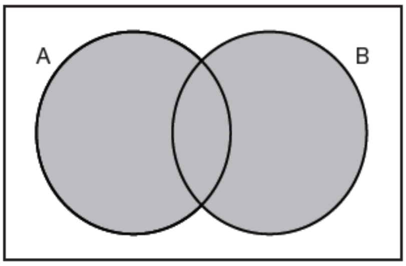
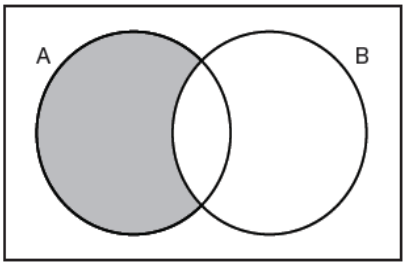
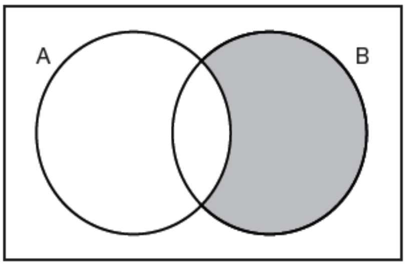
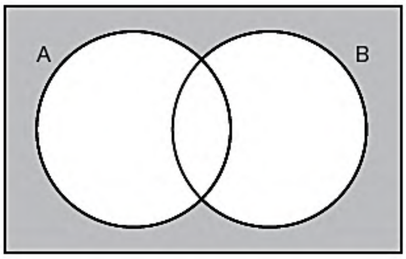
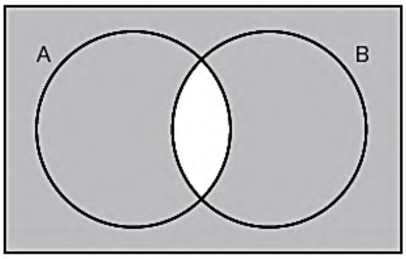
NOTES

- Use the table below to summarise and revise Venn diagram concepts. Shade in the appropriate areas.

Intersection (A and B)	Union (A or B)
	
A only	**B only**
	
not (A or B)	**not (A and B)**
	

3. Use the table to discuss the symbols used for some of these terms. Add each symbol as you discuss it and tell learners to do the same.

intersection	union
$A \cap B$	$A \cup B$
A and B	A or B

not A	not B	not (A or B)
A'	B'	$(A \text{ or } B)'$

Make sure that you understand the definitions of union and intersection:
Intersection of two sets: All elements belonging to both of the sets.
Union of two sets: All the elements that are in either one of the two sets.

- The identity linked to Venn diagrams is: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

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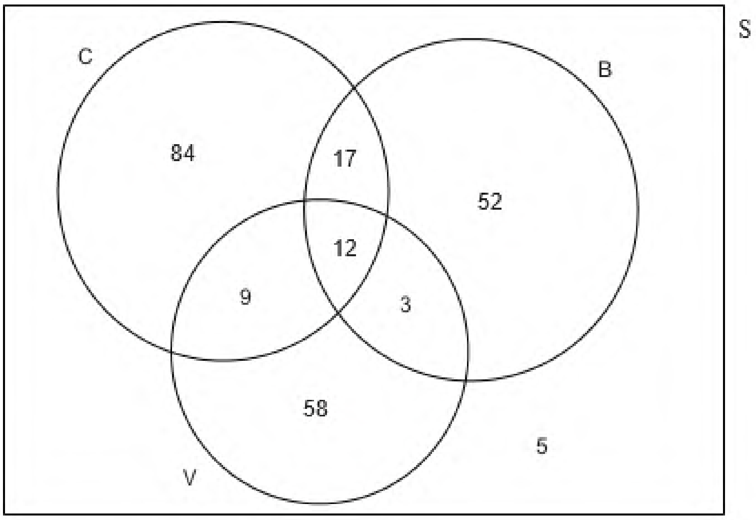
EXAMPLE:

DBE November 2014:



240 customers were surveyed at a fast food outlet. The diagram shows the number of customers who bought cheese burgers (C), bacon (B) and vegetarian burgers (V).

How many customers:

- Didn't buy any of the three burgers mentioned? (5)
- Bought all three types of burgers mentioned? (12)
- Bought ONLY cheese burgers? (84)
- Bought cheese burgers and bacon burgers? (**29** = 17 +12)
- Bought vegetarian burgers and bacon burgers? (**15** = 12 + 3)
- Bought vegetarian burgers? (82 = 58 + 9 + 12 + 3)

ACTIVITIES /ASSESSMENT

Activity 4.3.1:

1. A survey was conducted amongst 100 learners at a school to establish their involvement in three codes of sport, soccer, netball and volleyball. The results are shown below:
   - 55 learners play soccer (S)
   - 21 learners play netball (N)
   - 7 learners play volleyball (V)
   - 3 learners play netball only
   - 2 learners play soccer and volleyball
   - 1 learner plays all three sports.

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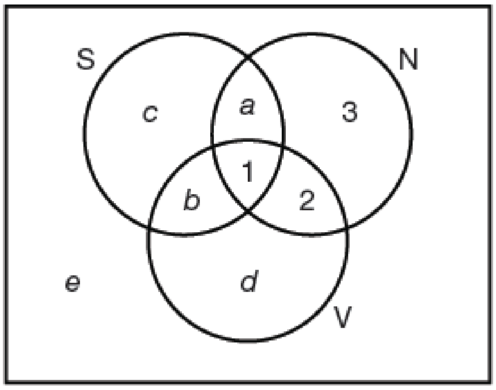
The Venn diagram below shows the information above.



1.1 Determine the values of *a*, *b*, *c*, *d* and *e*.

1.2 What is the probability that one of the learners chosen at random from this group plays netball or volleyball?

2. Question 9 from **DBE Gr. 11 November 2017:**

A survey was done among 80 learners on their favourite sport.
The results are shown below.

- 52 learners like rugby (R)
- 42 learners like volleyball (V)
- 5 learners like chess (C) only
- 14 learners like rugby and volleyball but not chess
- 12 learners like rugby and chess but not volleyball
- 15 learners like volleyball and chess but not rugby
- x learners like all 3 types of sport
- 3 learners did not like any sport

9.1 Draw a Venn diagram to represent the information above. (5)

9.2 Show that $x = 8$. (2)

9.3 How many learners like only rugby? (1)

9.4 Calculate the probability that a learner, chosen randomly, likes at least TWO different types of sport. (3)

[11]

3. William writes a Mathematics examination and an Accounting examination. He estimates that he has a 40% chance of passing the Mathematics examination. He estimates that he has a 60% chance of passing the Accounting examination. He estimates that he has a 30% chance of passing both.

Determine the probability that William will fail Mathematics and Accounting.

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TOPIC : PROBABILITY Lesson 4	**Weighting**	15/150 in Paper 1	**Grade**	11

RESOURCES

Grade 11 textbooks.
Past question papers.

NOTES

- Two events are **dependent** if the outcome of the first event affects (influences) the outcome of the second event. For example: the event of studying for an exam and the event of passing the exam are dependent, i.e. if you study for the exam, if will influence the probability of you passing the exam!

- Two events are **independent** if the outcome of the first event does not affect (influence) the outcome of the second event. For example, owning a pet and wearing a blue shirt. Whether you own a pet or not has no effect on the probability of you wearing a blue shirt.

- **Example 1:**

  First Scenario:
  A bag contains 4 red, 3 green and 2 blue balls.
  2 balls are drawn from the bag.
  Balls are drawn from the bag without looking – they are randomly drawn.
  <u>After the first ball is drawn, it is replaced</u>.
  The second ball is drawn.
  What is the probability of getting a blue ball, and then a green ball?
  $P(Blue, Green) = \frac{2}{9} \times \frac{3}{9} = \frac{6}{81} = \frac{2}{27}$
  Because the ball was replaced, the sample space remained the same size.

  Second Scenario:
  A bag contains 4 red, 3 green and 2 blue balls.
  2 balls are drawn from the bag.
  Balls are drawn from the bag without looking – they are randomly drawn.
  <u>After the first ball is drawn, it is not replaced</u>.
  The second ball is drawn.
  What is the probability of getting a blue ball, and then a green ball?
  $P(Blue, Green) = \frac{2}{9} \times \frac{3}{8} = \frac{6}{72} = \frac{1}{12}$
  Because the ball was not replaced, the sample space needed to be adjusted.

  Notice how the probability was calculated for the first example (the independent events). Probability was calculated by multiplying the probability of one event (drawing a blue ball) by the probability of the other event (drawing a green ball).

$$P\ (A \text{ and } B) = P\ (A) \times P\ (B)$$

This is always true if events are independent of each other.

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Example 2:

Given:

$P(A) = 0,2$

$P(B) = 0,5$

$P\ (A\ or\ B) = 0,\ 6$

Where A and B are two different events.

a) Calculate $P\ (A\ and\ B)$

b) Are the events A and B independent? Show your calculations.

Solution:

a) $P(A\,or\,B) = P(A) + P(B) - P(A\,and\,B)$

$0,6 = 0,2 + 0,5 - P(A\ and\ B)$

$-0,1 = -\ P(A\ and\ B)$

` $P(\ A\ and\ B) = 0,\ 1$

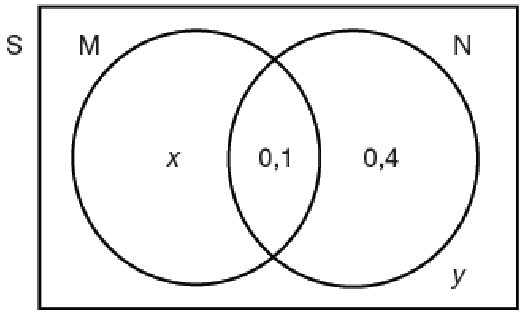
b) $P(A\ and\ B) = 0,1$

$P(A) \times P(B) = 0,\ 2 \times 0,\ 5 = 0,\ 1$

the events are independent

Example 3:

The Venn diagram below shows two independent events, M and N.



Determine the values of probabilities x and y. Show all calculations.

EC 2015

Solution:

$P(M\ and\ N) = P(M) \times P(N)$

$0,1 = P(M) \times 0,5$

$P(M) = 0,2$

$\therefore x = 0,1$

$0,1 + 0,1 + 0,4 + y = 1$

$\therefore y = 0,4$

ACTIVITIES /ASSESSMENT

Activity 4.4.1:

1. Question 8.2 from November 2017 DBE Gr. 11:

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8.2 A and B are two events. The probability that event A will occur is 0,4 and the probability that event B will occur is 0,3. The probability that either event A or event B will occur is 0,58.

8.2.1 Are events A and B mutually exclusive? Justify your answer with appropriate calculations. (3)

8.2.2 Are events A and B independent? Justify your answer with appropriate calculations. (3)

2. The events C and D are independent.

P(C) = 0,4 and P(D) = 0,5. Determine:

2.1 P(C and D) (2)

2.2 P(C or D) (2)

2.3 P(not C and not D) (3)

3. It is given that two events, E and F, are independent. P(E) = $\frac{2}{5}$ and P(F) = 0,35.

Calculate P(E or F). (4)

TOPIC : PROBABILITY Lesson 5	**Weighting**	15/150 in Paper 1	**Grade**	11	

RESOURCES

Grade 11 textbooks.
Past question papers.

NOTES

		Body Image			
		About Right	Overweight	Underweight	**Total**
Gender	Female	560	163	37	760
	Male	395	72	73	440
	Total	855	235	110	1200

- A contingency table shows the distribution of one variable in rows and another in columns. It is used to study the correlation between the two variables.

- Look at the table in more detail.

What was the survey about and who participated?
(The survey was about how people feel about their body image. Males and females were questioned, and their answers were listed separately).

- *How many females were surveyed?* (760)
- *How many people were surveyed altogether?* (1200)
- *How many people in total consider themselves underweight?* (110)
- *How many of those people are male?* (73)
- *How many males consider themselves overweight?* (72)

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- *How many people in total think their weight is just right?* (855).

- Contingency tables can be used when calculating probabilities. Use the same table to work through some probability questions:

There is no need to simplify the fractions. The important idea here is to know where the numbers are being read off for both the numerator and denominator.

Determine the probability that a person chosen at random from the participants in the survey is:

male overweight	$\frac{440}{1200}$ $\frac{235}{1200}$
happy with their weight	$\frac{855}{1200}$
female	$\frac{760}{1200}$
underweight	$\frac{110}{1200}$
a female who thinks she is overweight	$\frac{163}{1200}$
a male who thinks he is underweight	$\frac{73}{1200}$
a female who is happy with her weight	$\frac{560}{1200}$
a female **or** underweight	$\frac{833}{1200}$

ACTIVITIES /ASSESSMENT

Activity 4.5.1:

1. A survey was conducted amongst 60 boys and 60 girls in Grade 8 relating to their participation in sport. 20 girls did not participate in any sport and 50 boys did participate in a sport.

1.1 Complete a two-way contingency table for the above survey.

	Boys	Girls	
Sport	50		
No sport		20	
	60	60	120

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1.2 What is the probability that if a Grade 8 learner is chosen at random that:

1.2.1 it is a girl and participates in sport?

1.2.2 the learners does not participate in sport and is not female?

2. Question 9.1 from November 2018 DBE Gr. 11:

9.1 On a flight, passengers could choose between a vegetarian snack and a chicken snack. The snacks selected by the passengers were recorded. The results are shown in the table below.

SNACK	MALE	FEMALE	TOTAL
Vegetarian	12	20	32
Chicken	55	63	118
TOTAL	67	83	150

Was the choice of snack on this flight independent of gender? Motivate your answer with the necessary calculations. (5)

TOPIC Gr. 12 Probability Lesson 6	**Weighting**	15/150 in Paper 1	**Grade**	12

RESOURCES

Grade 12 Mathematics Textbooks, e.g.: Mind Action Series, Maths Handbook and Study Guide, Platinum Mathematics.

NOTES

- The Fundamental Counting Principle is a rule used to count the total number of possible outcomes in a situation.
  It states that: If there are n ways of doing something and m ways of doing another thing, then there are $n \times m$ ways to perform both actions.
  In the above, one is choosing one of the 3 items from A and one of the 4 items from B.
  The number of ways of choosing from **A and B** is $3 \times 4 = 12$ ways.

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Example 3 from p. 309 of Gr. 12 Mind action Series.

Consider the word PARKTOWN. You are required to form different eight-letter word arrangements using the letters of the word PARKTOWN. An example of a word arrangement would be the word APKROTWN. This arrangement of the letters need not make any sense. How many possible word arrangements can be made if:

(a) the letters may be repeated?

(b) the letters may not be repeated?

Solution:

(a) An example of an eight-letter word arrangement where the letters may be repeated is AAPKWANO. This means that for the first letter of the word arrangement, any of the eight letters in the word PARKTOWN can be used. For the second letter, any of the eight letters may be used again. In the word arrangement AAPKWANO, the letter A is used as the first letter and then again as the second letter as well as the sixth letter.
In forming a word arrangement, there are 8 possible letters that can be used for the first letter. For the second letter, we can still use 8 letters (repeating the letters is allowed). For the third letter there are still 8 possible letters available to use. From the fundamental counting principle, there are:
$8\times8\times8\times8\times8\times8\times8\times8=8^8=16\ 777\ 216$ word arrangements.

(b) An example of an eight-letter word arrangement where the letters may not be repeated is NOTWARPK. This means that for the first letter of the word arrangement, any of the eight letters in the word PARKTOWN can be used. However, for the second letter, only seven letters may be used since the first letter may not be used again. In the word arrangement NOTWARPK, the letter N is used as the first letter but not again as the second letter.
In forming a word arrangement, there are 8 possible letters that can be used for the first letter. For the second letter, we can use 7 letters (repeating the letters is not allowed). For the third letter, there will be 6 possible letters available to use. For the eighth letter, there will only be one choice. This will be the last remaining letter that was not used. From the fundamental counting principle, there are:
$8\times7\times6\times5\times4\times3\times2\times1=40\ 320$ word arrangements.

- Example 3 from p. 302 of Maths Handbook and Study Guide:

<u>Example 3</u>: A security system requires a user to choose a security code. The code consists of 4 digits. How many different codes can be chosen if:

a) The system allows repetition of digits.

b) The system does not allow repetition of digits.

a) There are 10 possible choices for each digit in the code, the numbers 0 through to 9. As repetition is allowed there are **10 choices for the 1st digit, 10 choices for the 2nd digit, 10 choices for the 3rd digit** and **10 choices for the 4th digit**.

Code: <u>10 choices</u> <u>10 choices</u> <u>10 choices</u> <u>10 choices</u>

$\therefore$ Number of different codes $=10\times10\times10\times10=10^4$

b) If repetition is not permitted, there are **10 choices for the 1st digit, 9 choices for the 2nd digit, 8 choices for the 3rd digit and 7 choices for the 4th digit**.

Code: <u>10 choices</u> <u>9 choices</u> <u>8 choices</u> <u>7 choices</u>

$\therefore$ Number of different codes $=10\times9\times8\times7=5040$

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ACTIVITIES /ASSESSMENT

Activity 4.6.1:

Exercise 1 from Gr. 12 Mind Action Series p. 310:

1. A party pack of three items can be made up by selecting one item from each of the following choices:
   Choice 1: Smarties, Astros, Jelly Tots, Wine Gums
   Choice 2: Coke, Fanta, Sprite, Ginger Beer, Crème Soda
   Choice 3: Doughnut, Chelsea Bun, Cheese Roll
   How many different party packs can be made?
2. Consider the word FLORIDA. You are required to form different seven-letter word arrangements using the letters of the word FLORIDA.
   How many possible word arrangements can be made if:
   (a) the letters may be repeated?
   (b) the letters may not be repeated?
3. Consider the word RANDOM. You are required to form different six-letter word arrangements using the letters of the word RANDOM.
   How many possible word arrangements can be made if:
   (a) the letters may be repeated?
   (b) the letters may not be repeated?
4. How many different ways are there of predicting the results of six soccer matches where each match can end in either a win or a lose?
5. A password is to be made up using the format XXXYY where X represents any digit from 0 to 9 and Y represents any letter of the alphabet. How many different passwords can be formed in each of the following cases?
   (a) The digits may be repeated as well as the letters of the alphabet.
   (b) The digits may not be repeated including the letters of the alphabet.
   (c) The digits and letters may be repeated but the number 0 and the vowels must be excluded.
   (d) The digits and letters may not be repeated and the number 0 and the vowels must be excluded.
   (e) The digits may be repeated but must be prime numbers and the letters may be repeated excluding the first five letters and the last five letters.
6. There are four bus lines between town A and town B and three bus lines between town B and town C.
   (a) In how many ways can a person travel by bus from town A to town C by way of town B?
   (b) In how many ways can a person travel return trip by bus from town A to town C by way of B?
   (c) In how many ways can a person travel return trip by bus from town A to town C by way of town B, if this person doesn't want to use a bus line more than once?

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TOPIC: Probability Gr. 12 Lesson 7	**Weighting**	15/150 in Paper 1	**Grade**	12

RESOURCES

Grade 12 Mathematics Textbooks, e.g.: Mind Action Series, Maths Handbook and Study Guide, Answer Series, Platinum Mathematics.

NOTES

Example from Gr. 12 Mind Action Series:

Example 5:

Consider the word PARKTOWN. You are required to form different eight-letter word arrangements using the letters of the word PARKTOWN. How many possible word arrangements can be made if the letters may not be repeated?

Solution

From Example 4, the solution to this question is $8\times7\times6\times5\times4\times3\times2\times1 = 40\ 320$ word arrangements.

We can write this in factorial notation:

$8\times7\times6\times5\times4\times3\times2\times1 = 8!$

The calculator keys you need to press, are: **8 SHIFT x! =**

- **Example 8 from p. 312 of Gr. 12 Mind Action Series:**

  There are 12 different singers that are hoping to occupy the first three places in SA Idols. In how many different ways can the first three places be occupied?

  Solution

  In this example there are 12 people to be arranged in 3 different ways.

  The number of possible arrangements will be:

  $12\times11\times10 = 1320$

ACTIVITIES/ASSESSMENT

Activity 4.7.1:

Exercise 2 Mind Action Series p 312 – 313

1. (a) In how many ways can 8 vacant places be filled by 8 different people?
   (b) In how many ways can 5 vacant places be filled by 15 different people?
2. Find the number of ways that a judge can award first, second and third places in a competition with ten contestants.
4. Consider the word ORANGES.
   (a) How many seven-letter word arrangements can be made if the letters may be repeated?
   (b) How many seven-letter word arrangements can be made if the letters may not be repeated?
   (c) How many four-letter word arrangements can be made if the letters may be repeated?
   (d) How many four-letter word arrangements can be made if the letters may not be repeated?

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6. The digits 0, 1, 2, 3, 4, 5, 6, 7 and 8 are used to make 4 digit codes.
   (a) How many different codes are possible if the digits may be repeated?
   (b) How many different codes are possible if the digits may not be repeated?
   (c) How many codes are numbers that are greater than or equal to 4 000 and are exactly divisible by 2? The digits may be repeated.
   (d) How many codes are numbers that are greater than 4 000 and are exactly divisible by 2?

TOPIC: Probability Lesson 8	**Weighting**	15/150 in Paper 1	**Grade**	12

RESOURCES

Grade 12 Mathematics Textbooks, e.g.: Mind Action Series, Maths Handbook and Study Guide, Answer Series, Platinum Mathematics.

NOTES

- An OBJECT is a material thing that can be seen and touched. Unlike numbers and letters, objects CANNOT be repeated.
- The following examples show how objects are arranged in a row:
  Maths Handbook and Study Guide, p 303 – 304, Examples 1 – 4(b).

Example 1: **A Grade 12 learner has an Accounting, Science, Zulu and Mathematics textbook. How many different ways can they be arranged on a bookshelf?**

4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4

Number of ways the books can be arranged $= 4! = 4 \times 3 \times 2 \times 1 = 24$

Example 2: **In how many different ways can the letters of the word MARKS be arranged (repetition of letters is not permitted)?**

5 choices	4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4	space 5

Number of ways the letters can be arranged $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

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Example 3: **2 different History books, 3 different Geography books and 2 different Science books are placed on a book shelf.**

a) How many different ways can they be arranged?

b) How many ways can they be arranged if books of the same subject must be placed together?

a) Number of different arrangements $= 7! = 5040$

7 choices	6 choices	5 choices	4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4	space 5	space 6	space 7

b) In order to solve this problem we will treat each subject as one space:

History

2 choices	1 choice
space 1	space 2

Geography

3 choices	2 choices	1 choice
space 1	space 2	space 3

Science

2 choices	1 choice
space 1	space 2

This means that there are $2!$ ways the history books can be arranged **and** $3!$ ways that the geography books can be arranged **and** $2!$ ways the science books can be arranged. You also have to take into account that the subjects can be arranged in any order. This means there are $3!$ ways the subjects can be arranged.

$\therefore$ Number of ways the books can be arranged $= 2! \times 3! \times 2! \times 3! = 144$

Example 4: **In a school there are 8 prefects, 5 girls and 3 boys. They are to be seated on the stage for an assembly.**

a) How many different ways can they be seated on stage?

b) How many ways can they be seated if all the boys and girls are to sit next to each other.

c) If the prefects are randomly seated what is the probability of all the boys and girls being seated next to each other? (In other words all the boys together and all the girls together.)

a) Number of different seating arrangments $= 8! = 40320$

8 choices	7 choices	6 choices	5 choices	4 choices	3 choices	2 choices	1 choice
seat 1	seat 2	seat 3	seat 4	seat 5	seat 6	seat 7	seat 8

b) In order to solve this problem we will treat the boys as one seat and the girls as another seat

Boys

3 choices	2 choices	1 choice
boy 1	boy 2	boy 3

Girls

5 choices	4 choices	3 choices	2 choices	1 choice
girl 1	girl 2	girl 3	girl 4	girl 5

This means that there are $3!$ Ways that the boys can be arranged **and** $5!$ ways that the girls can be arranged. You also have to take into account that the boys and girls can also swap places. In other words the girls could be seated on the left and the boys could be seated on the right. This means there are $2!$ ways the groups of boys and girls can be arranged

$\therefore$ Number of ways boys and girls can be seated together $= 3! \times 5! \times 2! = 1440$

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ACTIVITIES/ASSESSMENT

Activity 4.8.1:

Exercise 3 Mind Action Series p 315 - 316

3. (a) In how many ways can five boys and four girls sit in a row?
   (b) In how many ways can they sit in a row if a boy and his girlfriend must sit together?
   (c) In how many ways can they sit in a row if the boys and girls are each to sit together?
   (d) In how many ways can they sit in a row if just the girls are to sit together?
   (e) In how many ways can they sit in a row if just the boys are to sit together?
   (f) In how many ways can they sit in a row if the boys and girls are to alternate?
4. Four History books and three Geography books must be placed on a shelf.
   (a) In how many different ways can you arrange the books on the shelf?
   (b) If all the History books must be placed next to each other and all the Geography books must be placed next to each other, in how many ways can you arrange the books on the shelf?
   (c) If just the History books are to be together, in how many ways can you arrange the books on the shelf?
5. In how many ways can four Mathematics books, three History books, three Science books and two Biology books:
   (a) be arranged on a shelf?
   (b) be arranged on a shelf so that all books of the same subject are together?
7. In how many ways can three South Africans, four Americans, four Italians and two British citizens be arranged so that those of the same nationality sit together if they sit in a row?

TOPIC: Probability Lesson 9	**Weighting**	15/150 in Paper 1	**Grade**	12

RESOURCES

Grade 12 Mathematics Textbooks, e.g.: Mind Action Series, Maths Handbook and Study Guide, Answer Series, Platinum Mathematics.

NOTES

- The following equation shows how word arrangements with identical letters is actually done.

  $$Number\ of\ arrangements = \frac{Factorial\ of\ the\ total\ number\ of\ letters}{Product\ of\ the\ factorials\ of\ the\ number\ of\ each\ of\ the\ identical\ letters}$$

  In the word SUCCESS, the number of possible arrangements = $\frac{7!}{3!\times 2!}$ = 420
- Do example 12 from Gr. 12 Mind Action Series p. 318:

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Consider the letters of the word NEEDED.

(a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?

(b) How many word arrangements can be made with this word if the repeated letters are treated as identical?

(c) How many word arrangements can be made with this word if the word starts and ends with the same letter?

Solutions

(a) There are 6 letters in the word NEEDED.
The total possible word arrangements (repeated letters are treated as different) is:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \qquad \text{(Rule 2)}$$

(b) The total possible word arrangements (repeated letters are treated as identical) is:

$$\frac{6!}{3! \times 2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60 \qquad \text{(Rule 4)}$$

The numerator represents the 6 letters in the word NEEDED.
The denominator represents the three E's and the two D's.

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(c) The only possibilities with the word NEEDED if you start and end with the same letter are:

Option 1

D					D

Option 2

E					E

With the first option the letters in between the two D's will be NEEE. The possible word arrangements will then be:

$$\frac{4!}{3!}=\frac{4\times3\times2\times1}{3\times2\times1}=4 \qquad \text{(Rule 4)}$$

The numerator represents the 4 letters in the word NEEE (ignore the two D's). The denominator represents the three E's.

With the second option E the letters in between the two E's will be NDED.

The possible word arrangements will then be:

$$\frac{4!}{2!}=\frac{4\times3\times2\times1}{2\times1}=12 \qquad \text{(Rule 4)}$$

The numerator represents the 4 letters in the word NDED (ignore the two E's). The denominator represents the two D's.

Therefore the total number of possible word arrangements that can be made if the word starts and ends with the same letter will be:

$$\frac{4!}{3!}+\frac{4!}{2!}=4+12=16$$

ACTIVITIES /ASSESSMENT

Activity 4.9.1:
Exercise 4 Mind Action Series p 319 – 320

1. Consider the word WINNERS.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) How many word arrangements can be made with this word if the word starts with W and ends with the S?

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3. Consider the word TECHNOLOGY.
   - (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   - (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   - (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   - (d) How many word arrangements can be made with this word if the word starts with the letter O?
   - (e) How many word arrangements can be made with this word if the word ends with the letter N?

4. Three Mathematics books and five Science books are to be arranged on a shelf.
   - (a) In how many ways can these books be arranged if they are treated as separate books?
   - (b) In how many ways can these books be arranged if they are treated as identical books?

5. There are six pool balls on a pool table. Some are red and some are blue. The red balls are identical to each other as well as the blue balls. The balls are removed from the table, one by one. How many different results can happen if there are:
   - (a) five red balls?
   - (b) four blue balls?
   - (c) three of each colour?

TOPIC: Probability Gr. 12 Lesson 10	**Weighting**	15/150 in Paper 1	**Grade**	12

RESOURCES

Grade 12 Mathematics Textbooks, e.g.: Mind Action Series, Maths Handbook and Study Guide, Answer Series, Platinum Mathematics.

NOTES

- Determining the probability of an event happening, as learnt in grade 10 is given by.

  $$P(E)=\frac{n(E)}{n(S)}$$

  P(E) refers to the probability of an event happening
  n(E) refers to the number of ways the event can take place
  n(S) refers to the sample space

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Example 14:

Consider the letters of the word DREAMS. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will start with D and end with S?

Solution

The number of ways that the six letters can be arranged is 6!

Let event E be defined as the event that the word formed will start with D and end with S.

The number of arrangement for event E is $\frac{4!}{0!} = 4!$

Therefore the probability of event E happening is:

$$\frac{4!}{6!} = \frac{4!}{6 \times 5 \times 4!} = \frac{1}{30}$$

Example 15:

A combination to a lock is formed using three letters of the alphabet, excluding the letters O, Q, S, U, V and W and using any three digits. The numbers and letters can be repeated. Calculate the probability that a combination, chosen at random:

(a) starts with the letter X and ends with the number 6.

(b) has exactly one X.

(c) has one or more number 6 in it.

Solutions

(a) Let A be the event that a number plate starts with the letter X and ends with the number 6.
Since 20 letters and 10 digits can be used, the number of plates possible will be:
$20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8\ 000\ 000$

For event A, the number of possibilities is reduced to:

$1 \times 20 \times 20 \times 10 \times 10 \times 1 = 40\ 000$

Therefore, the probability of event A happening is: $\frac{40\ 000}{8\ 000\ 000} = \frac{1}{200}$

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(b) Let B be the event of choosing exactly one X.

The number of possible ways of event B happening is:

$$(1\times19\times19\times10\times10\times10)+(19\times1\times19\times10\times10\times10)+(19\times19\times1\times10\times10\times10)$$
$$=1\ 083\ 000$$

Therefore, the probability of event B happening is: $\dfrac{1\ 083\ 000}{8\ 000\ 000}=\dfrac{1\ 083}{8\ 000}$

(c) Let C be the event of at least one 6 being chosen.

Method 1

Total number of possible combinations:
$20\times20\times20\times10\times10\times10=8\ 000\ 000$

Number of combinations without 6:
$20\times20\times20\times9\times9\times9=5\ 832\ 000$

Number of combinations with at least one 6:
$$20\times20\times20\times10\times10\times10-20\times20\times20\times9\times9\times9$$
$$=8\ 000\ 000-5\ 832\ 000$$
$$=2\ 168\ 000$$

The probability of at least one 6 is:

$$\frac{2\ 168\ 000}{8\ 000\ 000}=\frac{217}{1\ 000}$$

Method 2

The probability of event C happening can determined by using the fact that $P(C)=1-P(\text{not }C)$.

$$\therefore P(C)=1-\frac{20\times20\times20\times9\times9\times9}{8\ 000\ 000}$$
$$\therefore P(C)=1-\frac{729}{1\ 000}$$
$$\therefore P(C)=\frac{271}{1\ 000}$$

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Example 16:

Consider the letters of the word NEEDED. What is the probability that the word arrangement formed will start and end with the same letter? The repeated letters are identical.

Solution

The only possibilities with the word NEEDED if you start and end with the same letter are:

Option 1

D					D

Option 2

E					E

With the first option the letters in between the two D's will be NEEE. The possible word arrangements will then be:

$$\frac{4!}{3!} = \frac{4\times3\times2\times1}{3\times2\times1} = 4 \qquad \text{(Rule 4)}$$

The numerator represents the 4 letters in the word NEEE (ignore the two D's).
The denominator represents the three E's.

With the second option E the letters in between the two E's will be NDED.

The possible word arrangements will then be:

$$\frac{4!}{2!} = \frac{4\times3\times2\times1}{2\times1} = 12$$

Therefore the total number of possible word arrangements that can be made if the word starts and ends with the same letter will be:

$$\frac{4!}{3!} + \frac{4!}{2!} = 4 + 12 = 16$$

However, the sample space in this example (the total possible word arrangements where repeated letters are treated as identical) is:

$$\frac{6!}{3!\times2!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 60$$

Therefore the probability that the word arrangement formed will start and end with the same letter is:

$$\frac{\frac{4!}{3!} + \frac{4!}{2!}}{\frac{6!}{3!\times2!}} = \frac{16}{60} = \frac{4}{15}$$

FOR CONVENIENCE: Use the short form to describe any given event happening, e.g.
What is the probability that a word arrangement will start with letter S, if the word LOVERS is given?
SOLUTION: Let the event of the word starting with S be A.

$n(A) = 5! = 120$

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$n(S) = 6! = 720$

$$P(A) = \frac{120}{720} = \frac{1}{6}$$

ACTIVITIES /ASSESSMENT

Activity 4.10.1:

Exercise 5 Mind Action Series p. 324:

1. Consider the letters of the word KNIGHT. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with K and end with T?
   (b) start with the letter N?

3. A password is formed using three letters of the alphabet, excluding the letters A, E, I, O and U and using any three digits, excluding 0. The numbers and letters can be repeated. Calculate the probability that a password, chosen at random:
   (a) starts with the letter B and ends with the number 4.
   (b) has exactly one B.
   (c) has at least one 4.

4. Determine the probability of getting a ten digit cell-phone number if the first digit is even, none of the first three must be 0 and none of the digits may be repeated.

5. Consider the letters of the word WINNERS. The repeated letters are identical. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with W and end with S?
   (b) start with the letter N?

8. Four Mathematics books, three History books, three Science books and two Biology books are arranged randomly on a shelf. What is the probability that:
   (a) all books of the same subject land up next to each other?
   (b) just the History books will be together?

9. Seven boys and six girls are to be seated randomly in a row. What is the probability that:
   (a) the row has a boy at each end?
   (b) the row has boys and girls sitting in alternate positions?
   (c) two particular girls land up sitting next to each other?
   (d) all the girls sit next to each other?

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5. TRIGONOMETRY

5.1 REDUCTION FORMULA

- Trigonometric ratios of any angles are reduced to an acute angle using the following reduction formulae $(180° \pm; 360° \pm \text{and } 90° \pm)$
- For function values $(90° \pm\ )$, each function change to their co- functions.
- The **quadrant** should be identified and then the **sign** of the trigonometric ratio.
- Two fundamental identities to be used: Quotient Identity:
  - ✓ $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
  - ✓ Square identity: $\sin^2\theta + \cos^2\theta = 1$
- Special angles should be identified and considered.

Reduction formulae for $(180° \pm$ **and** $360° \pm)$

S

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

$180° -$

A

$\sin\theta = \sin\theta$
$\cos\theta = \cos\theta$
$\tan\theta = \tan\theta$

θ

$360° + \theta$

$180° +$

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

T

$360° -$

$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

C

Reduction formulae for $(90° -$ **and** $90° +)$

S

$\sin(90° + \theta) = \cos\theta$
$\cos(90° + \theta) = -\sin\theta$

$90° +$

A

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$

$90° -$

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Examples.

Simplify the following:

1. $$\frac{\sin(180^\circ + x).\cos(180^\circ - x)}{\cos(360^\circ - x).\sin(180^\circ - x)}$$

2. $$\frac{\sin^2(360^\circ - \theta)}{\sin(360^\circ + \theta).\sin(180^\circ - \theta)}$$

3. $$\frac{\sin(180^\circ - \alpha).\cos(90^\circ - \alpha) + \tan(180^\circ + \alpha).\sin(90^\circ + \alpha)}{\cos(360^\circ - \alpha)}$$

Solutions

1. $$\frac{\sin(180^\circ + x).\cos(180^\circ - x)}{\cos(360^\circ - x).\sin(180^\circ - x)}$$
$$= \frac{(-\sin x).(-\cos x)}{\cos x.\sin x}$$
$$= 1$$

2. $$\frac{\sin^2(360^\circ - \theta)}{\sin(360^\circ + \theta).\sin(180^\circ - \theta)}$$
$$= \frac{(-\sin\theta)^2}{\sin\theta(\sin\theta)}$$
$$= 1$$

3. $$\frac{\sin(180^\circ - \alpha).\cos(90^\circ - \alpha) + \tan(180^\circ + \alpha).\sin(90^\circ + \alpha)}{\cos(360^\circ - \alpha)}$$
$$= \frac{\sin\alpha.\sin\alpha + \tan\alpha.\cos\alpha}{\cos\alpha}$$
$$= \frac{\sin^2\alpha + \frac{\sin\alpha}{\cos\alpha}.\cos\alpha}{\cos\alpha}$$
$$= \frac{\sin^2\alpha + \sin\alpha}{\cos\alpha}$$
$$= \frac{\sin\alpha(\sin\alpha + 1)}{\cos\alpha}$$
$$= \tan\alpha(\sin\alpha + 1)$$

Activity 5.1

5.1.1 $$\frac{\tan(180^\circ + \theta).\sin(360^\circ - \theta)}{\tan(360^\circ - \theta)}$$

5.2.2 $$\frac{\tan(180^\circ + x).\cos(360^\circ + x)}{\sin(180^\circ + x).\cos(90^\circ + x) + \cos^2(360^\circ - x)}$$

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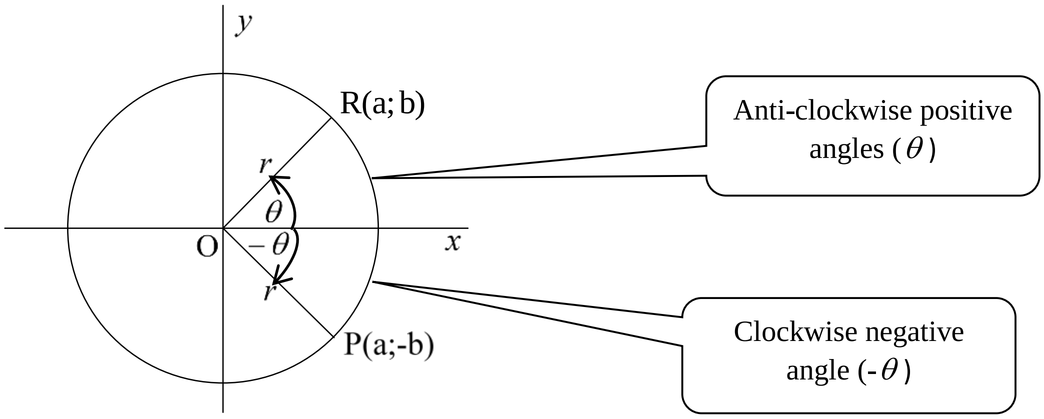
5.2.3 $\dfrac{\sin(90^0 - x).\tan(180^0 - x)}{\cos(-x).\sin(180 + x)}$

5.2 NEGATIVE ANGLES



- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$

NB: Remember to change negative angles (-θ) to positive angles (θ)

Examples

Simplify the following:

1. $\cos(\alpha - 90°)$
2. $\tan(\theta - 180°)$
3. $\sin(-\beta - 360°)$
4. $\dfrac{\cos(x - 360°).\sin(360° - x)\cos(90° - x)}{\sin^2 .\sin(-x)}$

Solutions

1. $\cos(\alpha - 90°) = \cos(-90° + \alpha)$
$= \cos[-(90° - \alpha)]$
$= \cos(90° - \alpha) = \sin\alpha$

2. $\tan(\theta - 180°) = \tan(-180° + \theta)$
$= \tan[-(180° - \theta)]$
$= -\tan(180° - \theta)$
$= -(-\tan\theta)$
$= \tan\theta$

OR $\tan(\theta - 180°) = \tan(\theta - 180° + 360°)$
$= \tan(180° + \theta)$
$= \tan\theta$

3. $\sin(-\beta - 360°) = \sin(-360° - \beta)$ OR $\sin(-\beta - 360°) = \sin(-\beta - 360° + 360°)$

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$$= \sin[-(360° + \beta)] \qquad\qquad = \sin(-\beta)$$
$$= -\sin(360° + \beta) \qquad\qquad = -\sin\beta$$
$$= -\sin\beta$$

4. $$\frac{\cos(x-360°)\sin(360^0 - x)\cos(90^0 - x)}{\sin^2(90^0 - x)\sin(-x)} = \frac{\cos x.(-\sin x).\sin x}{\cos^2 x.(-\sin x)}$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$

Activity 5.2

5.2.1 $\dfrac{\cos(90° - \theta)}{\sin(180° - \theta)} - \sin(-\theta)$

5.2.2 $\dfrac{\cos(-A).\tan(180° - A).\sin(180° - A)}{\sin(360° - A)\cos(90° - A)}$

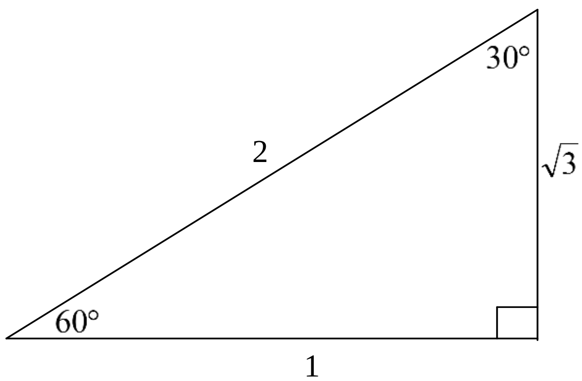
5.2.3 $\cos^2(180° + x) + \tan(x - 180°).\sin(720° - x).\sin(90 - x)$

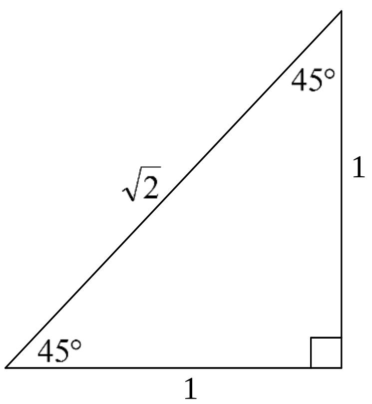
5.2.4 $\dfrac{\cos(x - 180°).\cos(90° - x)}{\sin(90° + x).\sin(-x - 180)}$

5.2.5 $\dfrac{\cos(\alpha - 90°).\tan(-\alpha)}{\sin(-\alpha).\tan(720° - \alpha)}$

5.2.6 $\dfrac{\sin(\beta - 180°).\tan(-\beta - 180°).\cos(180° + \beta)}{\cos(-\beta).\sin(360° + \beta)}$

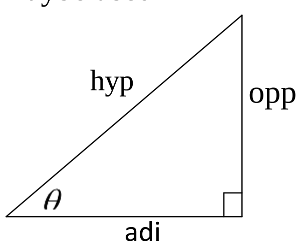
5.3 SPECIAL ANGLES:





- For angles $0°; 90°; 180°; 270°; 360°$ trigonometric functions maybe used
- In a right angle triangle,

a) $\sin\theta = \dfrac{opp}{hyp}$ b) $\cos\theta = \dfrac{adj}{hyp}$ c) $\tan\theta = \dfrac{opp}{adj}$



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- **NB:** Whenever the angle is greater than $360°$, keep subtracting $360°$ from the angle until you get angle in the interval $[0°;360°]$.

Example: **(a)** $\cos 510° = \cos 150°$ ------------ $\boxed{510° - 360° = 150°}$

(b) $\tan 1\,290° = \tan 210°$ --------- $\boxed{1\,290° - 360° - 360° - 360° = 210°}$

Examples

Evaluate without the use of a calculator:

1. a) $\cos 60°$ b) $\tan 30°$ c) $\sin 45°$
2. $\sin 150°$
3. $\cos(-300°)$
4. $\dfrac{\sin(-740°)}{\cos 70°}$
5. $\dfrac{\cos 180^0.\sin(-225^0).\cos 80^0}{\sin 170^0.\tan(-135^0)} = \dfrac{(-1).(-\sin 225°)\cos 80°}{\sin(90° + 80°).(-\tan 135°)}$

Solutions

1. a) $\cos 60° = \dfrac{1}{2}$ b) $\tan 30° = \dfrac{1}{\sqrt{3}}$ c) $\sin 45° = \dfrac{1}{\sqrt{2}}$

2.
$$\begin{aligned} \sin 150° &= \sin(180° - 30°) \\ &= \sin 30° \\ &= \frac{1}{2} \end{aligned}$$

3.
$$\begin{aligned} \cos(-300°) &= \cos 300° \\ &= \cos(360° - 60°) \\ &= \cos 60° \\ &= \frac{1}{2} \end{aligned}$$

4.
$$\begin{aligned} \frac{\sin(-740°)}{\cos 70°} &= \frac{-\sin 20°}{\cos 70°} \\ &= \frac{-\sin(90° - 70°)}{\cos 70°} \\ &= \frac{-\cos 70°}{\cos 70°} = -1 \end{aligned}$$

$\boxed{740° - 360° - 360° = 20°}$

5.
$$\begin{aligned} \frac{\cos 180^0.\sin(-225^0).\cos 80^0}{\sin 170^0.\tan(-135^0)} &= \frac{(-1).(-\sin 225°)\cos 80°}{\sin(90° + 80°).(-\tan 135°)} \\ &= \frac{(-1)[-\sin(180° + 45°)].\cos 80°}{\cos 80°[-\tan(180° - 45°)]} \end{aligned}$$

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$$= \frac{(-1)[-(-\sin 45°)]}{-(\tan 45°)}$$

$$= \frac{\sqrt{2}}{2}$$

ACTIVITY 5.3

Evaluate the following expressions without the use of a calculator:

5.3.1 $\dfrac{\tan(-60°).\cos(156°).\cos 294°}{\sin 492°}$

5.3.2 $\sqrt{4^{\sin 150°} \times 2^{3\tan 225°}}$

5.3.3 $\dfrac{\cos 180°.\sin 225°.\tan 495°}{\sin 170°.\tan 225°}$

5.3.4 $\tan 315° - 2\cos(-300°) + \sin 210°$

5.3.5 $\dfrac{\sin 190° \cos 225° \tan 390°}{\cos 100°.\sin 135°}$

5.3.6 $\dfrac{\sin 150^{o}.\tan 225^{o}}{\sin\left(-30^{o}\right).\sin 420^{o}}$

5.4 USE OF DIAGRAMS TO DETERMINE THE NUMERICAL VALUES OF RATIOS FOR ANGLES FROM 0° TO 360°.

Pre-knowledge

- Definition of trig ratios
- Interpretation of interval notation
- Knowing in which quadrant a trig ratio is positive or negative (CAST rule- moving anticlockwise from the fourth quadrant)
- Application of Pythagoras theorem
- Reduction formulae (Grade 11)
- Compound and double angles (Grade 12)

Approach to determine the numerical values of ratios for angles from 0° to 360°.

1. Write the given equation in a form of a simple trig ratio, for example,

   $If\ \ b\cos\theta - a = 0\ then\ \ \cos\theta = \dfrac{a}{b};$

   $If\ \ b\sin\theta - a = 0\ then\ \sin\theta = \dfrac{a}{b}\ or$

   $If\ \ b\tan\theta - a\ then\ \ \tan\theta = \dfrac{a}{b}$

2. Draw the sketch in the correct quadrant, using the given interval/restriction and where the trig ratio holds, for example

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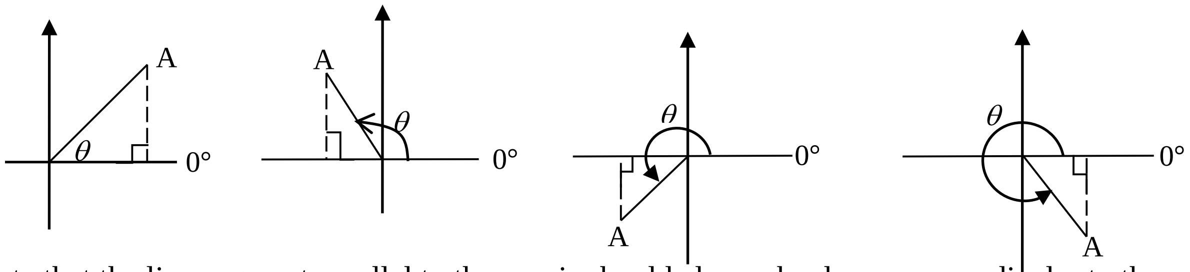
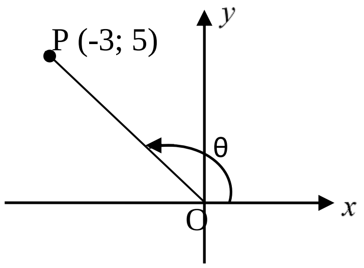


(Note that the line segment parallel to the y-axis should always be drawn perpendicular to the *x*-axis)

3. Fill in the known details in the diagram drawn in the correct quadrant.
4. Use Pythagoras theorem to calculate the value of the unknown side. Decide whether the value is positive or negative. Remember the radius is always positive
5. Use the diagram to answer the questions asked based on the diagram
6. Do not use a calculator.

Example 1

Use the figure to answer the following questions



1.1 Determine the length of OP.

1.2 Determine the value of

1.2.1 cos θ

1.2.2 sin θ

1.2.3 tan θ

Answers

1.1 $OP^2 = (-3)^2 + 5^2$

$\therefore OP = \sqrt{34}$

1.2 1.2.1 $\cos\theta = -\dfrac{3}{\sqrt{34}}$

1.2.2 $\sin\theta = \dfrac{5}{\sqrt{34}}$

1.2.3 $\tan\theta = -\dfrac{5}{3}$

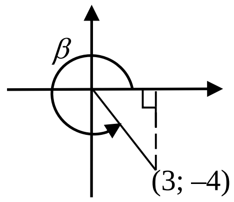
Example 2

If $4\tan\beta + 3 = 0$ and $\beta \in [180^\circ; 360]$. Calculate without the use of a calculator and with the aid of a diagram the value of: $2\sin\beta\cos\beta$

Answer

1. $\tan\beta = -\dfrac{3}{4}$

2. Draw the diagram for $\beta \in [180^\circ; 360]$, where tan β is negative.



3. Fill in the known details in the diagram(x = 3 and y = –4)

4 Use Pythagoras theorem to calculate the value of *r*.

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$r^2 = (3)^2 + (-4)^2$

$r = 5$

5 $\quad 2\sin\beta\cos\beta = 2 \times \frac{-4}{5} \times \frac{-4}{3}$

$= \frac{32}{15}$

If $\sin 12° = k$, in term of k, determine

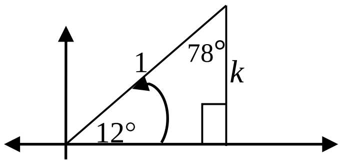
1. $\cos 12°$
2. $\tan 78°$
3. $\sin 192°$
4. $\sin 24°$

ANSWERS

1. $x^2 + k^2 = 1^2$

$x = \sqrt{1-k^2}$

$\cos 12° = \sqrt{1-k^2}$



OR

Using the square identity

$\cos^2 12° + \sin^2 12° = 1$

$\cos^2 12 = 1 - \sin^2 12$

$= 1 - k^2$

$\therefore \cos 12° = \sqrt{1-k^2}$

2. $\tan 78° = \frac{\sqrt{1-k^2}}{k}$

3. $\sin 192° = \sin(180° + 12°)$

$= -\sin 12°$

$= -k$

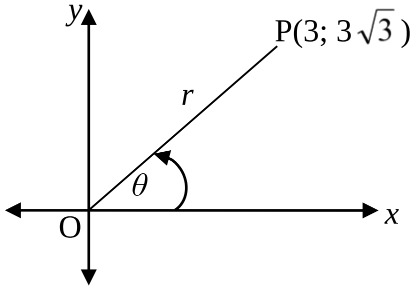
4. $\sin 24° = \sin 2(12°)$

$= 2\sin 12° \cos 12°$

$= 2k.\sqrt{1-k^2}$

ACTIVITY 5.4

5.4.1 Use the accompanying diagram to determine the following:



(a) the value of r

(b) $\sin\theta$

5.4.2 If $\tan A = \frac{2}{3}$ and $90^0 < A < 360^0$,

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determine the value of the following without the use of a calculator:

(a) sin A

(b) 2sinAcosA

5.4.3 If $\cos\theta = \frac{8}{17}$ and $\sin\theta < 0°$, find the value of:

(a) $\tan\theta$

(b) $\frac{\cos\theta}{\sin\theta}$

5.4.4 If 4tan A = – 3 where 0° < A < 180° and

13cos B – 12 = 0 where 180° < B < 360°

Calculate the value of

(a) sin A.sin B

(b) $\cos^2 B$

(c) tan B

5.4.5 If 4 tan θ + 5 = 0 and θ ∈ [0°; 180°]

Determine, without the use of a calculator, the value of $\sqrt{41}\cos\theta - 4\sin(150°).\cos180°$

5.4.6 Given: $\sin\alpha = \frac{3}{5}$ and $90^{\circ} < \alpha < 270^{\circ}$

With the aid of a sketch and **without using a calculator**, determine:

(a) tan α

(b) cos (90° + α)

5.4.7 If $\cos 21° = t$,

determine, without the use of a calculator, the value of sin 66° in terms of t.

5.4.8 If 13 sin x + 5 = 0 and tan x > 0,

Determine without the use of a calculator, the value of cos2x.

5.4.9 If sin 36° = a,

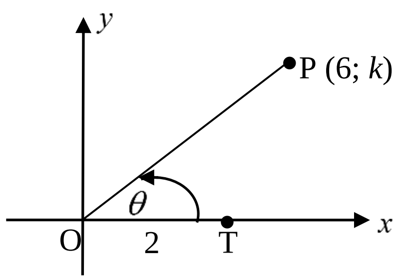
express the following in terms of a

(a) cos 54°

(b) cos 36°

(c) cos 72°

5.4.10 In the diagram, P (6 ; k) is a point in the first quadrant. $\hat{POT} = \theta$ and $OT = 2$



It is further given that $\sqrt{5}\cos\theta - 2 = 0$

Determine, without the use of a calculator:

(a) $\tan\theta$ in terms of k

(b) The value of k

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5.4.11 Given: $\sin 38° = p$

determine the following in terms of *p*. (Without using a calculator.)

(a) $\cos\left(-38^{\circ}\right)$

(b) $\sin 76^{\circ}$

5.4.12 If cos 27° = *t*,

express the following in terms of *t*:

(a) cos (–387°)

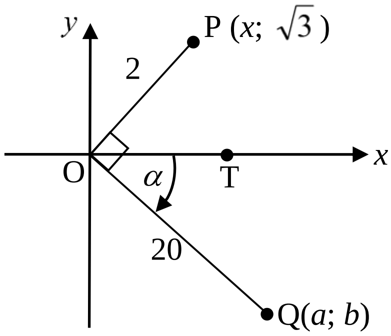
(b) sin 333°

(c) $\dfrac{\tan 153°}{\sin 207°}$

5.4.13 Grade 10 DBE November 2014

In the diagram below, $P(x;\sqrt{3})$ is a point on the Cartesian plane such that OP = 2.

Q(*a*; *b*) is a point such that $T\hat{O}Q = \alpha$, OQ = 20 and $P\hat{O}Q = 90°$



(a) Calculate the value of x

(b) Hence, calculate the size of α.

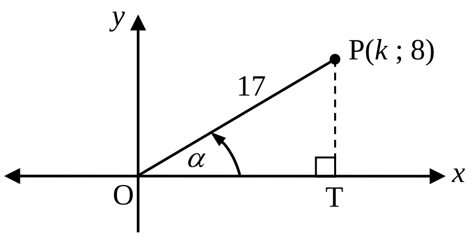
(c) Determine the coordinates of Q

5.4.14 Grade 12 DBE November 2012

The point P(*k* ; 8) lies in the first quadrant

such that OP = 17 units and $T\hat{O}P = \alpha$

as shown in the diagram alongside.



(a) Determine the value of *k*. (2)

(b) Write down the value of $\cos\alpha$. (1)

(c) If it is further given that $\alpha+\beta = 180°$, determine $\cos\beta$. (2)

(d) Hence, determine the value of $\sin(\beta-\alpha)$. (4)

5.4.15 Grade 11 November 2016

If sin 17° = *a*, WITHOUT using a calculator, express the following in terms of *a*:

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(a) tan 17° (3)
(b) sin 107° (2)
(c) $\sin^2 253° + \sin^2 557°$ (4)

5.4.16 If $\cos\alpha = \sqrt{t}$, where α is an acute angle,
express each of the following in terms of t:
(a) $\tan\alpha$
(b) $\sin(180° - \alpha)$
(c) $\sin 2\alpha$

5.5 TRIGONOMETRIC IDENTITIES

A. **Background knowledge:**

Definition: An identity is to work with only one side at a time and to show that one side equals the other. Sometimes it is necessary to first simplify one side of the identity, and then to simplify the other side in order to show that they are equal.

- An identity consists of two sides, namely, a left hand side (LHS) and a right hand side (RHS).
- If we have to prove an identity, we have to simplify one side until it is equal to the other side. (Sometimes it is necessary and sufficient to simplify both sides separately)
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is a trigonometric identity which is true for all values of $\boldsymbol{\theta}$, for which both sides are meaningful, i.e. for which both sides are defined, but, normally only simplify one side of the identity at a time.
- It is sometimes useful to write tan θ in terms of sin θ and cos θ
- Never write a trigonometric ratio without an angle. For example, $\tan = \dfrac{\sin}{\cos}$ it has no meaning.
- Reductions are used to simplify the LHS and/or the RHS i.e. replace function values of $90° \pm \theta$; $180° \pm \theta$ and $360° \pm \theta$, for example, with function values of θ.
- Sometimes it is necessary to multiply the expression (LHS or RHS) by one, e.g. by $\dfrac{1+\sin\theta}{1+\sin\theta}$
- Square identity is sometimes used where necessary
- Remember to write down restrictions:
  - the values for which any of the trigonometric ratios are not defined;
  - the values of the variable which make any of the denominators in the identity equal to zero.

B. Two fundamental identities to be used:

Quotient Identity : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Prove that: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

PROOF:

By definition: $\tan\theta = \dfrac{y}{x}$

Therefore, on dividing both numerator and denominator by r,

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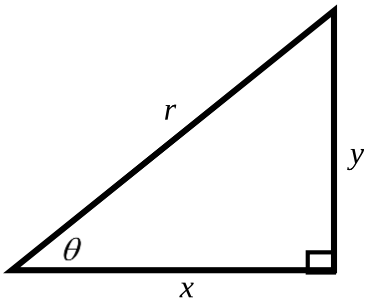
$$\tan\theta = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$$

Square identity: $\sin^2\theta + \cos^2\theta = 1$

PROOF:

Prove that: $\sin^2\theta + \cos^2\theta = 1$



$x^2 + y^2 = r^2$

Therefore dividing both sides by r^2

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} = 1$$

According to the definitions,

$\cos^2\theta + \sin^2\theta = 1$

OR

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2}$$
$$= \frac{y^2 + x^2}{r^2}$$
$$= \frac{r^2}{r^2} = 1$$
$$= 1$$

C. Compound angles identities:

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

D. Double angles identities:

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \cos^2\alpha \\ 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha \end{cases}$$

Examples

1. Prove that: $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$$

$(\cos\theta)(\cos\theta) = (1-\sin\theta)(1+\sin\theta)$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos^2\theta = \cos^2\theta$

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2. Prove that: $\frac{\cos\theta}{1+\sin\theta}=\frac{1-\sin\theta}{\cos\theta}$

$$RHS=\frac{1-\sin\theta}{\cos\theta}$$
$$=\frac{1-\sin\theta}{\cos\theta}\times\frac{1+\sin\theta}{1+\sin\theta}$$
$$=\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}$$
$$=\frac{\cos^2\theta}{\cos(1+\sin^2\theta)}$$
$$=\frac{\cos\theta}{1+\sin\theta}$$
$$=LHS$$

Restrictions: undefined where $\cos\theta=0$, $\sin\theta=-1$. So then $\theta\neq 90^\circ+k.180^\circ$ and $\theta\neq -90^\circ+k.360^\circ$ Therefore $\theta\neq 90^\circ+k.180^\circ,\quad k\in Z$

3. Prove that: $\frac{1}{\cos\theta}-\cos\theta\tan\theta=\cos\theta$

$$LHS=\frac{1}{\cos\theta}-\frac{\cos\theta\tan\theta}{1}$$
$$=\frac{1-\cos^2\theta\times\tan^2\theta}{\cos\theta}$$
$$=\frac{1-\left(\cos^2\theta\times\frac{\sin^2\theta}{\cos^2\theta}\right)}{\cos\theta}$$
$$=\frac{1-\sin^2\theta}{\cos\theta}$$
$$=\frac{\cos^2\theta}{\cos\theta}$$
$$=\cos\theta$$
$$=RHS$$

Restrictions: undefined where $\cos\theta=0^\circ$ and where $\tan\theta$ is undefined. Therefore $\theta\neq 90^\circ\ ;\ \ 270^\circ$.

4. Prove that

$$\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta-\frac{1}{\sin\theta+\cos\theta}$$

$$RHS=\frac{\sin^2\theta+\sin\theta\cos\theta+\cos\theta\sin\theta+\cos^2\theta-1}{\sin\theta+\cos\theta}$$
$$=\frac{1+2\sin\theta\cos\theta-1}{\sin\theta+\cos\theta}$$
$$=\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}$$

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$= LHS$

Restrictions: undefined where $\sin\theta = 0°$, $\cos\theta = 0°$, $\cos\theta = 0°$.

Therefore $\theta \neq 0°;90°;180°;\ 270°$ and $360°$.

Prove that: $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

$$LHS = \sin(A+B)-\sin(A-B)$$
$$LHS = \sin A\cos B+\cos A\sin B-[\sin A\cos B-\cos A\sin B]$$
$$LHS = \sin A\cos B+\cos A\sin B-\sin A\cos B+\cos A\sin B$$
$$LHS = 2\cos A\sin B$$
$$= RHS$$

ACTIVITY 5.5

5.5.1 Prove the following identity

$$\tan\theta.\sin\theta+\cos\theta+\frac{1}{\cos\theta}$$

5.5.2 Prove that $\dfrac{\cos(A-45°)}{\cos(A+45°)}=\dfrac{1+\sin 2A}{\cos 2A}$

5.5.3 Prove the identity:

$$\frac{\sin 3\theta}{\sin\theta}=3-4\sin^2\theta$$

5.5.4 Prove that: $\dfrac{\cos 3\theta}{\cos\theta}=2\cos 2\theta-1$

5.5.5 Prove:

$$\frac{2\sin^2 x}{2\tan x-\sin 2x}=\frac{1}{\tan x}$$

5.5.6 Prove that $\sin 2x+2\sin^2(45°-x)=1$ and hence deduce, without the use of a calculator,

5.5.7 Given the following identity: $\dfrac{\cos\theta-\sin\theta\sin 2\theta}{\cos 2\theta}=\cos\theta$

(a) Prove the identity.

(b) For which values of *x* is the identity undefined? Give your answer in general solution form.

5.5.8 Prove that:

(a) $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

(b) For which value(s) of *x* in the interval $0°\leq x\leq 180°$ is the above identity undefined?

5.5.9 Given the identity:

(a) Prove the identity $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{\cos\theta}{1+\sin\theta}=2\tan\theta$

(b) If $x\in[-180°;180°]$, give 2 values of *x* for which the identity is undefined?

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5.5.10 Given the identity: $\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}=\frac{2}{\cos\theta}$

(a) Prove the identity:

(b) For which values of x in the interval $0^\circ \le \theta \le 360^\circ$ will the identity be undefined?

5.6 GENERAL SOLUTION, SOLVING TRIG EQUATIONS

Hints for the solution of trigonometric equations

(a) Simplify the equation by using identities where possible.

(b) Simplify the equation as much as possible until you have a trigonometric ratio of one angle equal to a constant/value.

(c) Trigonometric ratios with compound angles and double angles should be simplified to trigonometric ratios with single angles where applicable

(d) Factorization, as in the solution of algebraic equations, is often used. Look out for:

- Common factor
- Difference of squares
- Quadratic trinomials
- Grouping of terms

(e) Take care not to divide by an unknown variable, example: if $\sin x\cos x=\sin x$, dividing by $\sin x$ on both sides is incorrect.
Make sure that the dividing value is not equal to zero.

(f) Use the CAST rules to determine in which quadrants the angle will be

Example 1

Solve for x if $\tan x = 2{,}22$; where $0^\circ \le x \le 90^\circ$.

$x=\tan^{-1}(2{,}22)$

$\quad =70^\circ$

ACTIVITY 5.6

5.6.1 Solve for x correct to ONE decimal place, where $0^\circ \le x \le 90^\circ$

(a) $5\cos x = 3$

(b) $\tan 2x = 1{,}19$

5.6.2 Solve for θ correct to TWO decimal places, if $\frac{4}{3}\sin\theta=\cos 37^\circ$

Example 2

Find the values of x between -180° and 180° if: $7\sin(x-30^\circ)+2=0$

$\sin(x-30^\circ)=-\frac{2}{7}$

$ref\ \angle = 16{,}6^\circ$

$x-30^\circ=180^\circ+16{,}6^\circ+k.360^\circ$ or $x-30^\circ=360^\circ-16{,}6^\circ+k.360^\circ$

$x=226{,}6^\circ+k.360^\circ$ or $x=373{,}6^\circ+k.360^\circ$

$x=-133{,}4^\circ$ or $x=13{,}6^\circ$

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Example 3

Consider: $\dfrac{1-\cos^2 A}{4\cos(90°+A)}$.

(a) Simplify the expression to a single trigonometric ratio

$$\frac{1-\cos^2 A}{4\cos(90°+A)} = \frac{1-\cos^2 A}{4(-\sin A)}$$
$$= \frac{\sin^2 A}{-4\sin A}$$
$$= -\frac{1}{4}\sin A$$

(b) Hence, determine the general solution of $\dfrac{1-\cos^2 2x}{4\cos(90°+2x)} = 0{,}21$.

$-\frac{1}{4}\sin 2x = 0{,}21$

$\sin 2x = -0{,}84$

$\text{ref } \angle = 57{,}14°$

$\therefore 2x = 237{,}14° + k.360°$ or $2x = 302{,}86° + k.360°;\ k \in Z$

$x = 118{,}57° + k.180°$ or $x = 151{,}43° + k.180°$

Example 4

Determine the general solution for $\sin(x-30°) = \cos 2x$

We do not divide by sine on both sides but we take arc sine on both sides

$\sin(x-30°) = \sin(90°-2x)$

$x - 30° = 90° - 2x + k.360°$ or $x - 30° = 180° - (90° - 2x) + k.360°;\ k \in Z$

$3x = 120° + k.360°$ or $-x = 120° + k.360°$

$x = 40° + k.120°$ or $x = -120° + k.360°$

OR

$\cos[90° - (x-30°) = \cos 2x]$

$90° - x + 30° = 2x + k.360°$ or $90° - x + 30° = 360° - 2x + k.360°;k \in Z$

$-3x = -120° + k.360°$ or $x = 360° - 120° + k.360°$

$x = -40° - k.120°$ or $x = 240° + k.360°$

ACTIVITY 5.7

5.7.1 Solve the following equation, rounded off to one decimal digit:
$\tan x = 2\cos 306{,}1°;\ 0° \le x \le 270°$

5.7.2 Determine the general solution for $2\sin x\cos x = \cos x$

5.7.3 Determine the general solution of $3\sin x = 2\tan x$

5.7.4 Determine the general solution for $\cos\frac{1}{2}x = \sin(x-30°)$

5.7.5 Determine the general solution for $3\cos^2 x + 10\sin x + 5 = 0$

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Example 5

Determine the general solution of $\sin x + 2\cos 2x = 1$

$\sin x + 2(1 - \sin^2 x) = 1$

$-2\sin^2 x + \sin x + 1 = 0$

$2\sin^2 x - \sin x - 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$\sin x = 1$

$x = 90° + k.360°; k \in Z$

or

$\sin x = -\frac{1}{2}$

$x = 210° + k.360°$ or $x = 330° + k.360°; k \in Z$

OR

$x = 210^0 + k.360°$ or $x = -30° + k.360°; k \in Z$

OR

$x = -150° + k.360°$ or $x = 330° + k.360°; k \in Z$

OR

$x = -150° + k.360°$ or $x = -30° + k.360°; k \in Z$

OR

$\sin x + 2\cos^2 x = 1$

$\sin x = 1 - 2\cos^2 x$

$\sin x = -\cos 2x$

$-\cos(90° - x) = \cos 2x$

$2x = 180° + (90° - x) + k.360°$ or $2x = 180° - (90° - x) + k.360°; k \in Z$

$3x = 270° + k.360°$ or $x = 90° + k.360°$

$x = 30° + k.360°$

ACTIVITY 5.8

5.8.1 Solve for A if $\tan A = \tan 135°$ and

(a) $180° \leq A \leq 360°$

(a) $360° \leq A \leq 720°$

5.8.2 Determine the general solution to $3\sin\theta.\sin 22° = 3\cos\theta.\cos 22° + 1$

5.8.3 If $\cos\theta = 2\sin 75°\sin 15°$; $\theta \in [-360°; 360°]$, determine θ without using a calculator.

5.8.4 Determine the general solution to $\tan\theta.\sin\theta + \cos\vartheta = \frac{3}{\sin\theta}$

5.8.5 Determine the general solution to $\frac{\sin 3\alpha}{\sin\alpha} = 2$

5.8.6 Determine the general solution of the equation $2\sin A.\cos A - 0,8 = 0$

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5.8.7 Calculate the values of x if $4\sin^2 x + 6\sin x.\cos x - 2\sin x - 3\cos x = 0$ for $-360° \leq x \leq 0°$. Round off the answer to 2 decimal digits, if necessary.

5.8.8 If $\theta \in [-180°;180°]$, determine the value(s) of θ:

(a) $\sin 5\theta \cos 20° - \cos 5\theta \sin 20° = 1$

(a) $2\cos 3\theta \cos 30° - 2\sin 3\theta \sin 30° = 1$

5.8.9 Calculate the value of x between 0° and 360° if: $\cos 2x + \sin x = 0$.

5.8.10 Determine the general solution of $\cos(x - 30°) = 2\sin x$

5.8.11 Consider: $\sin(2x + 40°)\cos(x + 30°) - \cos(2x + 40°)\sin(x + 30°)$

(a) Write as a single trigonometric term in its simplest form

(b) Determine the general solution of the following equation:

$$\sin(2x + 40°)\cos(x + 30°) - \cos(2x + 40°)\sin(x + 30°) = \cos(2x - 20°)$$

5.8.12 Determine the general solution of x if $2\cos x = 3\tan x$

5.9 BASIC FUNCTIONS

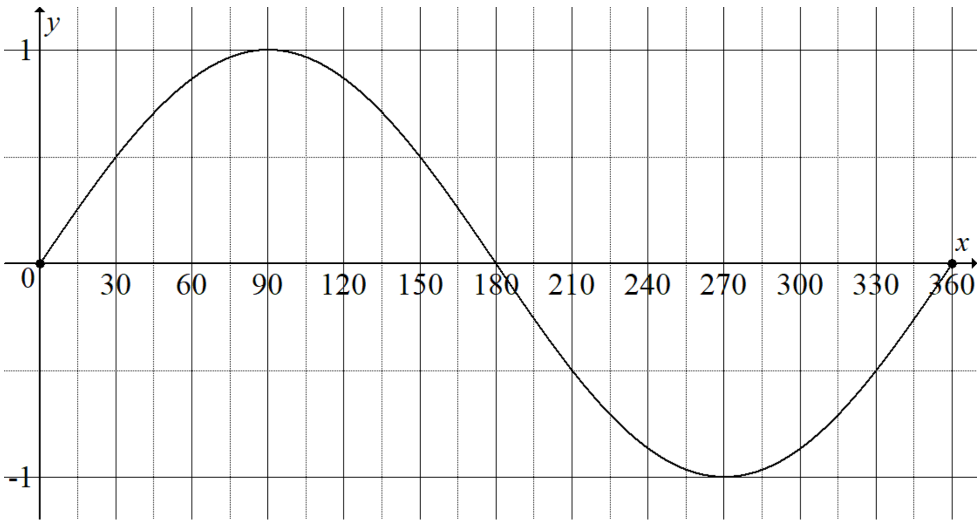
NOTE:

- Table and/or calculator is useful when plotting/ sketching the graphs.
- The maximum distance that the graph can be from the x-axis is called the **amplitude** of the graph.
- The set of all possible values that y **can** be is called the **range** of the function.
- The set of numbers that x **can** be substituted with is called the **domain** of the function.
- The number of degrees the graph takes to complete one cycle is called the **period** of the graph.

Let us start with Basic Functions:

- Sketch the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ on separate axes.
- For each graph, write down the amplitude (where possible), range (where possible), domain, period and asymptote(s) (where possible).

A.1: $y = \sin x$



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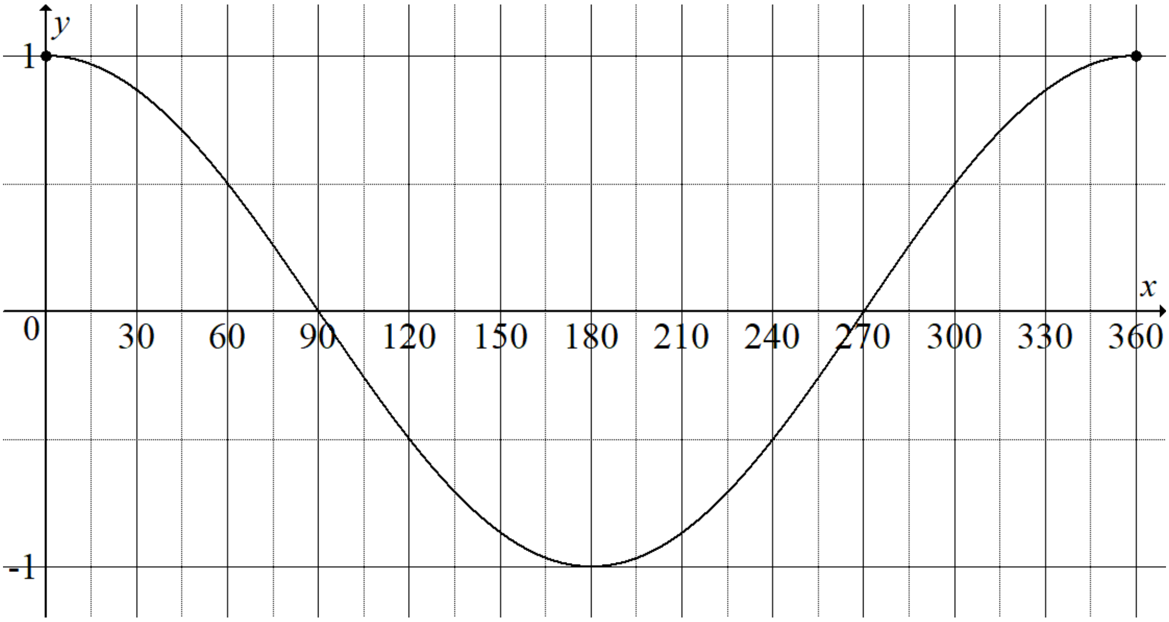
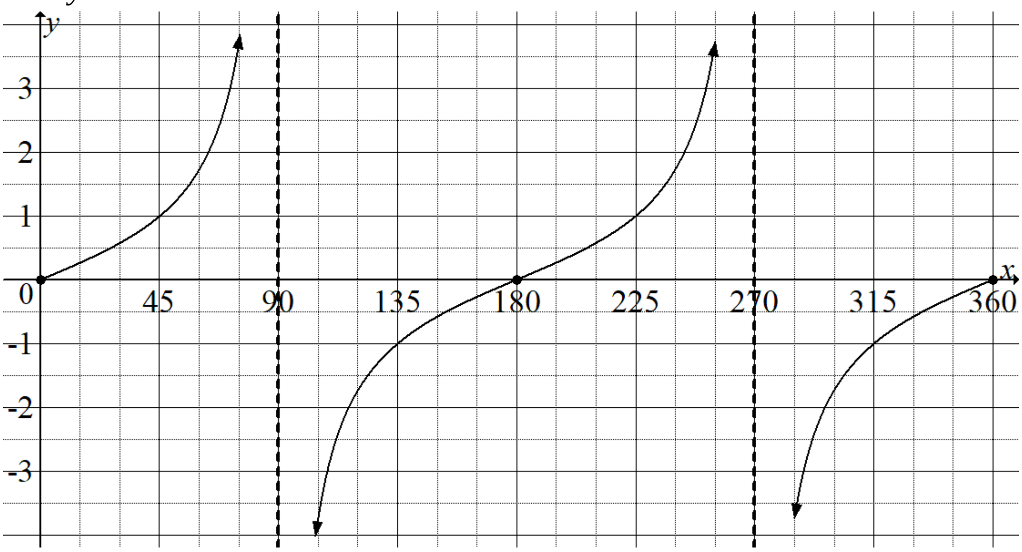
- Amplitude: 1
- Domain: $x \in [0°; 360°]$ or $0° \le x \le 360°$
- Range: $y \in [-1; 1]$ or $-1 \le y \le 1$
- Period: $\frac{360°}{1} = 360°$

A.2: $y = \cos x$



- Amplitude: 1
- Domain: $x \in [0°; 360°]$ or $0° \le x \le 360°$
- Range: $y \in [-1; 1]$ or $-1 \le y \le 1$
- Period: $\frac{360°}{1} = 360°$

A.3: $y = \tan x$



- Asymptotes: $x = 90°$ and $x = 270°$

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- Domain: $x\in[0;360°]$, $x \neq 90°$ / $x \neq 270°$ OR $0 \leq x \leq 360°$, $x \neq 90°$ / $x \neq 270°$
- Range: $y\in(-\infty;\infty)$ or $-\infty < y < \infty$
- Period: $\frac{180°}{1} = 180°$

ACTIVITY 5.9

5.9.1 Given $f(x) = 2\sin x$ and $g(x) = \frac{1}{2}\sin x$, $x\in[0°;360°]$

(a) Sketch the graphs of *f* and *g* on the same set of axes.
(b) Write down the amplitude, range, domain and period of *f* and *g*.
(c) The graphs are in the form $y = a\sin x$:
Compare the amplitude and range of *f*, *g* and $y = \sin x$ and comment on the effect of *a*.

5.9.2 Given $f(x) = \cos x + 2$ and $g(x) = \cos x - 1$, $x\in[0°;360°]$

(a) Sketch the graphs of *f* and *g* on the same set of axes.
(b) Write down the amplitude, range, domain and period of *f* and *g*.
(c) The graphs are in the form $y = \cos x + q$:
Compare the range of *f*, *g* and $y = \sin x$ and comment on the effect of *q*.

5.9.3 Given $f(x) = 2\tan x$ and $g(x) = \frac{1}{2}\tan x$, $x\in[0°;360°]$

(a) Sketch the graphs of *f* and *g* on the same set of axes.
(b) If there is a point with coordinates (45^o; *a*), in the graphs of *f*, *g* and $y = \tan x$, write down the value of a in the graphs of *f*, *g* and $y = \tan x$.
3.3 The graphs are in the form $y = a\tan x$:
Compare and comment on the values of *a* in *f*, *g* and $y = \tan x$.

5.9.4 Given $f(x) = \sin 2x$, $h(x) = \sin 4x$ and $g(x) = \sin \frac{1}{2}x$, $x\in[0°;360°]$

(a) Sketch the graphs of *f*, *h*and *g* on the same set of axes.
(b) Write down the period of *f*, *h* and *g*.
(c) The graphs are in the form $y = \sin kx$:
Compare the periods of *f*, *h*, *g* and $y = \sin x$ and comment on the effect of *k*.

5.9.5 Given: $f(x) = \cos(x - 60°)$, $h(x) = \cos(x + 30°)$ and $g(x) = \cos(x - 90°)$, $x\in[0°;360°]$

(a) Sketch the graphs of *f*, *h*and *g* on the same set of axes.

(b) The graphs are in the form $y = \cos(x + p)$:
Compare the graphs of *f*, *h*and *g* with $y = \cos x$ and comment on the shifting.

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5.9.6 Consider the functions: $f(x)=a\sin k(x+p)+q$

$h(x)=a\cos k(x+p)+q$ and

$g(x)=a\tan k(x+p)+q$

For each of the functions, summarise the effects of *a*, *k*, *p* and *q*.

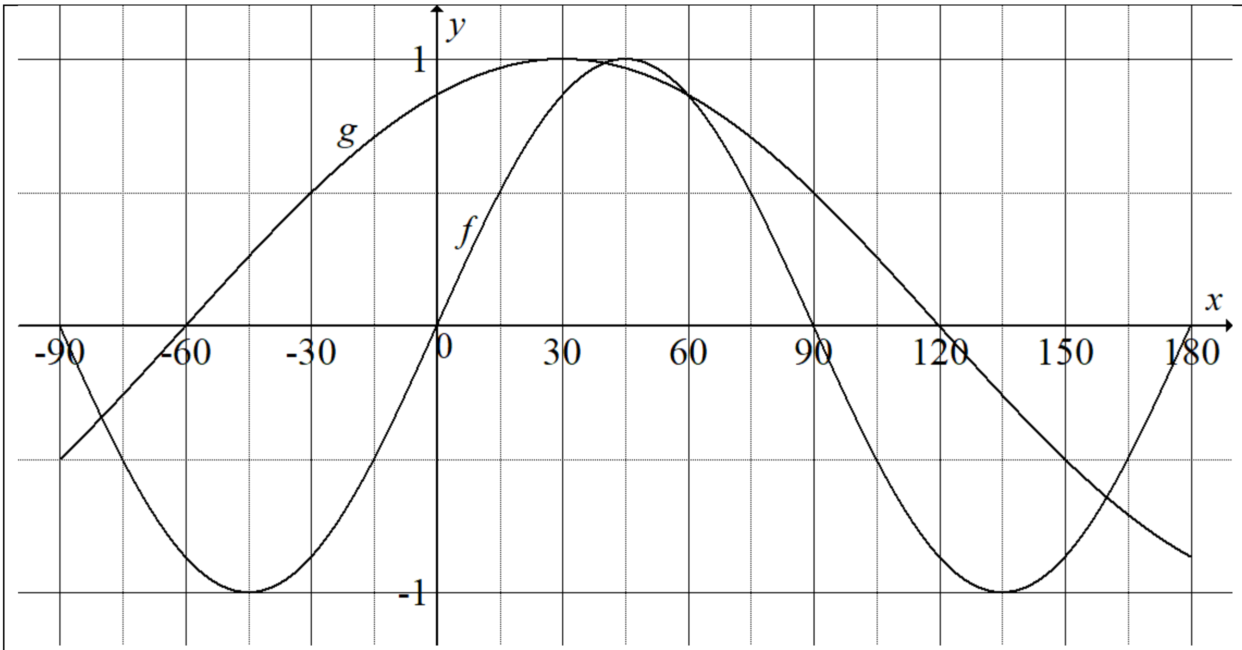
C. NOTE:

After sketching the graph(s) or when the graph is already drawn, interpretation can be done. The following may be asked to be determined/ calculated:

- Period
- Domain
- Range
- Determine equations
- Amplitude
- Asymptote(s) for tan graph
- Intersection between TWO graphs
- Increasing and decreasing graphs
- Inequalities
- Distance between curves
- Transformation of functions
- etc.

Example

The graphs of $f(x)=\sin 2x$ and $g(x)=\cos(x-30°)$, for $x\in[-90°;180°]$ are given below:



Determine the values of *x* for which:

1.1 $f(x)-g(x)=0$

1.2 $f(x)\geq g(x)$

1.3 $\dfrac{f(x)}{g(x)}<0$

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1.4 $f(x).g(x) \geq 0$

1.5 $f^{|}(x).g(x) > 0$

Solutions

1.1 $\sin 2x = \cos(x - 30°)$

$\cos(90° - 2x) = \cos(x - 30°)$

$90° - 2x = x - 30° + k.360°$ or $90° - 2x = 360° - (x - 30°) + k.360°$

$-3x = -120 + k.360°$ or $90 - 2x = 360° - 90 - x + 30° + k.360°$

$x = 40° + k.360°$ or $x = -300° + k.360°$

$\therefore x \in (-80°; 40°; 60°; 160°)$

1.2 $-90° \leq x \leq 80°$ or $40° \leq x \leq 60°$ or $160° \leq x \leq 180°$

1.3 $-60° < x < 0°$ or $90° < x < 120°$

1.4 $-90° \leq x \leq 60°$ or $0° \leq x \leq 90°$ or $120° \leq x \leq 180°$

1.5 $-90° < x < -60°$ or $-45° < x < 45°$ or $120° < x < 135°$

D. PRACTICE EXERCISES

1. Given $f(x) = \sin x$ and $g(x) = \cos x + 1$ where $x \in [-90°; 270°]$.

1.1 Draw sketch graph of $f(x) = \sin x$ and $g(x) = \cos x + 1$ where $x \in [-90°; 270°]$.
Indicate on the sketch the coordinates of all intercepts with the axes as well as the coordinates of the turning points

1.2 Calculate the value(s) of x, $x \in [-90°; 270°]$, if $\sin x = \cos x + 1$.

1.3 Use the sketch graphs drawn in **QUESTION 1.1** to answer the following questions:

1.3.1 Write down the range of *f*.

1.3.2 For which values of x, $x \in [-90°; 270°]$ is:

(a) $f(x) \leq g(x)$.

(b) $f(x).g(x) \leq 0$.

2. Given: $h(x) = \cos(x + 30°)$ and $g(x) = -2\sin x$

2.1 Draw sketch graphs of the curves $h(x) = \cos(x + 30°)$ and $g(x) = -2\sin x$
coordinates of the turning points, for $[-120°; 180°]$.

2.2 Determine the general solution, without the use of a calculator, if
$h(x) = g(x)$

2.3 Use the solutions obtained in **QUESTION 2.2** and the graphs drawn in **QUESTION 2.1**; to determine for which values of x:

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2.3.1 $2\sin x + \cos x.\cos 30° \geq \sin x.\sin 30°$ and $[-120°; 180°]$.

2.3.2 Determine if $h(x)$ and/or $g(x)$ increases for the interval $x \in [-120°; 0°]$

2.4 If the curve of h is shifted 2 units down, determine the resulting y –intercept.

3. Given: $f(x) = \cos x - \frac{1}{2}$ and $g(x) = \sin(x + 30°)$

3.1 Draw sketch graphs of f and g on the same set of axes, for $x \in [-120°; 60°]$. Indicate clearly all the intercepts with the axes, co-ordinates of the turning points and the end points.

3.2 Use the graphs drawn in **QUESTION 3.1**; to determine for which value(s) of x is:

3.2.1 $\cos(60° - x) < 0$?

3.2.2 $f(x) - g(x > 0?)$

3.2.3 $\frac{g(x)}{f(x)}$ – undefined?

4. Given: $f(x) = \frac{1}{2}\tan x$ and $g(x) = \sin 2x$

4.1 Determine the values of x if $\frac{1}{2}\tan x = \sin 2x$ for $x \in [-90°; 180°]$.

4.2 Draw sketch graphs of f and g on the same set of axes, where $x \in [-90°; 180°]$. Indicate clearly all the intercepts with the axes and the turning points.

4.3 Use the solutions obtained in **QUESTION 4.1** and the graphs drawn in **QUESTION 4.2**; to determine for which values of x, for $x \in [-90°; 180°]$:

4.3.1 $f(x) > g(x)$.

4.3.2 $f(x).g(x) \leq 0$.

4.3.3 both $f(x)$ and $g(x)$ are increasing as x increases.

5. Given: $g(x) = -\sin x$ and $f(x) = \tan x + 1$.

5.1 Draw sketch graphs of f and g on the same set of axes, where $x \in [-90°; 270°]$. Indicate clearly all the intercepts with the axes, asymptotes and the turning points.

5.2 Use the drawn graphs in QUESTION 5.1 to answer the following:

5.2.1 Indicate on the graphs the values of x for which, $1 + \tan x + \sin x = 0$, using letters A, B,......

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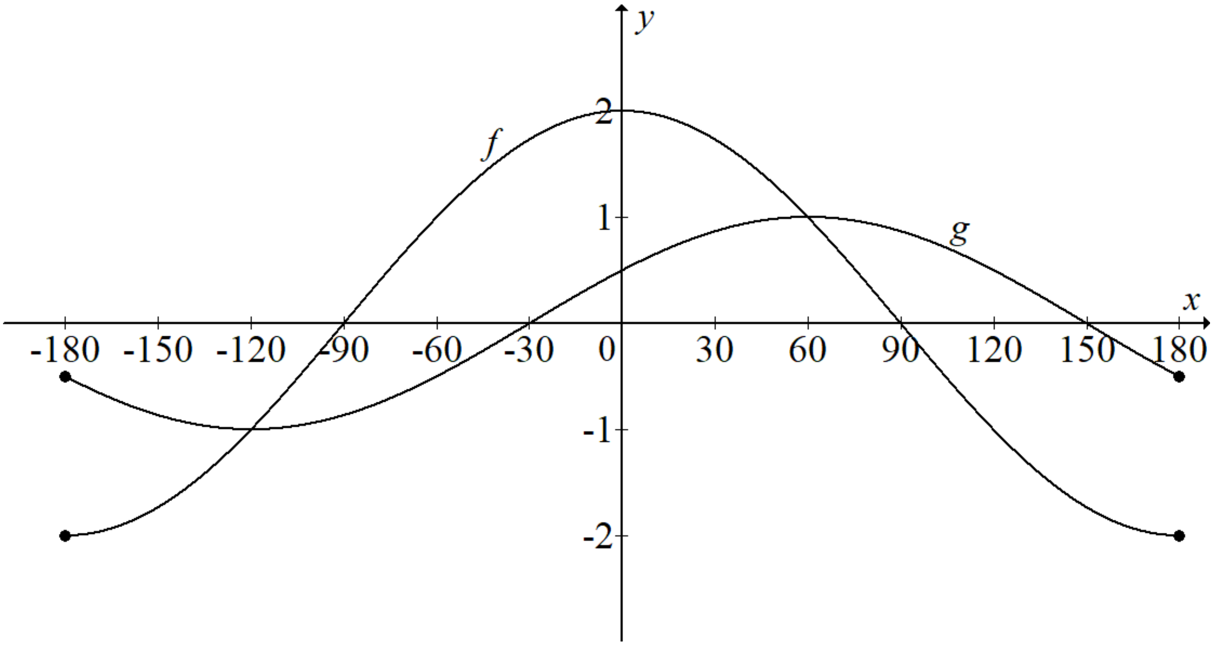
5.2.2 Determine the values of x, for $x \in [-90°;270°]$, for which $g(x) - f(x) = 1$.

5.2.3 Determine the values of x, for $x \in [-90°;90°]$, for which $f(x).g(x) \leq 0$.

5. The graphs of $f(x) = 2\cos x$ and $g(x) = \sin(x + 30°)$ are given for the interval. $-180° \leq x \leq 180°$. Use the graphs to answer the following questions:



6.1 What is the period of f?

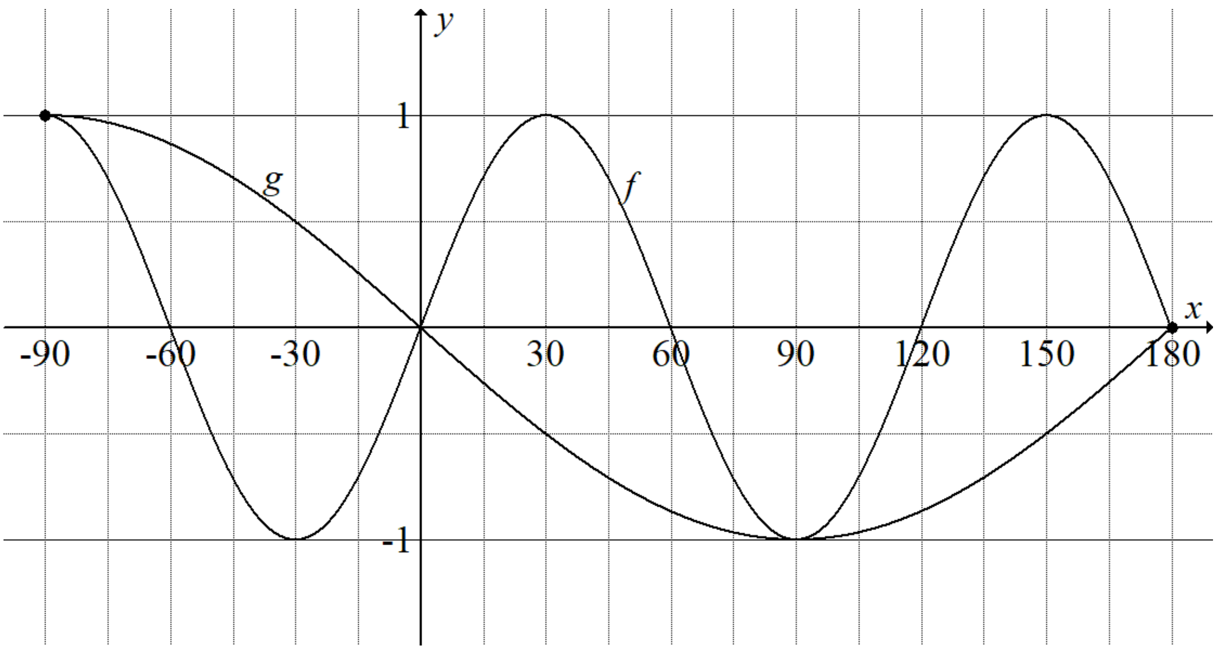
6.2 For which values of x does g increases in value as x increases?

6.3 Find the equation of the graph of g if it is translated $30°$ to the right.

6.4 Find the equation of the graph of f if the x-axis is translated five units upwards.

6.5 How many solutions are there for to the equation $f(x) = g(x)$ if $x \in [0°;180°]$?

7. Study the graphs $f(x) = a \sin kx$ and $g(x) = \cos(x + p)$ for the domain $x \in [-90°;180°]$ which are drawn on the same set of axes below.



7.1 Determine the values of a, k and p.

7.2 Graph g can also be written as a sine graph. Give this equation.

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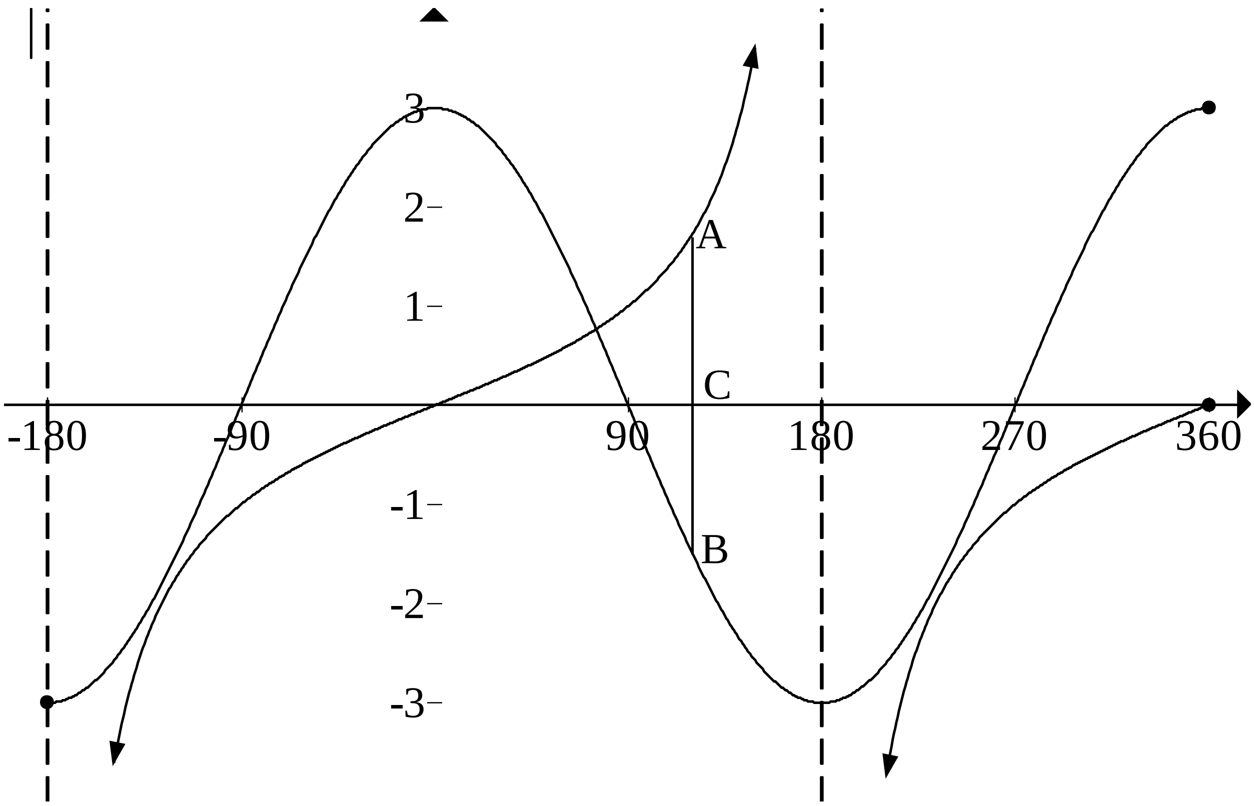
7.3 If graph g is translated to the left by $60°$, give the equation of the new graph in its simplest form.

7.4 For which values of x for the above domain will both $f(x)$ and $g(x)$ be increasing?

8. The graph shows the curves of $f=\{(x;y)/\,y=a\cos x\}$ and $g=\{(x;y)/\,y=\tan bx\}$ for $x\in[-180°;360°]$.

Answer the following questions with the aid of the graph given below:



4.1 Determine the values of a and b

4.2 Write down the equation (s) of the asymptotes of g.

4.3 Determine the maximum value of $f(x) - g(x)$ in the interval $0° \leq x \leq 90°$

4.4 For which values of x will $f(x).g(x) \leq 0$ for $x\in[-90°;270°]$

4.5 If for $x = 120°$, calculate the length of AB (in surd form) if AB || x-axes.

5.10 DIMENSIONAL AND 3 DIMENSIONAL PROBLEMS

DEFINITIONS:

- **ONE dimensional object:** object has only length (line segment)
- **TWO dimensional object:** object is plane, it has length and width. Examples are polygons, circles, etc.
- **THREE dimensional object:** object has length, breadth and height. 3D- involves 3 different planes. Examples are solids (cubes, pyramids, etc.)

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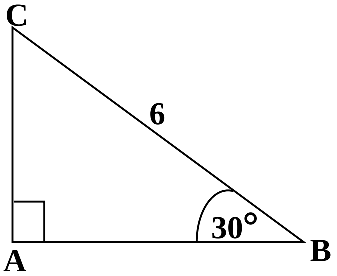
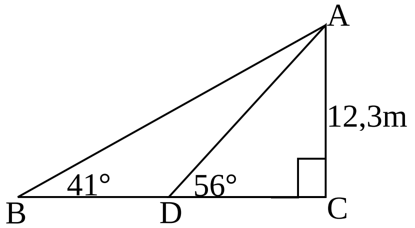
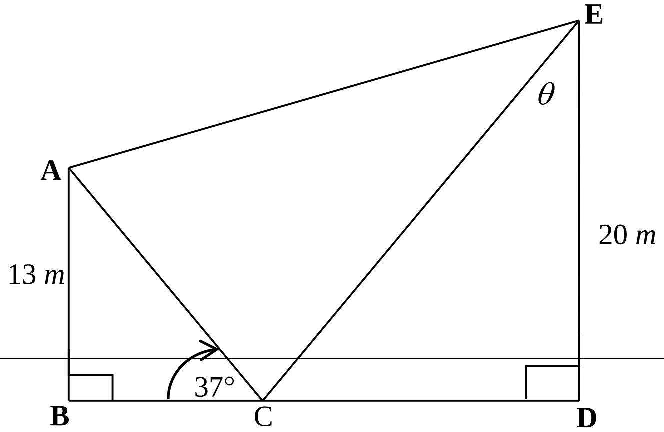
- **Angle of elevation:** The angle of elevation of an object is the angle which the eye has to be raised through from the horizontal in order to look at the object.
- **Angle of depression:** The angle of depression of an object is the angle which the eye has to be lowered through from the horizontal in order to look at the object.

1.1	2 DIMENSIONAL PROBLEMS		
	STRATEGIES FOR SOLVING 2 DIMENSIONS		
	• Fill in all the given information on the diagram. • Problems on 2D require you to solve a combinations of triangles in one plane • Identify the right angled-triangle first (try to use Pythagoras or trig ratios in this triangle) • Start to work in triangles with more information to find the common side • Use basic geometric results e.g. Exterior angle of a triangle, corresponding angle, co-interior angles, alternate angles, etc.		
1.	Use the diagram below to calculate the length of AB and AC  		
2.	In ΔABC, $AC \perp BC$, AB =12,3m, $\hat{B} = 41° \ and \ A\hat{D}C = 56°$  		
	2.1	Calculate the length of DC	
	2.2	Calculate the length of BC	
	2.3	Hence calculate the length of BD	
3.	AB and DE are two Towers in the same horizontal plane. From point C the angle of elevation of A, the top of tower AB, is 37°. AB is 13m high, DE = 20m high and CD = 21,4m.  		

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Calculate:

3.1	the length of AC	
3.2	the value of θ	
3.3	the area of ΔECD	

❖ **The sine rule**

The sine rule for triangle ABC is given by:

- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

  OR

- $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

NB: $a = \text{BC}$; $\text{b} = \text{AC}$ and $c = \text{AB}$

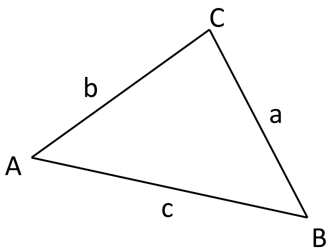
This formula is used to solve triangles when the following measurements are given:

1. two angles and a side
2. two sides and one angle (provided that angle given is not the included angle)

❖ **The cosine rule**

The cosine rule for triangle ABC is given by:

- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$



This formula is used to solve triangles when the following measurements are given:

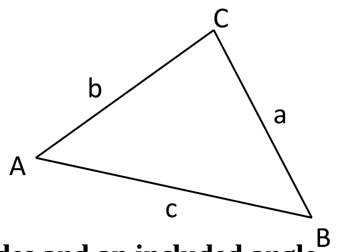
1. two sides and the included angle
2. three sides.

If the lengths of the three sides are given, the formula can be written in the following forms to find $\hat{A}; \hat{B}$ or $\hat{C}$ respectively:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

❖ **The area rule of a triangle**

- Area of Δ ABC = $\dfrac{1}{2}bc\sin A$
- Area of Δ ABC = $\dfrac{1}{2}ac\sin B$
- Area of Δ ABC = $\dfrac{1}{2}ab\sin C$



The area rule is half the product of any two sides and an included angle.

EXAMPLES OF 2D PROBLEMS

1. The angle of elevation on the top of a building B, from a point A on a level road is $36°$. At a point C 27 metres further down the road, the angle of elevation is $25°$. If D represents the base of the building, calculate the height of the building (BD in the diagram)

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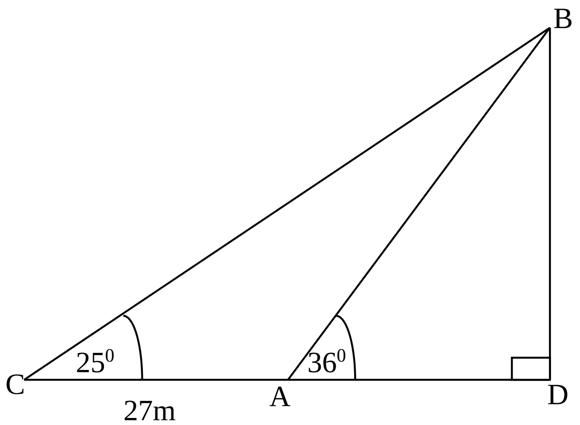
SOLUTION

Use the sine rule in Δ ABC



$\angle CBA = 36^0 - 25^0 = 11^0$

$$\frac{AB}{\sin 25^0} = \frac{27}{\sin 11^0}$$

$$AB = \frac{27 \sin 25^0}{\sin 11^0}$$

= 59,8m

Now in Δ ABD

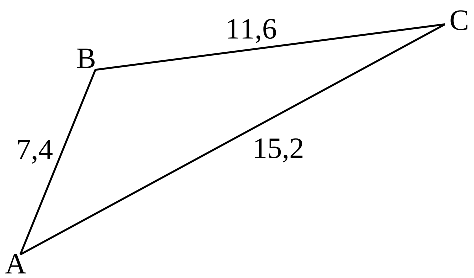
$$\sin 36^0 = \frac{BD}{AB}$$

$$BD = AB \sin 36^0$$

$$BD = 59.8 \sin 36^0$$

= 35,15m

2. In the diagram below, determine the size of angle B



Solution:

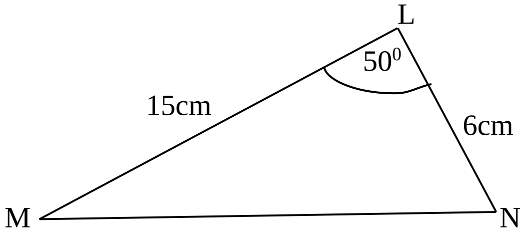
$$b^2 = a^2 + c^2 - 2ac \cos \mathrm{B}$$

$$(15{,}2)^2 = (11{,}6)^2 + (7{,}4)^2 - 2(11{,}6)(7{,}4) \cos \mathrm{B}$$

$$41{,}72 = -171{,}68 \cos \mathrm{B}$$

$$\cos \mathrm{B} = \frac{41{,}27}{-171{,}68}$$

$$\therefore \mathrm{B} = 104{,}1°$$



Determine the length of MN

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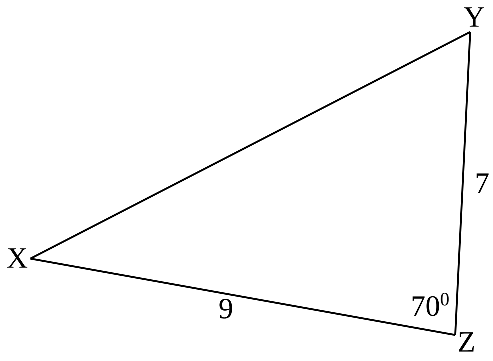
Solution:

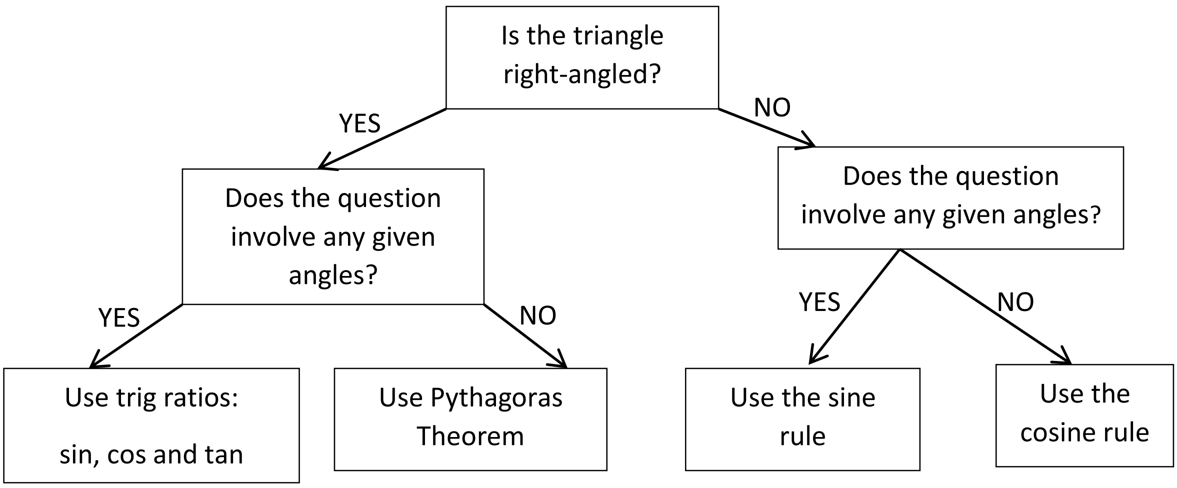
$$l^2 = m^2 + n^2 - 2mn\cos L$$
$$= (6)^2 + (15)^2 - 2(6)(15)\cos 50°$$
$$= 145{,}3$$
$$\therefore l = 12{,}1\text{cm}$$

3. Determine the area of ΔXYZ



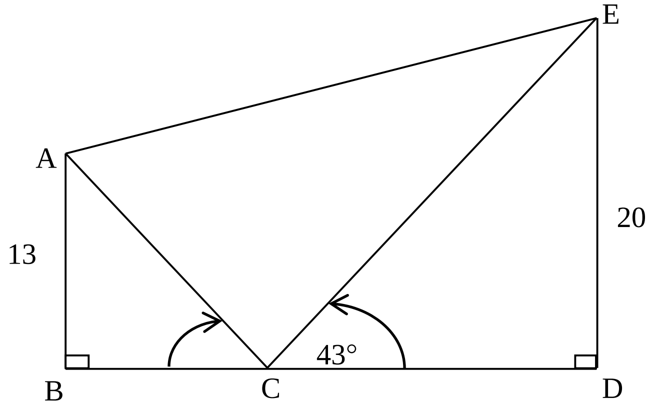
Solution:

$$\text{Area} = \frac{1}{2}xy\sin Z$$
$$= \frac{1}{2}(7)(9)\sin 70°$$
$$= 29{,}6\,\text{units}^2$$



Activities

5.10.1 AB and DE are two towers in the same horizontal plane. B, C and D are in the same plane. The angle of elevation from a man at point C to E, the top of tower DE is 43°. From point C the angle of elevation of A, the top of tower AB, is 37°. AB is 13m high and DE = 20 m.



5.10.1.1 Calculate the length of:

(a) AC

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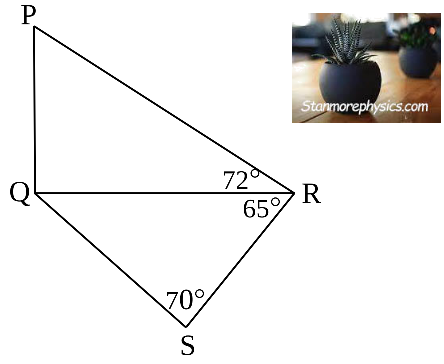
	(b)	CE
5.10.1.2	Hence, calculate the distance between the tops of the buildings, that is, AE.	
5.10.2	PQ is a mast with Q being the foot. R and S are two points in the same horizontal plane as Q. $Q\hat{R}S = 65°$, $Q\hat{S}R = 70°$ and $Q\hat{R}P = 72°$. If QS = 12m, Calculate the length of PQ	

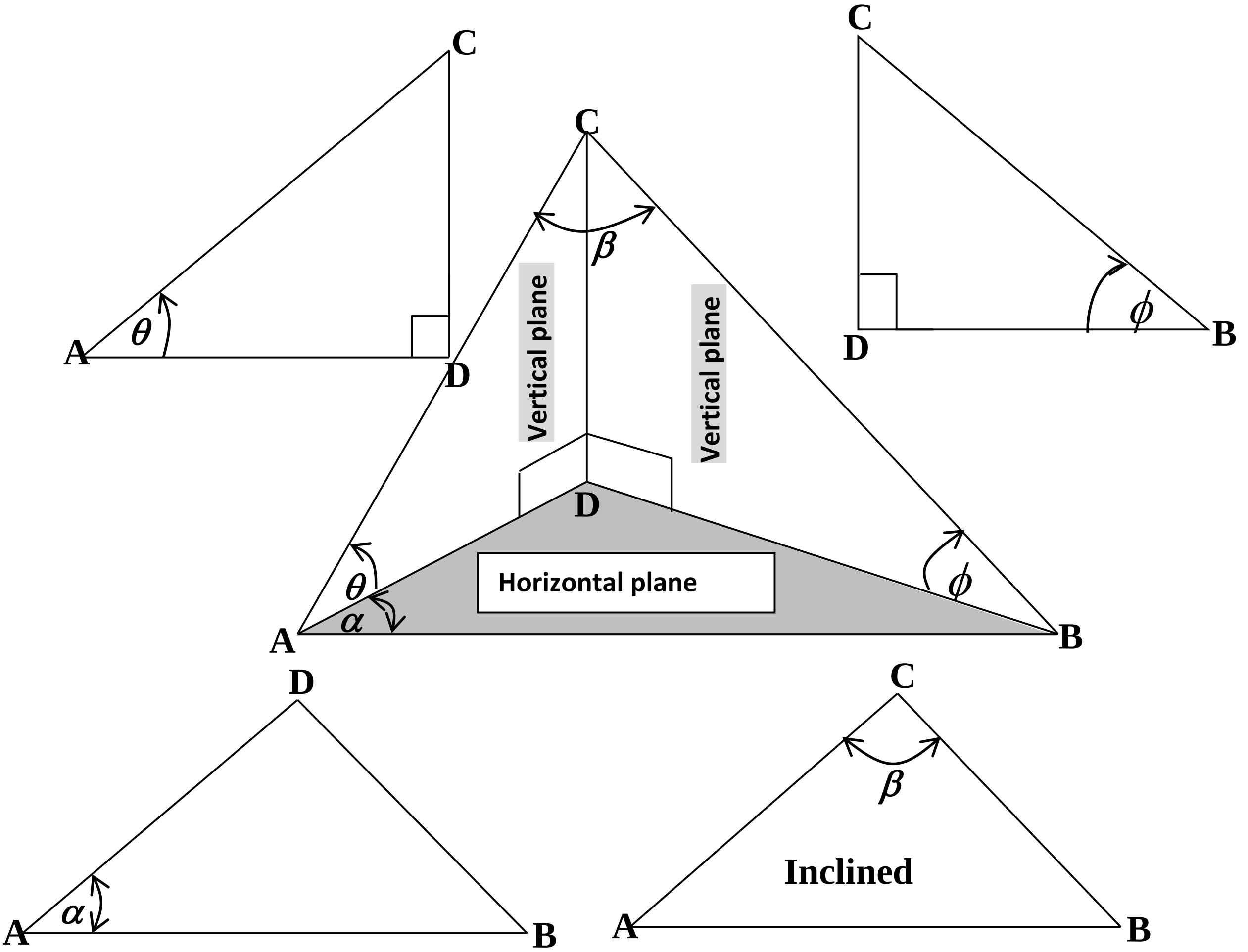


3 DIMENSIONAL PROBLEMS

STRATEGIESFOR SOLVING 3 DIMENSIONS

- Shade the horizontal plane
- The vertical line and horizontal plane are perpendicular (mark them as such)
- identify the right angled-triangle first
- In a right angled triangle, use definitions of trig ratios or Pythagoras theorem to solve the triangles
- If there are no right-angled triangles, start to work in triangles with more information.
- May draw various triangles separately (*do not lose focus of the whole*)
- Start to work in triangles with more information to find the common side.

NOTE that a 3D has a horizontal plane, vertical plane and an inclined plane, the teacher may use a model to demonstrate this to the learners, see the diagram below:



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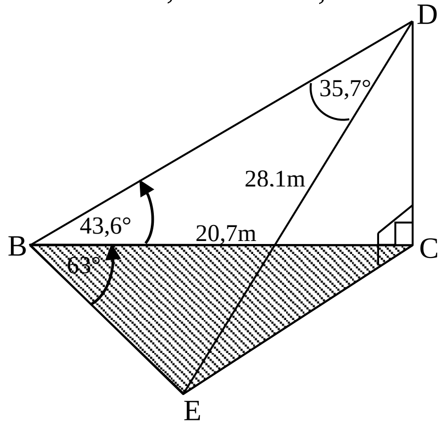
	EXAMPLES
1.	CD is a vertical mask. The points B, C and E are in the same horizontal plane. BD and ED are cables joining the top of mask to pegs on the ground. DE is 28,1 m and BC is 20,7 m. The angle of elevation of D from B, $D\hat{B}C = 43{,}6°$. $B\hat{D}E = 35{,}7°$ and $C\hat{B}E = 63°$. 

Give your answer correct to ONE decimal place in each of the following questions:		
	1.1	Calculate the length of BD.
	1.2	Show that the length of BE is 17,4 m.
	1.3	Calculate the area of ΔBEC.
Solutions:		
1.1	$\cos 43{,}6° = \dfrac{20{,}7}{BD}$ $\therefore BD = \dfrac{20{,}7}{\cos 43{,}6°}$ $= 28{,}6$	
1.2	$BE^2 = BD^2 + DE^2 - 2(BD)(DE)\cos B\hat{D}E$ $BE = \sqrt{(28{,}6)^2 + (28{,}1)^2 - 2 \times 28{,}6 \times 28{,}1 \times \cos 35{,}7°}$ $= 17{,}4$	
1.3	Area of $\Delta BEC = \dfrac{1}{2}(BE)(BC)\sin C\hat{B}E$ $= \dfrac{1}{2} \times 17{,}4 \times 20{,}7 \times \sin 63°$ $= 160{,}5 \text{ units}^2$	
2.	PQ is a vertical flagpole of length *x* metres, with Q at the foot of the flagpole. R, Q and S are three points on the same horizontal surface. If RQ = RS, $Q\hat{S}R = \alpha$ and $P\hat{S}Q = \theta$	

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	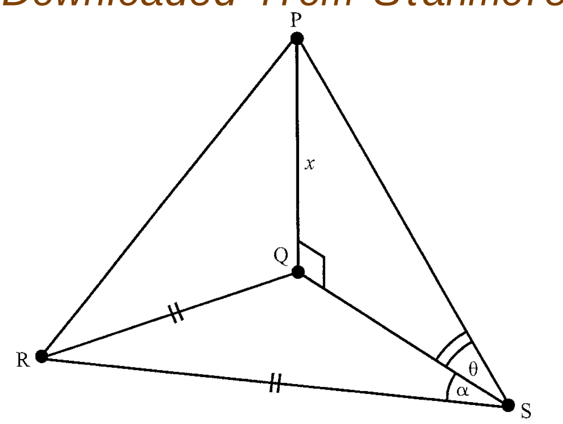
2.1	Show that: $QS = \dfrac{x}{\tan\theta}$
2.2	Prove that: $RS = \dfrac{x}{2\tan\theta\cos\alpha}$
2.3	If $\theta = 45°$ and $\alpha = 60°$ and $x = 4$ metres, calculate the length of RS.
Solutions:	
2.1	$\tan\theta = \dfrac{x}{QS}$ $\therefore QS = \dfrac{x}{\tan\theta}$
2.2	$R\hat{Q}S = \alpha$ (angles opp = sides) $\dfrac{\sin\alpha}{RQ} = \dfrac{\sin(180° - 2\alpha)}{QS}$ $RQ.\sin(180° - 2\alpha) = QS.\sin\alpha$ but $RQ = RS$ $\therefore RS = \dfrac{QS.\sin\alpha}{2\sin\alpha\cos\alpha}$ $= \dfrac{\frac{x}{\tan\theta} \times \sin\alpha}{2\sin\alpha\cos\alpha}$ $= \dfrac{x}{2\tan\theta\cos\alpha}$
	$RS = \dfrac{4}{2 \times \tan 45° \times \cos 60°}$ $= 4\,\text{metres}$
	ACTIVITIES/ASSESMENT
5.10.1	In the figure, A, B and C are three points in the same horizontal plane. DA is perpendicular to the horizontal plane at A and D is joined to C. $AB = \dfrac{1}{2}BC = a$ and $A\hat{C}D = \dfrac{1}{2}A\hat{B}C = \alpha$

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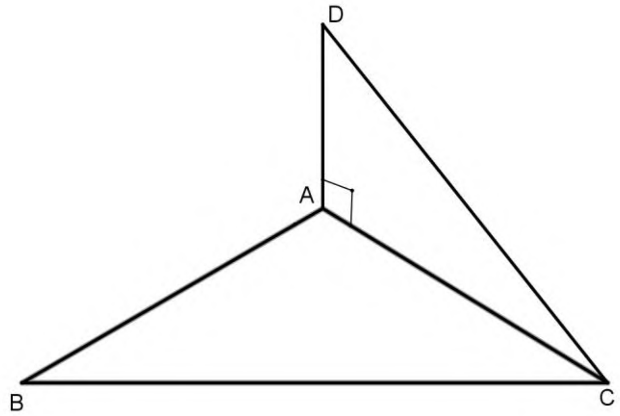
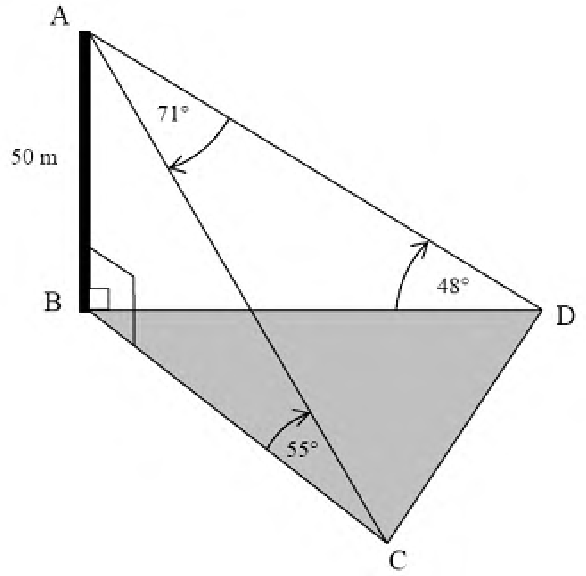


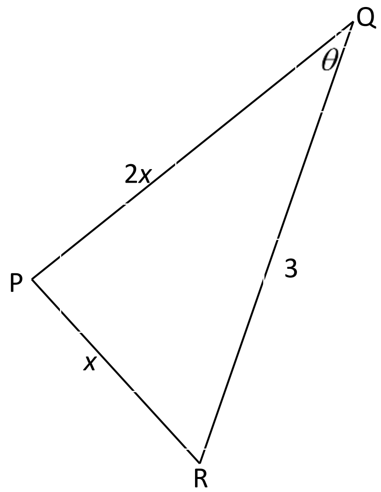
	(a)	Show that $AD = a.\tan\alpha\sqrt{1+8\sin^2\alpha}$
	(b)	Hence calculate the value of AD if $a = 89$ mm and $35°$ (Round the answer off to one decimal digit.)

5.10.2 In the diagram below Paul is standing at point A on the top of the building AB that is 50m high. He observed two cars, C and D that are in the same horizontal plane as B. The angle of elevation of C to A is $55°$ and the angle of elevation from D to A is $48°$. $C\hat{A}D = 71°$.



	(a)	Calculate the lengths of AC and AD, correct to 1 decimal place.	
	(b)	Calculate the distance between the two cars.	
	(c)	Calculate the area of ΔACD.	

5.10.3.1 In Δ PQR, QR = 3 units, PR = x units, PQ = $2x$ units and $P\hat{Q}R = \theta$ (Gr 11 P2 Nov 2017;DBE)



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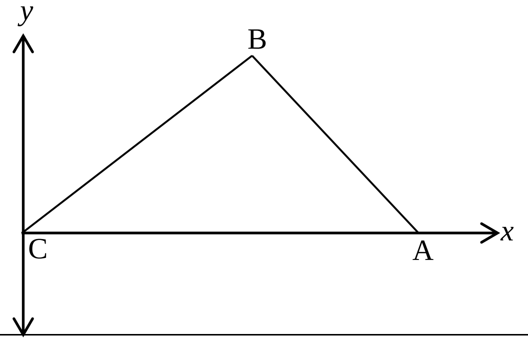
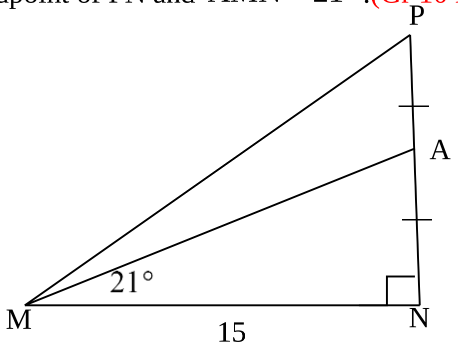
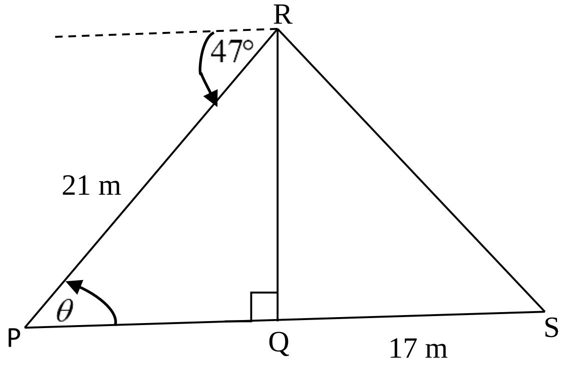
	Show that $\cos\theta = \dfrac{x^2+3}{4x}$		
5.10.3.2		If $x = 2{,}4$ units:	
	(a)	Calculate θ	
	(b)	Calculate the area of Δ PQR	
	(c)	Calculate the value of x for which the triangle exists.	
	In the figure below, acute-angled Δ ABC is drawn having C at the origin. (Gr 11 Nov P2 2016; DBE) 		
5.10.4	(a)	Prove that $c^2 = a^2 + b^2 - 2ab\cos C$	
	(b)	Hence, deduce that $1+\cos C = \dfrac{(a+b+c)(a+b-c)}{2ab}$	
5.10.5	In the sketch below, Δ MNP is drawn having a right angle at N and MN = 15units. A is the midpoint of PN and $\text{A}\hat{\text{M}}\text{N} = 21°$.(Gr 10 Nov P2 2016; DBE)  Calculate:		
	(a)	AN	
	(b)	$\text{A}\hat{\text{M}}\text{N}$	
	(c)	MP	
5.10.6	RQ is a vertical pole. The foot of the pole, Q is on the same horizontal plane as P and S. The pole is anchored with wire cables RS and RP. The angle of depression from the top of the pole to the point P is $47°$. PR is 21 *m* and QS is 17 *m*. $\text{R}\hat{\text{P}}\text{Q} = \theta$.(Gr 10 Nov P2 2017; DBE) 		
	(a)	Write down the size of θ .	

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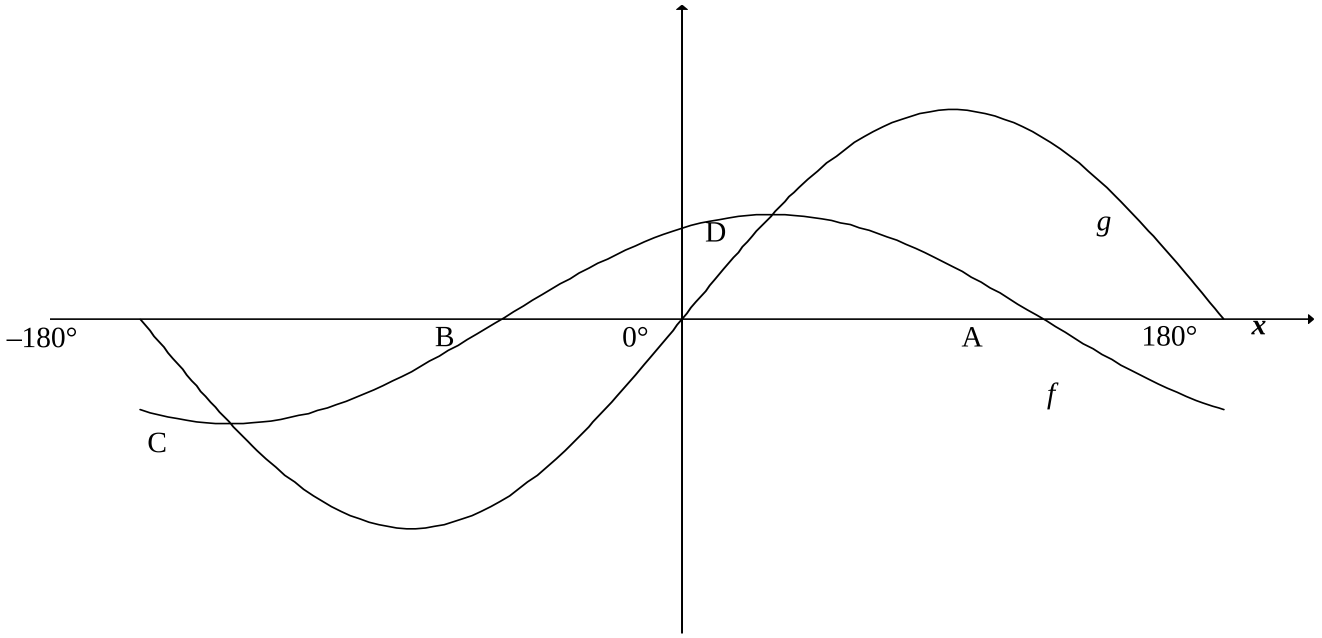
	(b)	Calculate the length of RQ.	
	(c)	Hence, calculate the size of $\hat{S}$	
	(d)	If P, Q and S lie in a straight line, how far apart the anchors of the wire cables?	

DBE May/June 2019

5.10.7 Determine the general solution of $\cos(x-30°)=2\sin x$.

5.10.8 In the diagram, the graphs of $f(x)=\cos(x-30°)$ and $g(x)=2\sin x$ are drawn for the interval $x\in[-180°;180°]$. A and B are the x–intercepts of f. The two graphs intersect at C and D, the minimum and maximum turning points respectively of f.



(a)	Write down the coordinates of:		
	(a)	A	(1)
	(b)	C	(2)

(b)	Determine the values of x in the interval $x\in[-180°;180°]$, for which:		
	(a)	Both graphs are increasing	(2)
	(b)	$f(x+10°)>g(x+10°)$	(2)
(c)	Determine the range of $y=2^{2\sin x+3}$		(5)
			[18]

In the diagram below, CGFB and CGHD are fixed walls that are rectangular in shape and vertical to the horizontal plane FGH. Steel poles erected along FB and HD extend to A and E respectively. ΔACE forms the roof of an entertainment centre.

BC = x, CD = $x+2$, $B\hat{A}C=\theta$, $A\hat{C}E=2\theta$ and $E\hat{C}D=60°$

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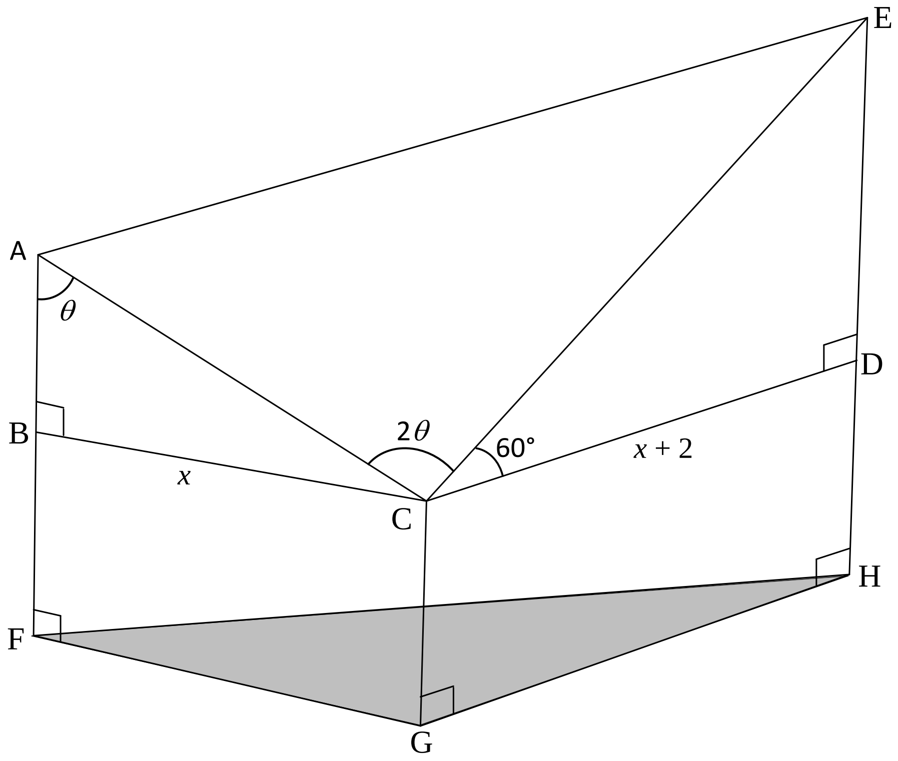

5.10.9.1	Calculate the length of:		
(a)	AC in terms of x and θ		(2)
(b)	CE in terms of x		(2)

5.10.9.2	Show that the area of the roof ΔACE is given by $2x(x+2)\cos\theta$.	(3)

5.10.9.3	If $\theta = 55°$ and BC = 12 metres, calculate the length of AE.	(4)
		[11]

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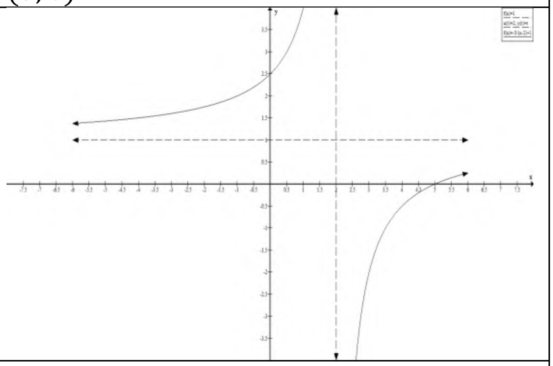
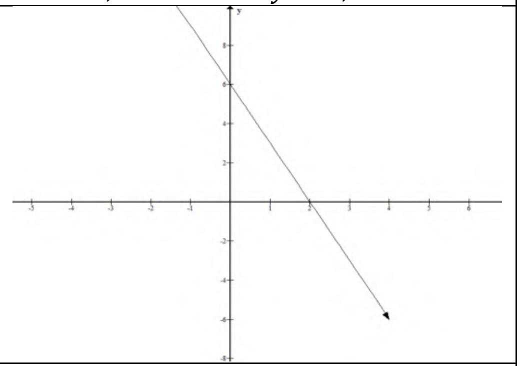
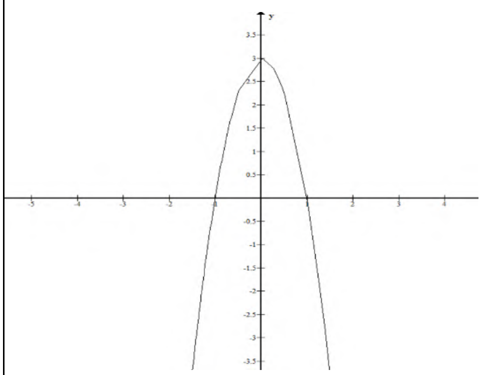
SOLUTIONS

TOPIC : FUNCTIONS TOPIC: Revision of grade 11 function

No.	SOLUTION
1.1.1	$p = -3, q = -2$
1.1.2	$a = -2$
1.1.3	$\left(0; -\frac{4}{3}\right)$
1.1.4	$x = 1$
1.1.5	$c = 1$
1.1.6	$\left(0; \frac{5}{2}\right)$
1.1.7	$(5; 0)$
1.1.8	
1.1.9	$y \in R;\ y \neq 1$
1.1.10	$h(x) = \frac{-3}{x-5} - 3$
1.1.11	From translation: $h(x) \leq -4$ $\therefore f(x) \leq 0$ (4 units up) If $f(x) \leq 0$, then $-2 < x \leq 5$ $\therefore$ for $h(x)$: $5 < x \leq 8$ (3 units to the right)
1.1.12	The asymptotes are $x = 1$ *and* $y = 3$

SUB TOPIC: Revision of grade 11 function

No.	SOLUTION
1.2.1	$x = -1$
1.2.2	$(-1; -8)$
1.2.3	4 units
1.2.4	$y = 2x^2 - 4x - 6$
1.2.5	$x < -3$ *or* $0 < x < 1$
1.2.6	The graph shifted by 3 units to the right

No.	SOLUTION
1.3.1	One-to-one
1.3.2	Many-to-one
1.3.3	One-to-many
1.3.4	One-to-one
1.3.5	Many-to-one
1.3.6	9,6 and 0
1.3.7	$x = -1, 0$ *and* 2 and $y = 0, 6$ *and* 9
1.3.8	
1.3.9	One-to-one
1.3.10	Linear
1.3.11	0, 3, 0
1.3.12	$x = -1, 0$ *and* 1 and $y = 0$ *and* 3
1.3.13	
1.3.14	Many-to-one
1.3.15	Quadratic function

SUB TOPIC: Inverse of one-to-one function

No.	SOLUTION
1.4.1	$Q\left(\frac{3}{2}; 0\right)$
1.4.2	$-7 \leq x \leq 5$

SUB TOPIC: Inverse of a many-to-one function

No.	SOLUTION
1.5.1	Yes. One–to-one relation
1.5.2	$R(-12; -6)$

SUB TOPIC: Inverse of exponential function

No.	SOLUTION
1.6.1	$C(0; 8)$
1.6.2	$g(x) = 2^{x+1} + 8$
1.6.3	$y > 8$ OR $y \in (8; \infty)$

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1.5.3	$a = -\frac{1}{3}$
1.5.4	$y = -\sqrt{-3x}\,;\ x \leq 0$
1.5.5	$v(x)$ is not a function, there are two different y values for each x. A vertical line test fails: line cuts the graph at more than one point.
1.5.6	$y \geq 0\ or\ y \leq 0$
1.5.7.1	$y < 0$
1.5.7.2	$0 < x \leq \frac{49}{100}$
1.5.8	$y = x^2;\ \ x \geq 0$ $y = x^2;\ \ x \leq 0$
1.5.9	$0 < y < 1$ $-1 < y < 0$

1.6.4	$D'(-1;7)$
1.6.5	Reflection about the $x - axis$, and a translation of 1 unit left and 18 units up. OR Reflection about the line $y = 9$ and a translation of 1 unit left.
1.6.6	$y = \log_{\frac{1}{3}} x$ **OR** $y = -\log_3 x$ **OR** $y = \log_3 \frac{1}{x}$
1.6.7	$y = \left(\frac{1}{3}\right)^x$ is a decreasing function ∴ the bigger the x −values the smaller the y −value maximum value of $f\ = 9$ minimum value: $y = \left(\frac{1}{3}\right)^{9-5}$ $y = \left(\frac{1}{3}\right)^{4}$ $y = \frac{1}{81}$

SOLUTIONS

TOPIC : GRADE 12 FINANCE

SUB TOPIC : simple and compound interest

No.2.1	SOLUTION
a)	$n = 3.09 \approx 4 years$
b)	P = R15 000
c)	P = R6000
d)	$i = 12\%$
e)	P = R 4890.41

SUB TOPIC : Compounding period

No.2.3	SOLUTION
a)	A = R6581.03
b)	Monthly
c)	$i = 8\%$
d)	A= R8728.07

SUB TOPIC : nominal and effective interest rates

No.2.5	SOLUTION
(a) i	$i = 10.25\%$
(a) ii	$i = 10.17\%$
(a) iii	$i = 10.04\%$
(b)	$i = 11.22\%$

SUB TOPIC : FUTURE VALUE

No.2.7	SOLUTION
(a)i	A = R15282.91
(a)ii	A = R17739.71

SUB TOPIC : simple and compound interest(decay

No.2.2	SOLUTION
a)	A = R 4915.20
b)	P = R122040.87
c)	$i = 18\%$

SUB TOPIC : Time line

No.2.4	SOLUTION
a)	A = R10684.96
b)	A = R73762.19
c)	A = R296977.00

SUB TOPIC : calculating n using logarithm

No.2.6	SOLUTION
a)	$n = 14.69 \approx 15 years$
b)	$n = 14.69 \approx 15 years$
c)	$n\ = 8\ years\ and\ 6\ months$
d)	$n = 2.37 years$
e)	$i = 12.12\%$

SUB TOPIC : Present value annuity

No.2.8	SOLUTION
2.8.1(a)	Selling price =R850000
2.8.1(b)	$x = R6729.95$
2.8.1(c)	A = R867188

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SUB TOPIC : Present value annuity continue

No.	SOLUTION
2.9.1(a)	A = R256289.06
2.9.1(b)	$x = R7359.79$
2.9.1(c)	$n = 10 monthssooner$
2.9.1(d)	OB = R3735.45 Final payment= R3782.14
2.9.2(a)	$n = 13.11686841 \times 12 = 158\, payments$
2.9.2(b)	R162503.51

2.8.1(d)	$85^{th} = R615509.74$
2.8.2(a)	$i = 7.95\% \rightarrow \therefore R12499.96$
2.8.2(b)	$OB = R885813.38$
2.8.3(a)	$F_v = R1674501.44$
2.8.3(b)	$n = 75.4 \approx 75\, fullmonths$
2.8.3(c)	$P_v = R1319260.60$

SUB TOPIC : Present value annuity continue

No.2.10	SOLUTION
2.10.1(a)	$x = R11986.33$
2.10.1(b)	OB = R816048.67
2.10.1(c)	$x = R12711.51$
2.10.2(a)	Loan = R793748.94
2.10.2(b)	$x = R8089.20$

SUB TOPIC : Sinking fund

No.2.11	SOLUTION
2.11.1(a)	$A = R567092.25$
2.11.1(b)	$i = 18.02\%$
2.11.1(c)	$x = R6031.89$
2.11.2(a)	A = R74883.86
2.11.2(b)	A = R168305.21
2.11.2(b)	$x = R1184.68$

SUB TOPIC : Comparing investments and loan

No.2.12	SOLUTION
2.12.1(a)	Kuda will have better investment
2.12.1(b)	A = R133929.25
2.12.1(c)1	$x = R3636.36$
2.12.2(2)2	Total payment =R196363.66

SOLUTIONS

TOPIC : DATA HANDLING

SUB TOPIC : Measures of central tendency and Dispersion

No.	SOLUTION
3.1.1	a) 6.73 b) 7 c) 19
3.1.2	a) 80 b) 150 c) Skewed to the left

SUB TOPIC: Variance and Standard deviation

No.	SOLUTION
3.2.1	a) Variance=256.10 b) Standard Dev = 16 c) Increase by 15 d) No effect
3.2.2	a) 18.83 b) 5 days

SUB TOPIC: Measures of central tendency for grouped data

No.	SOLUTION
3.3.1	a) i. $20 \leq x < 30$ ii. 35.4

	iii. 35 b) i. The modal interval is the most common age category ii. The mean tells us that if all the ages are added together and shared out equally, each person would be approximately 35,4 years old. iii. 50% of the people in this group are younger than or equal to 35 years. 50% of the peopleare older than or equal to 35 years.
3.3.2	a) i. $0 \leq x < 50000$ ii.R55343.51 iii. R25 000

SUB TOPIC: Histograms and Frequency Polygons

No.	SOLUTION
3.4.1	a)

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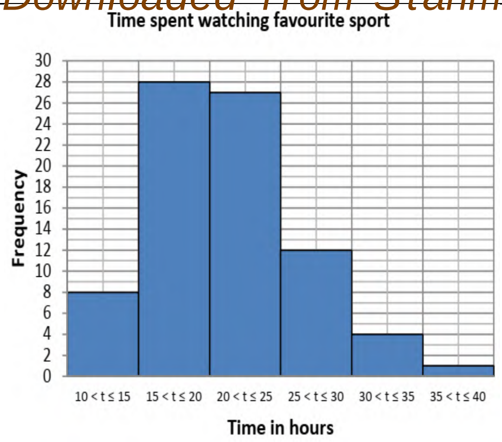
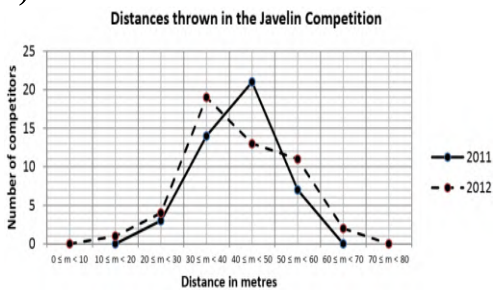
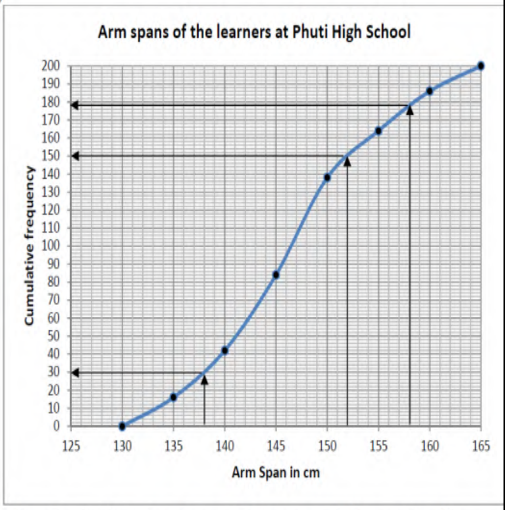
No.	SOLUTION
	 b) i. $15 < t \leq 20$ ii. 22.5 hours
3.4.2	a)  b) there were more people in 2012 than in 2011 who threw very far.

SUB TOPIC : Orgives

No.	SOLUTION
3.5.1	a) *Don't forget to add in another interval with a frequency of 0.*

Arm span in cm	Frequency	Cumulative Frequency	Ordered Pairs
$125 < h \leq 130$	0	0	(130; 0)
$130 < h \leq 135$	16	16	(135; 16)
$135 < h \leq 140$	26	42	(140; 42)
$140 < h \leq 145$	42	84	(145; 84)
$145 < h \leq 150$	54	138	(150; 138)
$150 < h \leq 155$	26	164	(155; 164)
$155 < h \leq 160$	22	186	(160; 186)
$160 < h \leq 165$	14	200	(165; 200)

b)



c) From the ogive, the number of learners with an arm span of 152 cm or less = 150

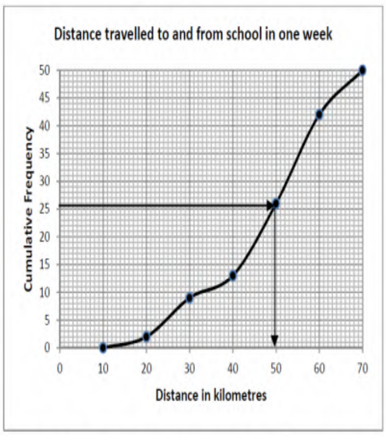
d) Approximately 178 learners are 158 cm or less.
Approximately 30 learners are 138 cm or less.
Number of learners between 138 cm and 158 cm ≈ 178 – 30 = 148

SUB TOPIC: Ogives

No.	SOLUTION
3.6.1	a) *Don't forget to add in another interval with a frequency of 0.*

Number of kilometres	Number of learners	Cumulative frequency	Ordered pairs
$0 < x \leq 10$	0	0	(10; 0)
$10 < x \leq 20$	2	2	(20; 2)
$20 < x \leq 30$	7	9	(30; 9)
$30 < x \leq 40$	4	13	(40; 13)
$40 < x \leq 50$	13	26	(50; 26)
$50 < x \leq 60$	16	42	(60; 42)
$60 < x \leq 70$	8	50	(70; 50)
	TOTAL = 50		

b)



c) 50 learners travelled by car. The median lies between the 25th learner and the 26th learner.
From the graph we can see that the median ≈ 50 km per week.

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1.6.2

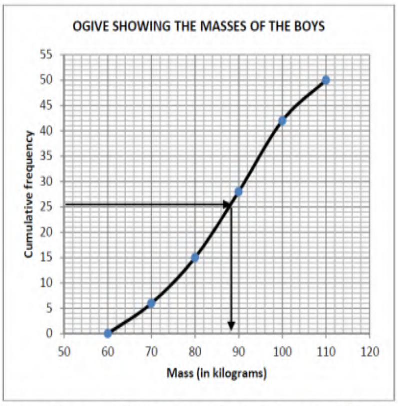
a) *Don't forget to add in another interval with a frequency of 0.*

	Frequency	Cumulative Frequency	Ordered pairs
$20 < x \leq 30$	0	0	(30; 0)
$30 < x \leq 40$	12	12	(40; 12)
$40 < x \leq 50$	18	30	(50; 30)
$50 < x \leq 60$	55	85	(60; 85)
$60 < x \leq 70$	57	142	(70; 142)
$70 < x \leq 80$	43	185	(80; 185)
$80 < x \leq 90$	11	196	(90; 196)
$90 < x \leq 100$	4	200	(100; 200)

1.6.3

a) From the graph:
Approximately 28 boys had a mass of 90 kg or less
Approximately 42 boys had a mass of 100 kg or less
Number of boys with a mass between 90 and 100 kg = 42 – 28 = 14

b)



The graph shows the mass of 50 boys. The median lies between the 25th and 26th boys.

The median ≈ 88 kg

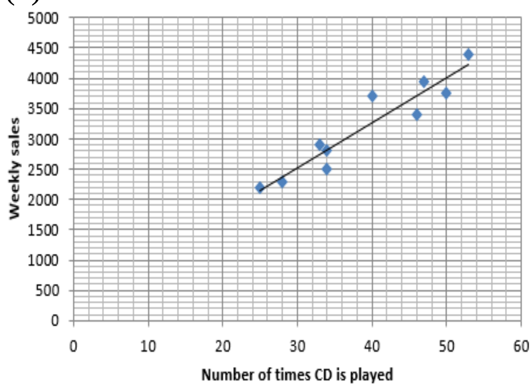
c) From the graph: Number of boys with a mass less than 80 kg ≈ 15

SUB TOPIC: Scatter Plot and Regression Line

No.	SOLUTION
3.7.1	Answers to activity

(a) Number of times CD is played

(b)



(c) $y = 293.06 + 74.28x$

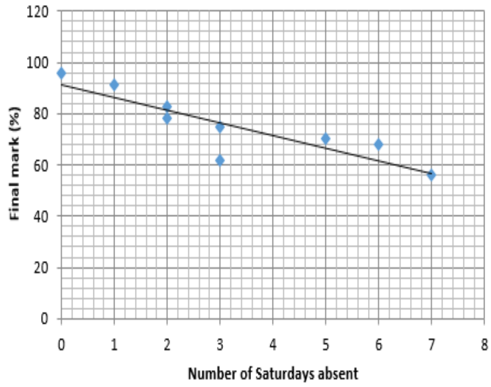
(d) 32 times.

SUB TOPIC: Correlation Coefficient

No.	SOLUTION
3.8.1	Answers to Activity

a) $y = 91{,}27 - 4{,}91x$

b)



c) Strong negative ($r = -0{,}87$) correlation

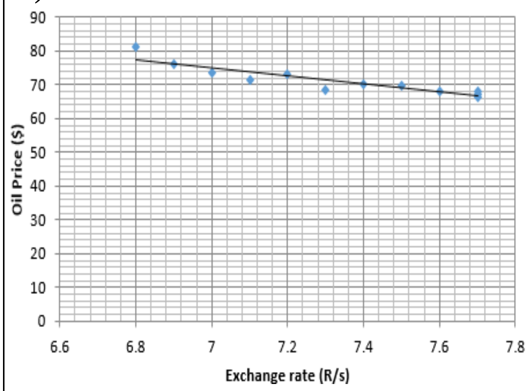
d) 72 %

SUB TOPIC: Exam type question

No.	SOLUTION
3.9.1	a)

$y = 158.67 - 11.96x$

b)



c) $r = -0.91$

d) Strong and negative correlation

e) $\bar{Y} = 71.05$

f) $\sigma_x = 0.31$

g) In December

3.9.2

a) 251 km/h

b)(i) $r = 0{,}52$ or C

(ii) The points are fairly scattered and the least squares regression line is increasing

c) There is a weak positive correlation, therefore there is no conclusive evidence that the height of a player will influence his/her tennis serve speed.

d) For (0 ; 27,07) it means that the player has a height of 0 m, but can

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	serve at a speed of 27,07 km/h – which is impossible.

TOPIC : PROBABILITY

SUB TOPIC: Fundamental Counting Principle Activity 4.6.1

No.	SOLUTION
1.	$4\times5\times3=60$
2.(a)	$7\times7\times7\times7\times7\times7\times7=7^7=823543$
2.(b)	$7!=5040$
2.(c)	$7!=5040$
3.(a)	$6^6=46656$
3.(b)	$6!=720$
4.	$2\times2\times2\times2\times2\times2=2^6=64$
5.(a)	$10\times10\times10\times26\times26=676000$
5.(b)	$10\times9\times8\times26\times25=468000$
5.(c)	$9\times9\times9\times21\times21=321489$
5.(d)	$9\times8\times7\times21\times20=211680$
5.(e)	$4\times4\times4\times16=16384$
6.(a)	$4\times3=12$
6.(b)	$4\times3\times4\times3=144$

SUB TOPIC: Arranging objects in a row Activity 4.8.1

No.	SOLUTION
3.(a)	9!
3.(b)	80 640
3.(c)	5 760
3.(d)	17 280
3.(e)	14 400
3.(f)	2 880
4.(a)	5 040
4.(b)	288
4.(c)	576
5.(a)	12!
5.(b) 7.	165 841 472

SUB TOPIC: Factorial notation Activity 4.7.1

No.	SOLUTION
1.(a)	19 958 400
1.(b)	360 360
2.	720
4.(a)	823 543
4.(b)	5 040
6.(a)	6 561
6.(b)	3 024

SUB TOPIC: Word arrangements with identical letters Activity 4.9.1

No.	SOLUTION
1.(a)	5 040
1.(b)	2 520
1.(c)	120
1.(d)	60
3.(a)	10!
3.(b)	1 814 400
3.(c)	120
3.(d)	362 880
3.(e)	181 440
5.(a)	6
5.(b)	15
5.(c)	20

SUB TOPIC : Activity 4.10

No.	SOLUTION
1.(a)	$\frac{1}{30}$
1.(b)	$\frac{1}{6}$
3.(a)	$\frac{1}{189}$
3.(b)	$\frac{400}{3087}$
3.(c)	$\frac{217}{729}$
4.	$\frac{14}{45}$
5.(a)	$\frac{1}{42}$
5.(b)	$\frac{2}{7}$
8.(a)	$\frac{1}{11550}$
8.(b)	$\frac{1}{22}$
9.(a)	$\frac{7}{27}$
9.(b)	$\frac{1}{1716}$
9.(c)	$\frac{2}{13}$
9.(d)	$\frac{2}{429}$

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SOLUTIONS

TOPIC : FUNCTIONS

SUB TOPIC:

No.	SOLUTION
5.1.1	$\sin\theta$
5.1.2	$\sin x$
5.1.3	$\frac{1}{\cos x}$

SUB TOPIC:

No.	SOLUTION
5.2.1	$1+\sin\theta$
5.2.2	1
5.2.3	$\cos 2x$
5.2.4	-1
5.2.5	-1
5.2.6	$-\tan\beta$

SUB TOPIC:

No.	SOLUTION
5.3.1	$-\frac{\sqrt{3}}{2}$
5.3.2	4
5.3.3	$-\frac{1}{\sqrt{2}\sin 10^\circ}$
5.3.4	$-\frac{5}{2}$
5.3.5	$\frac{1}{\sqrt{3}}$
5.3.6	$\frac{2}{\sqrt{3}}$

SUB TOPIC: Inverse of one-to-one function

No.		SOLUTION
5.4.1	*(a)*	$r = 6$
	(b)	$\frac{\sqrt{3}}{2}$
5.4.2	*(a)*	$-\frac{2}{\sqrt{13}}$
	(b)	$\frac{12}{13}$
5.4.3	*(a)*	$-\frac{15}{8}$
	(b)	$-\frac{8}{15}$
5.4.4	*(a)*	$-\frac{3}{13}$
	(b)	$\frac{144}{169}$
	(c)	$-\frac{5}{12}$
5.4.5		1
5.4.6	*(a)*	$-\frac{3}{4}$
	(b)	$-\frac{3}{5}$
5.4.7		$\frac{\sqrt{2}(\sqrt{1-t^2}+t)}{2}$
5.4.8		$-\frac{119}{169}$
5.4.9	(a)	a
	(b)	$\sqrt{1-a^2}$
	(c)	$1-2a^2$
5.4.10	*(a)*	$\frac{k}{6}$
	(b)	$k = 3$
5.4.11	*(a)*	$\sqrt{1-p^2}$
	(b)	$1-2p^2$
5.4.12	*(a)*	t
	(b)	$-\sqrt{1-t^2}$
	(c)	$\frac{1}{t}$
5.4.13	*(a)*	1
	(b)	30°
	(c)	$Q(10\sqrt{3};10)$
5.4.14	*(a)*	$2\sqrt{66}$
	(b)	$\alpha = 17.10$
	(c)	$-\frac{2\sqrt{66}}{17}$
	(d)	$0{,}56$
5.4.15	*(a)*	$\frac{a}{\sqrt{1-a^2}}$
	(b)	$\sqrt{1-a^2}$
	(c)	1

SUB TOPIC: Inverse of a many-to-one function

No.		SOLUTION
5.6.1	*(a)*	$x = 53{,}1^\circ$
	(b)	$x = 25^\circ$
5.6.2		$\theta = 36{,}87^\circ$

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SUB TOPIC: Inverse of exponential function

No.	SOLUTION
5.7.1	$x = 49{,}7^{\circ}$ *or* $x = 229{,}7^{\circ}$
5.7.2	$x = 30^{\circ} + k.360^{\circ}$ *or* $x = 150^{\circ} + k.360^{\circ}, k \in Z$
5.7.3	$x = 0^{\circ} + k.360^{\circ}$ *or* $x = 180^{\circ} + k.360^{\circ}, k \in Z$ $x = 48{,}2^{\circ} + k.360^{\circ}$ *or* $x = 131{,}8^{\circ} + k.360^{\circ}, k \in Z$
5.7.4	$x = 80^{\circ} + k.240^{\circ}$ *or* $x = 240^{\circ} + k.720^{\circ}, k \in Z$
5.7.5	$x = 221{,}8^{\circ} + k.360^{\circ}$ *or* $x = 318{,}2^{\circ} + k.360^{\circ}, k \in Z$

SUB TOPIC: Inverse of exponential function

No.		SOLUTION
5.8.1	*(a)*	$A = 315^{\circ}$
	(b)	$A = 495^{\circ};\ 675^{\circ}$
5.8.2		$\theta = 87{,}47^{\circ} + k.360^{\circ}$ $\theta = 228{,}53^{\circ} + k.360^{\circ}$, $k \in Z$
5.8.3		$\theta = -300^{\circ}; -60^{\circ}; 60^{\circ}; 300^{\circ}$
5.8.4		$\theta = 71{,}57^{\circ} + k.180^{\circ}$ $\theta = 251{,}57^{\circ} + k.180^{\circ}$, $k \in Z$
5.8.5		$\alpha = 0^{\circ} + k.360^{\circ}$ $\alpha = \pm 60^{\circ} + k.360^{\circ}$, $k \in Z$
5.8.6		$A = 26{,}57^{\circ} + k.180^{\circ}$ $A = 63{,}43^{\circ} + k.180^{\circ}$, $k \in Z$