KWAZULU-NATAL PROVINCE

EDUCATION
REPUBLIC OF SOUTH AFRICA

**CURRICULUM GRADE 10 -12
DIRECTORATE**

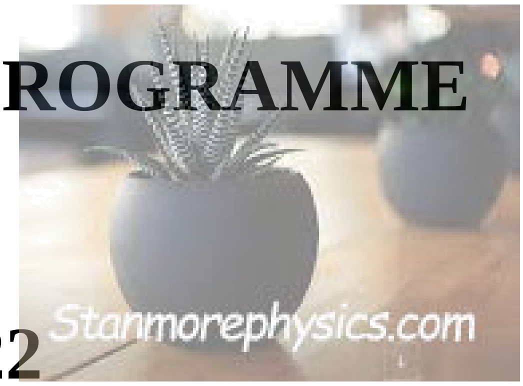
NCS (CAPS)

LEARNER SUPPORT DOCUMENT

GRADE 12

MATHEMATICS

STEP AHEAD PROGRAMME



2022

This document has been compiled by the KZN FET Mathematics Subject Advisors.

PREFACE

This support document serves to assist Mathematics learners on how to deal with curriculum gaps and learning losses as a result of the impact of COVID-19 since 2020. It also captures the challenging topics in the Grade 10 – 12 work. The exercises should be used in conjunction with the 2022 Recovery Annual Teaching Plans. Activities should serve as a guide on how to assess topics dealt with in this document. It will cover the following:

TABLE OF CONTENTS

TOPICS		PAGE NUMBERS
1.	**NUMBER PATTERNS**	**3 - 29**
2.	**EUCLIDEAN GEOMETRY**	**30 - 60**
3.	**TRIGONOMETRY**	**61 - 93**

TOPIC: PATTERNS, SEQUENCES AND SERIES					
LESSON 1: Grade 11 Revision					
Term	1	**Week**	1	**Grade**	12
Duration	1HR	**Weighting**	25/150 Marks	**Date**	
Sub-topics	Quadratic sequences				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Pattern: a repetitive regular arrangement of things.
- Sequence: the order in which related number/things follow one another.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Linear sequence
- Solving quadratic equations

NOTES

- What a quadratic sequence is?
  - Quadratic number sequences are sequences whose second difference is constant, and the first differences form a linear sequence.
  - Their general term (T_n) is given in the form $T_n = an^2 + bn + c$.
- How the general term of a quadratic sequence is derived?
  - In general, the n-th term of a quadratic number pattern is given by $T_n = an^2 + bn + c$.

    If we use this formula to calculate the first four terms, we get:

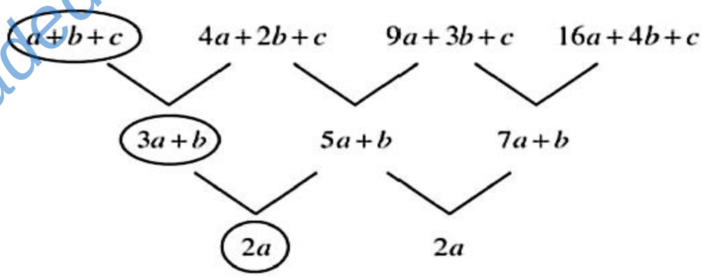
    $T_1 = a(1)^2 + b(1) + c = a + b + c$

    $T_2 = a(2)^2 + b(2) + c = 4a + 2b + c$

    $T_3 = a(3)^2 + b(3) + c = 9a + 3b + c$

    $T_4 = a(4)^2 + b(4) + c = 16a + 4b + c$

    Let us calculate the first and second differences between the terms:



Note that the first term is $a+b+c$, the first of the first differences is $3a+b$, and the constant second difference is $2a$.

In a **quadratic** number pattern:

First term $= a + b + c$

First of the first differences $= 3a + b$

Second difference $= 2a$

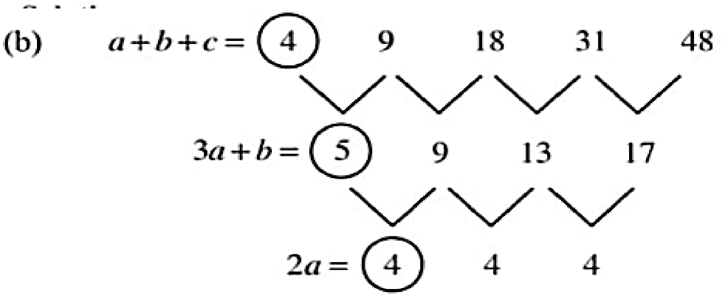
General term: $T_n = an^2 + bn + c$

- It is also worth noting that quadratic sequences will have a minimum or maximum value, since they assume the shape of a parabola when sketched.

- Worked examples:

1. Given the quadratic number pattern: 4; 9; 18; 31; …
   (a) Determine the next term (T_5). (b) Determine the n-th term (general term).

(b)



$2a=4$ $\therefore a=2$

$3a+b=5$

$\therefore 3(2)+b=5$ $\therefore b=-1$

$a+b+c=4$

$\therefore 2+(-1)+c=4$ $\therefore c=3$

$T_n=an^2+bn+c$

$\therefore T_n=2n^2-n+3$

remains constant

$T_5=48$

2. Determine the maximum value of a quadratic sequence given by $T_n = -n^2 + 26n - 170$.
   Solution:
   By completing a square, $T_n = -n^2 + 26n - 170 = -(n-13)^2 - 1$, which implies that the maximum value of the sequence is -1.

   Also, by differentiation, the maximum term of the sequence can be determined, i.e.
   From $T_n = -n^2 + 26n - 170$
   $\Rightarrow$ $T'_n = -2n + 26$
   At maximum $-2n + 26 = 0$
   Thus, $n = 13$, which means that T_{13} is the maximum term.
   Finally, $T_{13} = -(13)^2 + 26(13) - 170 = -1$

ACTIVITIES/ASSESSMENTS

- **Classwork/Homework**

1. Consider the following number pattern: 6 ; 13 ; 22 ; 33 ; …
   1.1 Write down the next term of the number pattern.
   1.2 Determine the n^{th} of the sequence.
   1.3 For what value/s of n is T_n equal to 2 497?

2. Given the quadratic pattern: -9 ; -6 ; 1 ; x ; 27 ; . .
   2.1 Calculate the value of x.
   2.2 Determine the general term of the pattern.
   2.3 Determine T_{100}.
   2.4 Which term in the sequence will be equal to 397?
3. Consider the quadratic number pattern: -145 ; -122 ; -101 ; . . .
   3.1 Write down the next term of the sequence.
   3.2 Show that the general term of the pattern is $T_n = -n^2 + 26n - 170$.
   3.3 Between which two terms of the quadratic pattern will there be a difference of -121?

3.4 What value must be added to each term of the number pattern so that the value of the maximum term in the new number pattern thus formed will be 1.

TOPIC: PATTERNS, SEQUENCES AND SERIES

LESSON 2: Arithmetic (Linear) Sequences

Term	1	**Week**	1	**Grade**	12
Duration	1 Hour	**Weighting**	25/150	**Date**	
Sub-topics	Arithmetic Sequences				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Pattern: a repetitive regular arrangement of things.
- Sequence (progression): the order in which related number/things follow one another.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Linear sequence
- Solving linear equations

NOTES

- What an arithmetic sequence is?
  - Arithmetic number sequences are sequences in which the difference between consecutive(successive) terms is constant.
- How the general term of an arithmetic sequence is derived?
  - Algebraically, an arithmetic sequence is written as follows:

    $T_1 = a$

    $T_2 = a + d$

    $T_3 = a + d + d = a + 2d$

    $T_4 = a + d + d + d = a + 3d$

    $T_5 = a + d + d + d + d = a + 4d = a + (5-1)d$

    $T_6 = a + d + d + d + d + d = a + 5d = a + (6-1)d$

    .

    .

    .

    $T_n = a + (n-1)d$, the general term, where:

    T_n is term number n ($\mathbb{N}$ *only*) in the sequence,

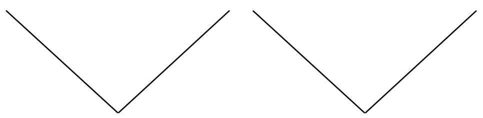
    a is the first term,

    $d = T_n - T_{n-1}$ is the constant difference between successive terms, and,

    n is the position of the term in the sequence, which can only be a natural number.

- Since the difference between terms is constant, it can be concluded that if three consecutive terms are given, as shown below:

T_1 ; T_2 ; T_3 ; . . .



$T_2 - T_1$; $T_3 - T_2$

Then, it follows that $T_2 - T_1 = T_3 - T_2$. This can be applied to any three successive terms of any arithmetic sequence.

- The general term, T_n, can also be used to determine the value of n, the position of any term in an arithmetic sequence.

- Worked examples:

1. Consider the arithmetic sequence: x ; $4x + 5$; $10x - 5$; . . .

   1.1 Determine the value of x.

   1.2 Write the numerical values of the first three terms of the sequence.

   1.3 Determine the general term of the sequence.

   1.4 What is the twentieth term of the sequence?

   1.5 Which term of the sequence will be equal to 1945.

Solutions:

1.1 $T_2 - T_1 = T_3 - T_2$

$4x + 5 - x = 10x - 5 - (4x + 5)$

$x = 5$

1.2 $T_1 = 5$
$T_2 = 4(5) + 5 = 25$
$T_3 = 10(5) - 5 = 45$

1.3 $T_n = a + (n-1)d$
$T_n = 5 + 20(n-1)$
$T_n = 20n - 15$

1.4 $T_{20} = 20(20) - 15 = 385$

1.5 $1945 = 20n - 15$
$20n = 1960$
$n = 98$

ACTIVITIES/ASSESSMENTS

- **Classwork/Homework**

1. p ; $2p + 2$; $5p + 3$; ... are the first three terms of an arithmetic sequence.

   (a) Calculate the value of p.

   (b) Determine the sequence.

   (c) Find the 49th term.

   (d) Which term of the sequence is 100.5?

2. Determine an expression for the n^{th} term for an arithmetic sequence whose 6^{th} term is 13 and

the 14^{th} term is 33.

3. Consider the following arithmetic sequence: $-11 \ ; \ 2\sin x \ ; \ 15 \ ; \ . \ . \ .$
   Determine the values of x in the interval $[0° ; 90°]$ for which the sequence will be arithmetic.

TOPIC: PATTERNS, SEQUENCES AND SERIES					
LESSON 3: Geometric Sequences					
Term	1	**Week**	1	**Grade**	12
Duration	60 minutes	**Weighting**	25	**Date**	
Sub-topics	Geometric Sequences				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Pattern: a repetitive regular arrangement of things.
- Sequence (progression): the order in which related number/things follow one another.
- Geometric: something that relates to geometric shapes or figures.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Laws of exponents
- Solving exponential equations

NOTES

- What a geometric sequence is?
  - Geometric number sequences are sequences in which the ratio between consecutive(successive) terms is common.
- How the general term of a geometric sequence is derived?
  - The general term for a geometric sequence can be derived as follows:

$T_1 = a$
$T_2 = a \times r = ar$
$T_3 = a \times r \times r = ar^2$
$T_4 = a \times r \times r \times r = ar^3$
$T_5 = a \times r \times r \times r \times r = ar^4 = ar^{5-1}$
$T_6 = ar^{6-1}$
.
.
.
$T_n = ar^{n-1}$ the general term, where:

T_n is term number n ($\mathbb{N}\ only$) in the sequence,
a is the first term,
$r = \frac{T_n}{T_{n-1}}$ is the common ratio between successive terms, and,
n is the position of the term in the sequence, which can only be a natural number.

- Since the ratio between terms is common, it can be shown that if three consecutive terms are given:

  $T_1 \quad ; \quad T_2 \quad ; \quad T_3 \quad ; \quad . \quad . \quad .$

  $\frac{T_2}{T_1} \quad ; \quad \frac{T_3}{T_2}$

  Then, it implies that $\frac{T_2}{T_1} = \frac{T_3}{T_2}$. This can be applied to any three successive terms of any geometric sequence.

- The general term, T_n, can also be used to determine the value of n, the position of any term in any geometric sequence.

- Worked examples:

1. $x - 4$; $x + 2$ and $3x + 1$ respectively represent T_4 , T_5 and T_6 of a geometric sequence.

   1.1 Determine the value of x.

   1.2 Determine the numerical values of the first three terms of the sequence.

   1.3 Determine the general term of the sequence.

   1.4 What is the tenth term of the sequence?

   1.5 Which term of the sequence will be equal to $\frac{3125}{8}$?

Solutions:

1.1 $\frac{T_5}{T_4} = \frac{T_6}{T_5}$

$\therefore \frac{x+2}{x-4} = \frac{3x+1}{x+2}$

$\therefore (x+2)(x+2) = (3x+1)(x-4)$

$\therefore x^2 + 4x + 4 = 3x^2 - 11x - 4$

$\therefore -2x^2 + 15x + 8 = 0$

$\therefore 2x^2 - 15x - 8 = 0$

$\therefore (2x+1)(x-8) = 0$

$\therefore x = -\frac{1}{2}$ or $x = 8$

1.2 $x=-\frac{1}{2}$ or $x=8$

$\therefore T_4=-\frac{9}{2}$ or $T_4=4$

$\therefore T_5=\frac{3}{2}$ or $T_5=10$

$\therefore T_6=-\frac{1}{2}$ or $T_6=25$

$\therefore r=\frac{-\frac{1}{2}}{\frac{3}{2}}=-\frac{1}{3}$ or $r=\frac{25}{10}=\frac{5}{2}$

$T_6=ar^5$ $T_6=ar^5$

$\therefore -\frac{1}{2}=a\left(-\frac{1}{3}\right)^5$ or $\therefore 25=a\left(\frac{5}{2}\right)^5$

$\therefore a=\frac{243}{2}$ or $\therefore a=\frac{32}{125}$

$\therefore$ The sequence is $\therefore$ The sequence is

$\frac{243}{2};-\frac{81}{2};\frac{27}{2};...$ or $\frac{32}{125};\frac{16}{25};\frac{8}{5};...$

1.3 $T_n=ar^{n-1}$

$T_n=\frac{243}{2}\left(-\frac{1}{3}\right)^{n-1}$ or $T_n=\frac{32}{125}\left(\frac{5}{2}\right)^{n-1}$

1.4 $T_{10}=\frac{243}{2}\left(-\frac{1}{3}\right)^{10-1}=-\frac{1}{162}$

or

$T_{10}=\frac{32}{125}\left(\frac{5}{2}\right)^{10-1}=\frac{15625}{16}$

1.5 From $T_n=\frac{32}{125}\left(\frac{5}{2}\right)^{n-1}$

$\frac{3125}{8}=\frac{32}{125}\left(\frac{5}{2}\right)^{n-1}$

$\frac{390625}{256}=\left(\frac{5}{2}\right)^{n-1}$

$$\left(\frac{5}{2}\right)^{8} = \left(\frac{5}{2}\right)^{n-1}$$

$$8 = n - 1$$

$$\therefore n = 9$$

ACTIVITIES/ASSESSMENTS

- **Classwork/Homework**

1. $k+1$; $k-1$; $2k-5$; ... are the first three terms of a geometric sequence, where $k < 0$:
   (a) Calculate the value of k.
   (b) Determine the sequence.
   (c) Determine the general term of the sequence.
   (d) Find the 10th term.
   (e) Which term of the sequence is $-59\ 049$?
2. Determine an expression for the n^{th} term for a geometric sequence whose 3^{th} term is -20 and the 6^{th} term is 160.
3. $x-4$; $x+2$; $3x+1$; ... are the first three terms of a geometric sequence. Determine the general term of the sequence in terms of x.

TOPIC: PATTERNS, SEQUENCES AND SERIES					
LESSON 4: Combined sequences					
Term	1	**Week**	1	**Grade**	12
Duration	1HR	**Weighting**	25	**Date**	
Sub-topics	Combined sequences				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Pattern: a repetitive regular arrangement of things.
- Sequence (progression): the order in which related number/things follow one another.
- Geometric: something that relates to geometric shapes or figures.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Laws of exponents
- Solving equations

NOTES

- The teacher leads discussion of the worked example:

  Worked example:

  1. Given below is the combination sequence of an arithmetic and a geometric pattern:

     $3 \; ; \; 8 \; ; \; 6 \; ; \; 5 \; ; \; 12 \; ; \; 2 \; ; \; . \; . \; .$

     1.1 If the pattern continues, write down the next two terms.

     1.2 Determine the 15th and the 16th term of the given sequence.

Solutions:

1.1 Geometric sequence: $3 \; ; \; 6 \; ; \; 12 \; ; \; . \; . \; .$, the common ratio is 2, which implies that the next term is 24.
Arithmetic sequence: $8 \; ; \; 5 \; ; \; 2 \; ; \; . \; . \; .$, the constant difference is -3, which implies that the next term is -1.

1.2 Geometric sequence: $a = 3$ and $r = 2$
All odd-numbered terms belong to the geometric sequence, thus:

$$T_n = 3(2)^{n-1}$$

$$\Rightarrow \quad T_{15} = 3(2)^{15-1} = 49\,152$$

All even-numbered terms belong to the arithmetic sequence, hence, from

$: \; 8 \; ; \; 5 \; ; \; 2 \; ; \; . \; . \; .$

$a = 8$ and $d = -3$

$\therefore \; T_{16} = a + 15d$

$\Rightarrow \ T_{16} = 8 + 15(-3) = -37$

ACTIVITIES/ASSESSMENTS

- **Classwork/Homework**

1. Consider the sequence: 12 ; 4 ; 14 ; 7 ; 18 ; 10 ; …

   1.1 Write down the next TWO terms if the given pattern continues.

   1.2 Calculate the value of the 50th term of the sequence.

   1.3 Write down the value of 131th term of the sequence.

2. The following sequence is a combination of an arithmetic and a geometric sequence:

   3 ; 3 ; 9 ; 6 ; 15 ; 12 ; ...

   2.1 Write down the next TWO terms.

   2.2 Calculate $T_{22} - T_{21}$.

   2.3 Prove that ALL the terms of this infinite sequence will be divisible by 3.

TOPIC: PATTERNS, SEQUENCES AND SERIES					
LESSON 5:					
Term	1	**Week**	2	**Grade**	12
Duration	1 HR	**Weighting**	25	**Date**	
Sub-topics	Series and Sigma Notation				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Sum – is a result on an addition.
- Series – is the sum of a sequence.
- Sigma notation (Σ) – Is the mathematical symbol which is used as the symbol for summing a series.
- $\sum_{k=1}^{n} T_k$ - This is read as follows: The sum of all the terms T_k (general term) from $k=1$ up to and including $k=n$, where $n \in \mathrm{N}$.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Learners should be able to:

- Substitute
- Determine the common difference/ ratio between consecutive terms
- Determine the number of terms in a finite series.
- Solving exponential equations.

NOTES

Example 1:

1.1 Calculate: $\sum_{i=2}^{8}(4i-1)$

Last value to be substituted into the general term to obtain the last term.

$\sum_{i=2}^{8}(4i-1)$ ← General term/ nth term

First value to be substituted into the general term to obtain the

Note! 7 terms not 8

No. of terms = Top – bottom

$= [4(2)-1]+[4(3)-1]+[4(4)-1]+[4(5)-1]+[4(6)-1]+[4(7)-1]+[4(8)-1]$
$= 7+11+15+19+23+27+31$
$= 133$

Example 2:

2.1 Calculate: $\sum_{k=0}^{7} 3.2^{1-k}$

Important! There are 8 terms not 7.

Number of terms = 7 – 0+ 1 = 8

Solution: $= [3.2^{1-0}]+[3.2^{1-1}]+[3.2^{1-2}]+[3.2^{1-3}]+[3.2^{1-4}]+[3.2^{1-5}]+[3.2^{1-6}]+[3.2^{1-7}]$

$= 6+3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+\frac{3}{64}$

$$=\frac{765}{64}$$

Example 3:

3.1 Write the following series in sigma notation: $-3+1+5+.........+313$

Solution

The above series is arithmetic with the $a=-3$ and $d=4$.

Advisable that learners determine the nth term of the series.

Note: Use ***k*** since ***n*** represents the total number of terms!

$T_n = a+(n-1)d$

$\therefore T_k = a+(k-1)d$

$T_k = -3+(k-1)(4)$

$= 4k-4-3$

$\therefore T_k = 4k-7 \qquad (general \ \ term)$

Bottom:

$-3=4k-7$

$4=4k$

$\therefore k=1$

Top:

$313=4k-7$

$320=4k$

$\therefore k=80$

$$\therefore -3+1+5+...............+313=\sum_{k=1}^{80}(4k-8)$$

Example 4:

4.1 Write the series in sigma notation: $3+\ 6+\ 12+\ 2+...............+6\ \ 144$

Solution: The above series is geometric.

$r=3$

$T_k = 3(2)^{k-1}$ nth term

Bottom:

$3=3(2)^{k-1}$

$1=2^{k-1}$

$2^0=2^{k-1}$

$\therefore k=1$

Top:

$6\ \ 144=3(2)^{k-1}$

$2\ \ 048=2^{k-1}$

$2^{11}=2^{k-1}$

$\therefore k=12$

Answer: $3+\ 6+\ 12+\ 2+...............+6\ \ 144\ =\sum_{k=1}^{12}3(2)^{k-1}$

ACTIVITIES/ASSESSMENTS

1.1 Expand and then calculate each of the following:

a) $\sum_{r=4}^{13}3$ b) $\sum_{k=2}^{8}(1-2k)$ c) $\sum_{k=0}^{7}2\left(\frac{1}{2}\right)^k$

1.2 Write each of the following series in sigma notation:

a) $5+\ 1\ -3+..............\ -83\ -87$

b) $2+\ 6+\ 18+.............$ to 50 terms

c) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+......\ +\frac{24}{25}$

TOPIC: PATTERNS, SEQUENCES AND SERIES					
LESSON 6:					
Term	1	**Week**	2	**Grade**	12
Duration	1 hour	**Weighting**		**Date**	
Sub-topics	Summing the terms of an arithmetic series				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Sum – is a result of an addition
- Series – is the sum of a sequence.
- Sigma notation (Σ)– Is the mathematical symbol which is used as the symbol for summing a series.

$\sum_{k=1}^{n} T_k$ - This is read as follows: The sum of all the terms T_k (general term) from $k=1$ up to and including $k=n$, where $n \in$ N.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Learners should be able to:

- Substitute
- Determine the common difference between consecutive terms
- Determine the number of terms in a finite series.

NOTES

We will now look at summing a large number of terms. We can calculate the sum of the first ***n*** terms of an arithmetic series by using the following formula.

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

S_n is the sum of the first n terms.
n is the number of terms.
a is the first term.
d is the constant difference.

How the formula for the sum on an arithmetic series is derived?

Proof

Let the first term of an arithmetic series be a and the constant difference d.
$\therefore S_n = a + (a+d) + (a+2d) + ... + T_n$, where $T_n = a + (n-1)d$.

$$\begin{array}{ccccccccccccccc}
S_n & = & a & + & (a+d) & + & (a+2d) & + & \dots & + & (T_n - 2d) & + & (T_n - d) & + & T_n \\
S_n & = & T_n & + & (T_n - d) & + & (T_n - 2d) & + & \dots & + & (a+2d) & + & (a+d) & + & a \\
\hline
\therefore 2S_n & = & (a+T_n) & + & (a+T_n) & + & (a+T_n) & + & \dots & + & (a+T_n) & + & (a+T_n) & + & (a+T_n)
\end{array}$$

$\therefore 2S_n = n(a+T_n)$

$\therefore S_n = \frac{n}{2}(a+T_n)$

But $T_n = a+(n-1)d$

$\therefore S_n = \frac{n}{2}[a+a+(n-1)d]$

$\therefore S_n = \frac{n}{2}[2a+(n-1)d]$

Worked example:

1.1 Calculate the sum of the first 30 terms of the sequence: $3;\ 11;\ 19;\ 27;...$

1.2 Calculate the sum of the following finite series: $-14\ -11\ -8+.......+103$

1.3 Calculate $\sum_{k=1}^{251}(7k-5)$

1.4 Given: $2+\ 5+\ 8+..........n$ terms $=72710$. Calculate the number of terms in the series.

1.5 Determine m if $\sum_{i=0}^{m}(1-3i)=-671$

Suggested solutions:

1.1 In this particular question, the number of terms to be added are specified.
The sequence is arithmetic with $d=8,\ a=3$ and $S_{30}=?$

$S_n = \frac{n}{2}[2a+(n-1)d]$ → Encourage learners to copy their formula from the information sheet.

$S_{20} = \frac{20}{2}[2(3)+(20-1)(8)]$ → Substitution

$\therefore S_{20} = 1580$ → Encourage learners to work out their answers using a calculator!

1.2 The number of terms to be added are not known at this stage. It is necessary to first determine the number of terms in the series before calculating the sum.

$a=-14,\ d=3$ and $T_n=103$

$T_n = a+(n-1)d$ ....(from the information sheet)

$103 = -14+(n-1)(3)$

$103 = -14+3n-3$

$\therefore n = 40$

There are 40 terms in the series.

$S_n = \frac{n}{2}[2a+(n-1)d]$ or $S_n = \frac{n}{2}[a+l]$ (where l is the last term)

$S_{40} = \frac{40}{2}[2(-14)+(40-1)(3)]$ $\quad S_{40} = \frac{40}{2}[-14+103]$

$\therefore S_{40} = 1780$ $\quad \therefore S_{40} = 1780$

1.4 1.3 Generate at least the first three terms to identify the type of the series.

$2+\ 9+\ 16+.......+1752$

The series is arithmetic since $d=7$

$a = 2, \; n = 251$

$S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + l]$

$S_{251} = \frac{251}{2}[2(2) + (251-1)(7)]$ $S_{251} = \frac{251}{2}[2 + 1752]$

$\therefore S_{251} = 220127$ $\therefore S_{251} = 220127$

1.5 72 710 is a sum of certain number of terms not the value of a term.

$a = 2, \; d = 3$ and $S_n = 72710$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$72710 = \frac{n}{2}[2(2) + (n-1)(3)]$

$145420 = n[4 + 3n - 3]$

$145420 = n[3n + 1]$

$3n^2 + n - 145420 = 0$

$(n - 660)(n + 661) = 0$ ⟷ A quadratic formula maybe used.

$n = 660$ or $n \neq -661$

1.6 Generate at least the first three terms to identify the type of the series.

$1 \; -2 \; -5 + \ldots\ldots\ldots$ the series is arithmetic

$a = 1; \; d = -3; \; S_n = -671$

$n = m - 0 + 1 = m + 1$……..calculate the number of terms ALWAYS. Do not assume any value.

$S_n = \frac{n}{2}[2a + (n-1)d]$

$-671 = \frac{m+1}{2}[2(1) + (m+1-1)(-3)]$

$-671 = \frac{m+1}{2}[2 - 3m]$

$-1342 = (m+1)(2-3m)$

$-1342 = -3m^2 - m + 2$

$3m^2 + m - 1344 = 0$

$(3m + 64)(m - 21) = 0$ ⟷ A quadratic formula maybe used.

$m = -\frac{64}{3}$ or $m = 21$

$\therefore m = 21$

ACTIVITIES/ASSESSMENTS

1.1 Calculate the sum of the series: 10 + 7 + 4 + ………….to 32 terms.

1.2 Calculate $\sum_{m=2}^{50}(5 - 2m)$

1.3 Given the following series: $-5 - 1 + 3 + 7 + \ldots\ldots\ldots + 35$. Calculate the sum of the series.

1.4 A job was advertised at a starting salary of R90 000 pa with an increase of R4 500.
Determine:

1.4.1 The employee's salary in the sixth year.

1.4.2 The total earnings after 10 years.

1.5 Calculate m if : $\sum_{k=1}^{m}(7k+5)=1287$

1.6 Determine the value of k for which :

$$\sum_{r=5}^{60}(3r-4)=\sum_{p=2}^{5}k$$

TOPIC: SEQUENCES, SERIES AND SIGMA NOTATION					
LESSON 7:					
Term	1	**Week**		**Grade**	12
Duration	1 hour	**Weighting**		**Date**	
Sub-topics	Summing the terms of a geometric series				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Sum – is a result of an addition.
- Series – is the sum of a sequence.
- Sigma notation (Σ)– Is the mathematical symbol which is used as the symbol for summing a series.

$\sum_{k=1}^{n} T_k$ - This is read as follows: The sum of all the terms T_k (general term) from $k=1$ up to and including $k=n$, where $n \in$ N.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Intuitive understanding and application of Laws of exponents.
- Determining the number of terms in a finite series.
- Determining the nth term of a geometric sequence.

NOTES

- A geometric series is the sum of a geometric sequence.

Educator will now facilitate the discussion on how to sum the large number of terms in a geometric series using one of the following formulae.

$$S_n = \frac{a(r^n - 1)}{r-1};\quad r \neq 1$$

or

$$S_n = \frac{a(1-r^n)}{1-r};\quad r \neq 1$$

S_n is the sum of the first n terms.
n is the number of terms.
a is the first term.
r is the constant ratio.

Note: The second formula is normally easier to use when $r < 1$.

The derivation of the said above formulae is as follows.

Proof

Let the first term of a geometric series be a and the constant ratio r.

$\therefore S_n = a + ar + ar^2 + ... + ar^{n-1}$

$\therefore rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n$

$$\begin{array}{lccccccccccccccc}
rS_n & = & & & ar & + & ar^2 & + & \dots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \\
S_n & = & a & + & ar & + & ar^2 & + & \dots & + & ar^{n-2} & + & ar^{n-1} & & \\
\hline
\therefore rS_n - S_n & = & -a & + & 0 & + & 0 & + & \dots & + & 0 & + & 0 & + & ar^n
\end{array}$$

$\therefore rS_n - S_n = ar^n - a$

$\therefore S_n(r-1) = a(r^n - 1)$

$\therefore S_n = \dfrac{a(r^n - 1)}{r-1}$

Educator will now facilitate the application of the above formula using the suggested examples below.

Worked examples:

1. Calculate the sum of the first 12 terms of the series: $\frac{2}{3} + 2 + 6 + ...$
2. Calculate the sum of the following finite series $1 + 4 + 16 + 64 + ... + 1073741824$
3. Calculate $\sum_{i=0}^{19} 3.(-2)^{i-1}$
4. How many terms of the geometric sequence $-1;\ 2;\ -4;\ 8;..........$ will add up to 349525?
5. Given: $\sum_{k=1}^{n} 3(2)^{1-k} = 5{,}8125$. Calculate the value of n.
6. The Constant ration of a geometric sequence is $-\frac{1}{2}$. The 8^{th} term of the same sequence is $-\frac{5}{32}$. Determine the sum of the first 8 terms.

Suggested solutions:

1. This is a geometric series with $a = \frac{2}{3};\ r = 3;\ S_{12} = ?$

   $S_n = \dfrac{a(r^n - 1)}{r-1}$ ....state the formula

   $S_{12} = \dfrac{3((3)^{12} - 1)}{3-1}$

   $\therefore S_{12} = 177146{,}\dot{6}$

2. The series is geometric with $a = 1;\ r = 4;\ T_n = 1073741824$

   $T_n = ar^{n-1}$

   $1073741824 = 1.(4)^{n-1}$

   $2^{30} = (2^2)^{n-1}$

   $2^{30} = 2^{2n-2}$

   $30 = 2n - 2$

$2n = 32$
$\therefore n = 16$

3. $\sum_{i=0}^{19} 3.(-2)^{i-1} = -\frac{3}{2} + 3 - 6 + \ldots\ldots\ldots + 786432$

The series is geometric with:

$a = -\frac{3}{2},\ r = -2,\ \ n = 19 - 0 + 1 = 20$

$S_n = \frac{a(r^n - 1)}{r - 1}$ …… state the formula

$S_{20} = \frac{-\frac{2}{3}\left[(-2)^{20} - 1\right]}{(-2) - 1} = 524287{,}5$

4. The series is geometric with $a = -1;\ r = -2$ and $S_n = 349525$

Important! Emphasize that 349 525 should not be confused with the value of the term.

$S_n = \frac{a(r^n - 1)}{r - 1}$

$349525 = \frac{(-1)\left[(-2)^n - 1\right]}{(-2) - 1}$

$349525 = \frac{(-1)\left[(-2)^n - 1\right]}{-3}$

$-1048575 = (-1)((-2)^n - 1)$

$1048576 = (-2)^n$

$(-2)^{20} = (-2)^n$

$n = 20$

The first 20 terms must be added to give 349 525.

5. $\sum_{k=1}^{n} 3(2)^{1-k} = 5{,}8125$

$3 + \frac{3}{2} + \frac{3}{4} + 24 + \ldots\ldots\ldots = 5{,}8125$

In this case a sum of the series is given. You are required to find the number of terms to be added to give a sum of 5,8125

The series is geometric with $a = 3;\ r = \frac{1}{2};\ S_n = 5{,}8125;\ n = n - 1 + 1 = n$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$5{,}8125 = \frac{3\left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1}$

$5{,}8125 \times -0{,}5 = 3\left[\left(\frac{1}{2}\right)^n - 1\right]$

$-\frac{93}{32} = 3\left[\left(\frac{1}{2}\right)^n - 1\right]$

$$-\frac{31}{32}=\left(\frac{1}{2}\right)^{n}-1$$

$$\left(\frac{1}{32}\right)=\left(\frac{1}{2}\right)^{n}$$

$$\left(\frac{1}{2}\right)^{5}=\left(\frac{1}{2}\right)^{n}$$

$\therefore n=5$

6. $\ldots;\ldots;\ldots;\ldots.;\ldots;\ldots;\ldots;-\frac{5}{32};\ldots\ldots..$

$r=-\frac{1}{2}$; $n=8$. It is important to first calculate the value of a.

$T_8=ar^7$ ……. refer to the general term discussed earlier.

$-\frac{5}{32}=a\left(-\frac{1}{2}\right)^{7}$ …… $\div\left(-\frac{1}{2}\right)^{7}$ on both sides

$\therefore a=20$

$$S_n=\frac{a\left(r^n-1\right)}{r-1}$$

$$S_8=\frac{20\left[\left(-\frac{1}{2}\right)^{8}-1\right]}{\left(-\frac{1}{2}\right)-1}=\frac{425}{32} / 13{,}28125$$

ACTIVITIES/ASSESSMENTS

1.1 Calculate the sum of the first 12 terms of the geometric sequence $\frac{1}{4};\ -\frac{1}{2};\ 1;\ldots\ldots.$

1.2 Evaluate $-9-6-4-\ldots\ldots\ldots.-1\frac{5}{27}$

1.3 Calculate $\sum_{m=3}^{11}8\left(\frac{1}{2}\right)^{m-4}$

1.4 Determine m if: $\sum_{p=1}^{m}(-8)\left(\frac{1}{2}\right)^{p-1}=-15\frac{3}{4}$

1.5 What is the least value of p for which the series $\sum_{k=1}^{p}\frac{1}{16}(2)^{k-2}>31$?

TOPIC: SEQUENCES, SERIES AND SIGMA NOTATION					
LESSON 8:					
Term	1	**Week**		**Grade**	12
Duration	1 hour	**Weighting**		**Date**	
Sub-topics	Sum to infinity of a converging geometric series				

RELATED CONCEPTS/ TERMS/VOCABULARY

- **Convergent series –** is one in which the sum approaches a specific value as ***n*** increases. Once the sum reaches this specific value (sum to infinity, S_∞), the sum does not go behind the value.
- **Divergent –** is one in which the sum becomes very large as ***n*** increases.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Determining the common ratio between consecutive terms.
- Generating terms of a series from sigma notation.

NOTES

Consider the following two geometric series:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}...$$

Educator to facilitate the development of the lesson.

$S_1 = 1$ $\quad\therefore S_1 = 1$

$S_2 = 1+\frac{1}{2} = 1\frac{1}{2}$ $\quad\therefore S_2 = 1{,}5$

$S_3 = 1+\frac{1}{2}+\frac{1}{4} = 1\frac{3}{4}$ $\quad\therefore S_3 = 1{,}75$

$S_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1\frac{7}{8}$ $\quad\therefore S_4 = 1{,}875$

$S_5 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 1\frac{15}{16}$ $\quad\therefore S_5 = 1{,}9375$

$S_6 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = 1\frac{31}{32}$ $\quad\therefore S_6 = 1{,}984375$

As n increases, S_n approaches 2.
Mathematically we say that:
if $n \to \infty$ then $S_n \to 2$.
$\therefore$ This series **converges** to 2.

$1+2+4+8+16+32+\ldots$

$S_1 = 1$ $\therefore S_1 = 1$

$S_2 = 1+2 = 3$ $\therefore S_2 = 3$

$S_3 = 1+2+4 = 7$ $\therefore S_3 = 7$

$S_4 = 1+2+4+8 = 15$ $\therefore S_4 = 15$

$S_5 = 1+2+4+8+16 = 31$ $\therefore S_5 = 31$

$S_6 = 1+2+4+8+16+32 = 63$ $\therefore S_6 = 63$

As n increases, S_n becomes very large.
Mathematically we say that:
if $n \to \infty$ then $S_n \to \infty$.
$\therefore$ This series **diverges**.

A geometric series will converge (the sum will approach a specific value), if the constant ratio is a number between -1 and 1.

Convergent geometric series :
$-1 < r < 1$

The value approached by a convergent geometric series is called the **sum to infinity** (S_∞) of the geometric series. We can calculate the sum to infinity of a convergent geometric series by using the following formula:

$$S_\infty = \frac{a}{1-r} \quad \text{where } -1 < r < 1$$

Where does this formula come from?

For r-values between -1 and 1:
If $n \to \infty$ then $r^n \to 0$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

Worked examples:

1. Consider the geometric series: $36 \ -18 \ +9+..........$
   1.1 Why does the sum to infinity exists for this series?
   1.2 Calculate the S_∞.
2. Calculate : $\sum_{m=1}^{\infty} 8(2)^{-2m}$
3. Consider the series $3x+3x(x-2)+3x(x-2)^2+...$
   (a) For which values of x will the series converge?
   (b) If x is a value for which the series converges, calculate the sum to infinity of the series in terms of x.
4. The first term of a geometric series is 124. The sum to infinity is 64. Determine the common ratio.

Suggested solutions:

1.1 The series is convergent (sum to infinity exist), because the common ration is between -1 and 1.

$-1 < r < 1$

$-1 < -\frac{1}{2} < 1$ ……the common ratio (r) = $-\frac{1}{2}$

1.2 $S_\infty = \frac{a}{1-r}$

$= \frac{36}{1-\left(-\frac{1}{2}\right)}$

$= 24$

2. $\sum_{m=1}^{\infty} 8(2)^{-2m} = 2 + \frac{1}{2} + \frac{1}{8} + ...$ Emphasize generating the terms of a series.

$r = \frac{1}{4}; a = 2; S_\infty = ?$

$S_\infty = \frac{a}{1-r}$

$= \frac{2}{1-\frac{1}{4}}$

$= \frac{8}{3}$

3.a) Geometric series ONLY converges when $-1 < r < 1$

$r = \frac{3x(x-2)^2}{3x(x-2)} = \frac{3x(x-2)}{3x} = (x-2)$

$\therefore -1 < x-2 < 1$

$\therefore 1 < x < 3$

$S_\infty = \frac{a}{1-r}$

b) $= \frac{3x}{1-(x-2)}$

$= \frac{3x}{-x+3}$

4. $a = 124; S_\infty = 64$

$S_\infty = \frac{a}{1-r}$

$64 = \frac{124}{1-r}$

$64 - 64r = 124$

$-64r = 60$

$\therefore r = -\frac{15}{16}$

ACTIVITIES/ASSESSMENTS

1.1 Consider the geometric series: $5(3x+1)+5(3x+1)^2+5(3x+1)^3+........$

a) For which value(s) of x will the series converge?

b) Calculate the sum to infinity of the series if $x = -\frac{1}{6}$.

1.2 Write the series in sigma notation: 2 + 0,2 + 0, 02 + …….

1.3 Given: $\sum_{k=1}^{\infty} 5(3^{2-k})$

a) Write down the value of the first TWO terms of the infinite geometric series.

b) Calculate the sum to infinity of the series.

1.4 In a geometric sequence, the second term is $-\frac{2}{3}$ and the sum to infinity of the sequence is $\frac{3}{5}$.

Determine the common ratio.

TOPIC: SEQUENCES, SERIES AND SIGMA NOTATION					
LESSON 9:					
Term	1	**Week**		**Grade**	12
Duration	1 hour	**Weighting**	25	**Date**	
Sub-topics	Determining terms from the sum formula				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Sum – is a result on an addition.
- Sequence - the order in which related number/things follow one another.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Substitute values in a given formula
- Determining the sum

RESOURCES

- Grade 12 text books (Mind Action Series)
- Previous papers from different provinces including national papers.

NOTES

An educator may adopt an investigative approach in facilitating discussions on how the terms of the sequence may be calculated using the sum formula.

Worked out examples:

Consider the arithmetic series $2+5+8+11+14+17+20+.........$ and answer the questions that follow.

Term 1 is equal to the sum of the 1st term

In this series, we define $T_1 = S_1 = 2$

1. Determine the values of:
   a) T_2; S_2; and S_1
   b) T_3; S_3; S_2
   c) T_4; S_4; and S_3
   What can you conclude?
2. Identify a relationship between T_n, S_n and S_{n-1} where $n > 1$ and $n \in \mathbb{N}$.
3. Determine the values of:
   a) T_5
   b) T_6
4. In an arithmetic sequence $S_n = n^2 - 2n$.
   Use your formula in 2 to determine the value of :
   a) T_7 b) T_{50}

Suggested solutions:

1. a) $T_2 = 5$ ....from the given series.
   $S_2 = 2+5 = 7$ ....sum of the first two terms of the series.

$S_1 = 2$ ……….adding only the first term.

$S_2 - S_1 = 7 - 2 = 5 = T_2$

b) $T_3 = 8$ ………from the given series.

$S_3 = 2 + 5 + 8 = 15$ ……..sum of the first 3 terms of the series.

$S_2 = 7$ …….sm of the first two terms of the series.

$S_3 - S_2 = 15 - 7 = 8 = T_3$

c) $T_4 = 11$ ……from the given series.

$S_4 = 2 + 5 + 8 + 11 = 26$ …….sum of the first 4 terms of the series.

$S_3 = 15$

$S_4 - S_3 = 26 - 15 = 11 = T_4$

Value of the term is equal to the difference between sums of two consecutive terms.

2. $T_n = S_n - S_{n-1}$
3. a) $T_5 = S_5 - S_4 = 40 - 26 = 14$ b) $T_6 = S_6 - S_5 = 57 - 40 = 17$
4. $T_7 = S_7 - S_6 = 35 - 24 = 11$ b) $T_{50} = S_{50} - S_{49} = 2400 - 2303 = 97$

ACTIVITIES/ASSESSMENTS

1.1

The sum of the first terms in an arithmetic series is given by: $S_n = n^2 - 2n$

Calculate:

a) the sum of the first 13 terms.

b) the 13^{th} term.

1.2

Given: $S_n = \dfrac{5\left(1-3^n\right)}{-2}$

a) Calculate S_4

b) Calculate T_5

1.3 Prove that: $\sum_{k=3}^{n}(2k-1)n = n^3 - 4n$

REVISION EXERCISES/EXPANDED OPPORTUNITIES	
1.	The sixth term of a geometric sequence is 80 more than the fifth term. 1.1 Show that $a = \dfrac{80}{r^5 - r^4}$. 1.2 If it is further given that sum of the fifth and sixth terms is 240, determine the value of the common ratio.

	KZN March 2019
2.	The following sequence represents a geometric progression: $x\,;\, x+2\,;\ \ldots\ldots\ldots\ldots$ 2.1 Write down the third term in terms of x. 2.2 Calculate the value of x if it is given that $S_\infty = -8$. EC Sept 2016
3.	Consider $4;\ \frac{3}{4}\,;\ 4\,;\ \frac{1}{4}\,;\ 4\,;\ \frac{1}{12}\,;\ \ldots$ which is a combination of 2 geometric patterns. 3.1 If the pattern continues in the same way, write down the next TWO terms in the sequence. 3.2 Calculate the sum of the first 25 terms of the sequence. Show all calculations. North West 2016
4.	Given the finite arithmetic sequence: $5\ ;\ 1\ ;\ -3\ ;\ \ldots\ ;\ -83\ ;\ -87$ 4.1 Write down the fourth term (T_4) of the sequence. 4.2 Calculate the number of terms in the sequence. 4.3 Calculate the sum of all the negative numbers in the sequence. 4.4 Consider the sequence: $5\ ;\ 1\ ;\ -3\ ;\ \ldots\ ;\ -83\ ;\ -87\ ;\ \ldots\ ;\ -4\,187$ Determine the number of terms in this sequence that will be exactly divisible by 5. Nsc Nov 2016
5.	Prove that the sum to n terms of the arithmetic series whose first is "a" and its common difference is "d" is given by $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ KZN Jun 2017
6.	Consider an arithmetic sequence which has the second term equal to 8 and the fifth equal 10. 6.1 Determine the common difference of this sequence. 6.2 Write down the sum of the first 50 terms of this sequence, using sigma notation. 6.3 Determine the sum of the first 50 terms of this sequence. Feb/March 2016

7.	The first term of a geometric sequence is 9. The ratio of the sum of the first eight terms to the sum of the first four terms is 97 : 81. Determine the first THREE terms of the sequence, if all terms are positive. GP Sep2018
8.	The first 24 terms of an arithmetic series are: $35 + 42 + 49 + \ldots + 196$. Calculate the sum of ALL natural numbers from 35 to 196 that are NOT divisible by 7. Nsc June 2018
9.	The sum of the first 3 terms of geometric series is $1\frac{8}{49}$. If the first term is 1, then calculate the value of the common ratio, r $(r > 0)$. KZN Jun 2019
10.	10.1 Prove that $\sum_{k=1}^{\infty} 4.3^{2-k}$ is a convergent geometric series. Show All your calculations 10.2 If $\sum_{k=p}^{\infty} 4.3^{2-k} = \frac{2}{9}$, determine the value of p Nsc Nov 2020
11.	The first two terms of a geometric sequence and an arithmetic sequence are the same. The first term is 12. The sum of the first three terms of the geometric sequence is 3 more than the sum of the first three terms of the arithmetic sequence. Determine TWO possible values for the common ratio, r, of the geometric sequence. (Feb/March 2011
12.	Without using a calculator, determine the value of: $\sum_{y=3}^{10} \frac{1}{y-2} - \sum_{y=3}^{10} \frac{1}{y-1}$ Nsc Nov 2019

TOPIC 2 : EUCLIDEAN GEOMETRY

LESSON 1:

Term	1	**Week**	3	**Grade**	12
Duration	2HR	**Weighting**	27% (40/150)	**Date**	

Sub-topics: Revision of previous Grades Euclidean Geometry

RELATED CONCEPTS/ TERMS/VOCABULARY

- Tangents, Secants, Segments, Circles, Arcs.
- Theorems, Corollaries, Converses, Axioms.

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

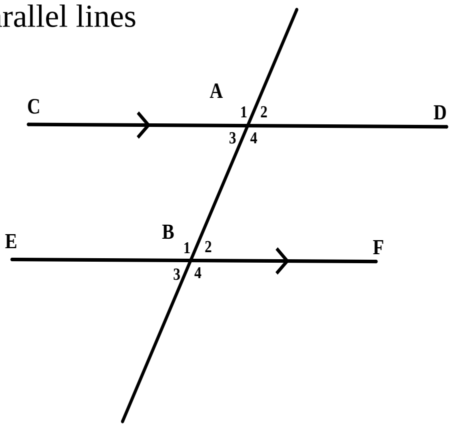
- Knowledge of all Grade 10 & 11 theorems & converses
- Grade 9 parallel lines and triangle geometry.
- Formal proofs of the 5 examinable Gr11 proofs.

RESOURCES

- KZN Provincial Euclidean Geometry document 2015
- Past Grade 11 Examination questions papers.
- Mind Action Series GR 12

NOTES

Parallel lines



Corresponding angles are equal

$\hat{A}_1 = \hat{B}_1$

$\hat{A}_2 = \hat{B}_2$

$\hat{A}_3 = \hat{B}_3$

$\hat{A}_4 = \hat{B}_4$

Alternate angles are equal

$\hat{A}_3 = \hat{B}_2$

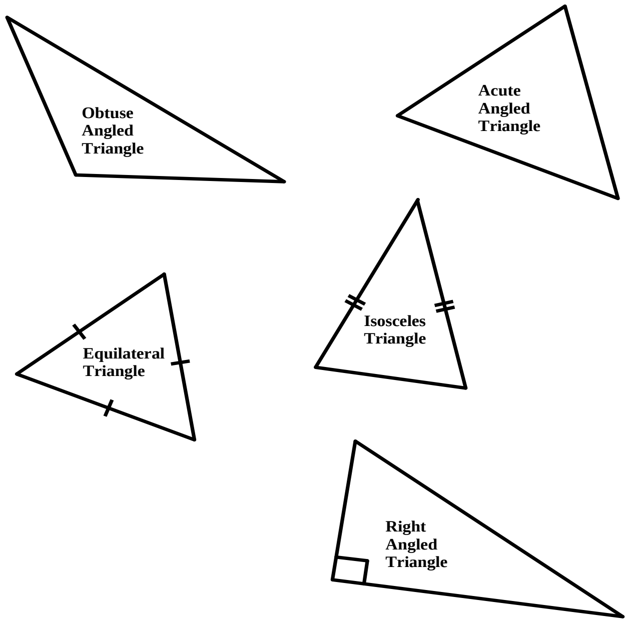
$\hat{A}_4 = \hat{B}_1$

Co-interior angles are supplementary

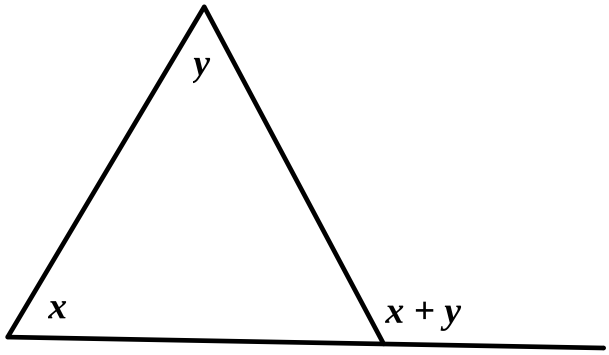
$\hat{A}_3 + \hat{B}_1 = 180°$

$\hat{A}_4 + \hat{B}_2 = 180°$

Types of triangles

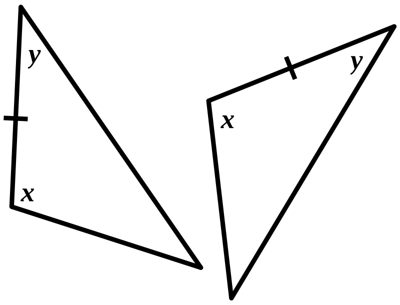


The exterior angle of a triangle equals the sum of its interior opposite angles

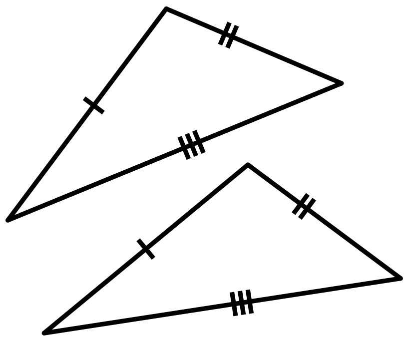


Cases of congruency

Angle / Side / Angle

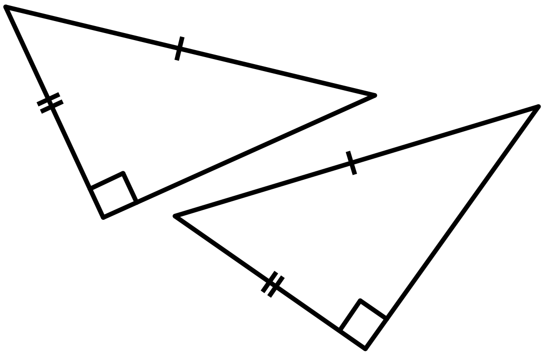


Side / Side / Side

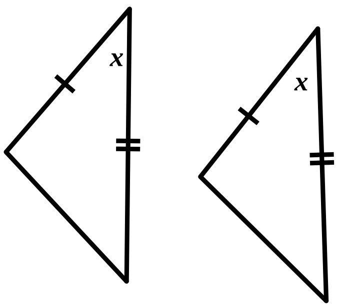


hypotenuse

90° / Hypoteneus / Side

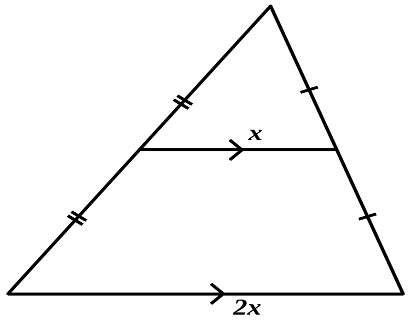


Side / Angle / Side

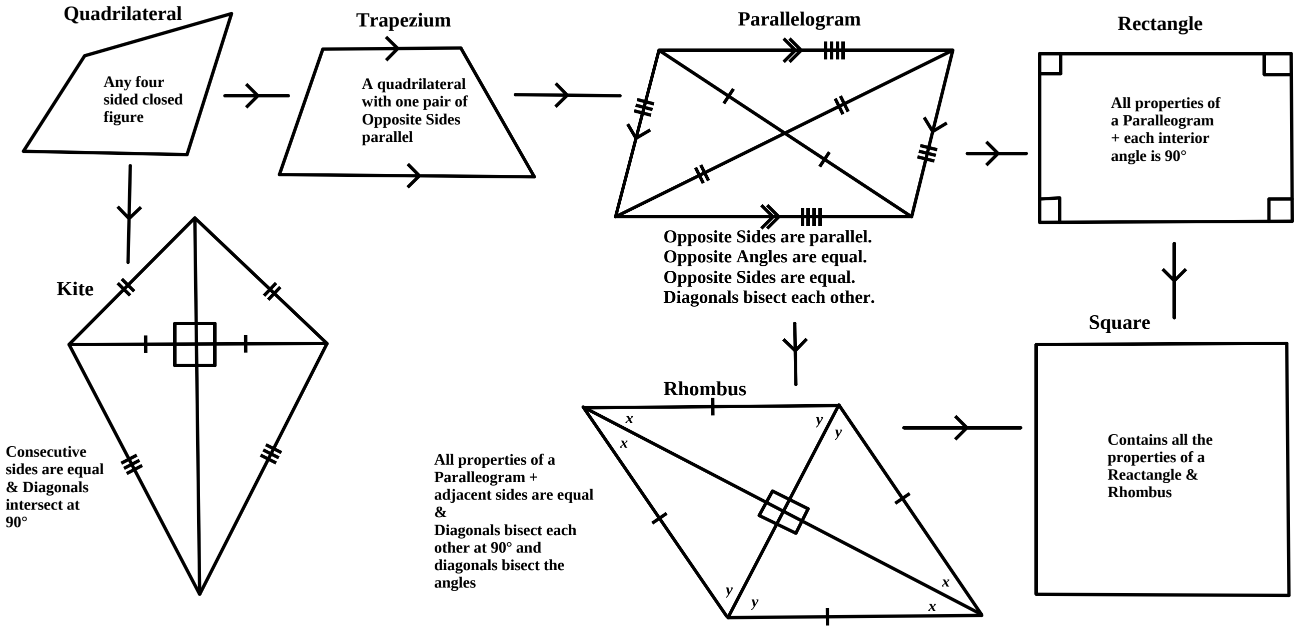


Midpoint Theorem

The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



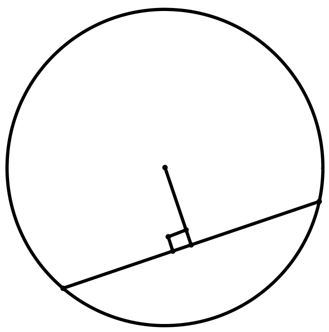
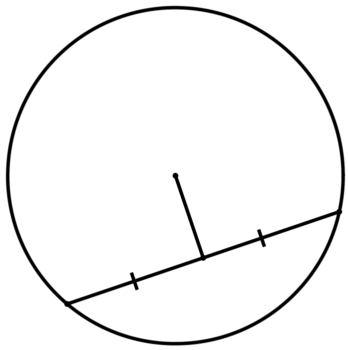
Grade 10 quadrilaterals

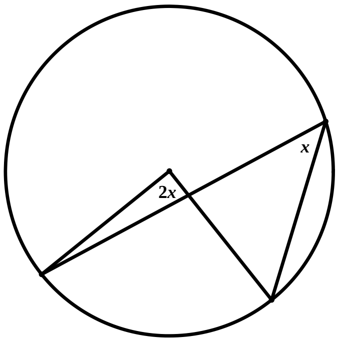
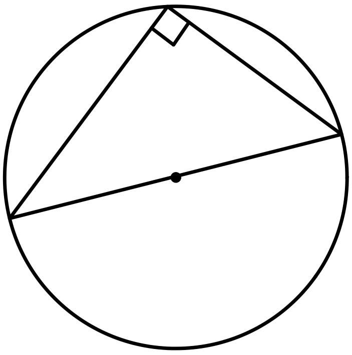
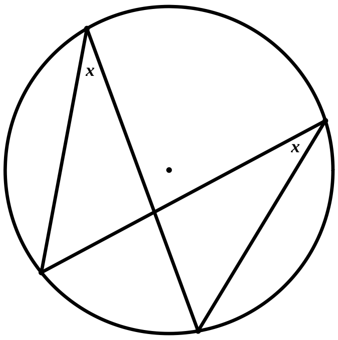
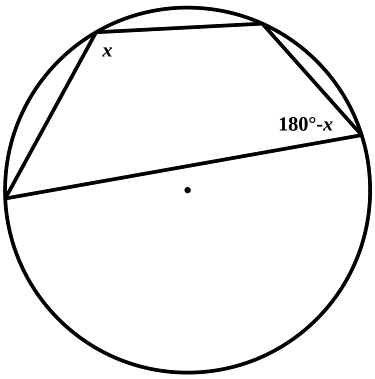
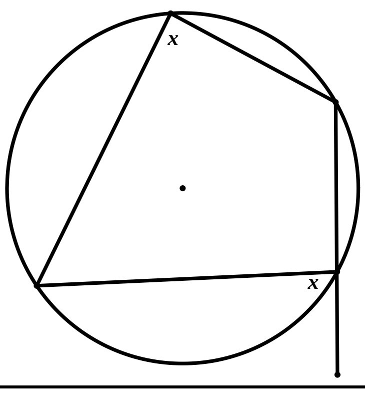


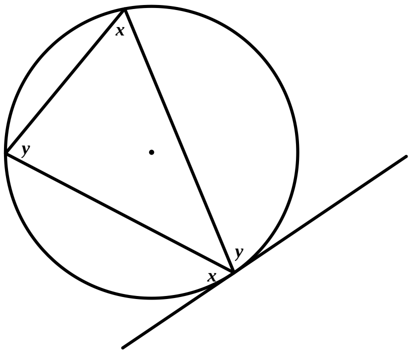
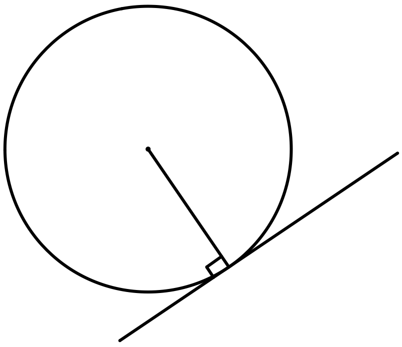
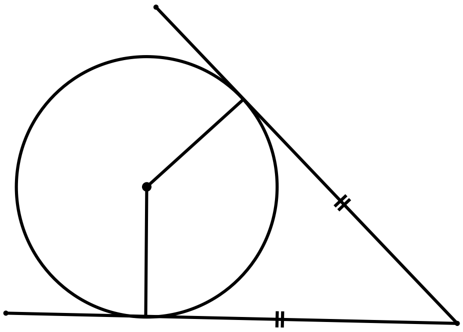
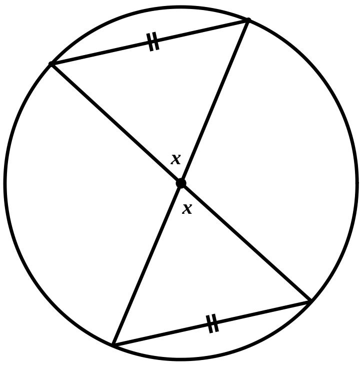
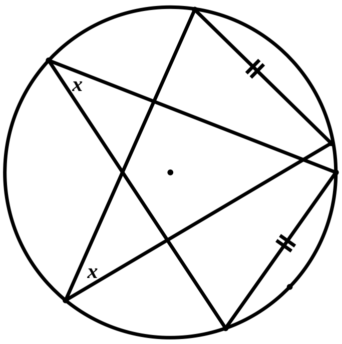
How to prove a quadrilateral is a parallelogram?

1. Prove that the opposite sides are parallel (definition) **or**
2. Prove that the opposite sides are equal. **or**
3. Prove that the opposite angles are equal. **or**
4. Prove that the diagonals bisect each other. **or**
5. Prove that one pair of opposite sides is equal and parallel.

Grade 11 Circle Geometry

No.	ILLUSTRATION	THEOREM OR COROLLARIES (Acceptable Reasons for Formal Proof is in brackets)
1.		*The line drawn from the centre of a circle perpendicular to a chord bisects the chord.* (line from centre ⊥ to chord)
2.		*The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.* (line from centre to midpt of chord)

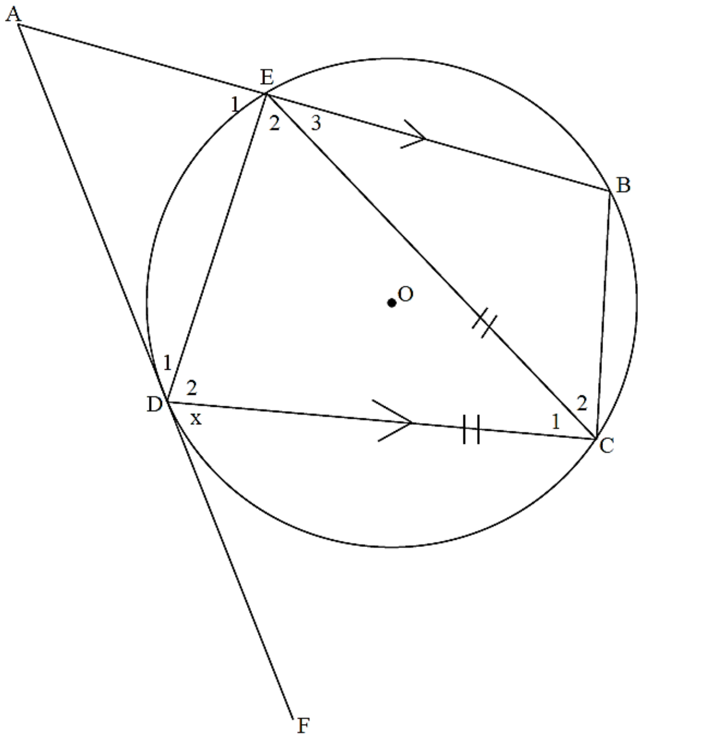
3.		*The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)* (∠ at centre = 2 ×∠ at circumference)
4.		*The angle subtended by the diameter at the circumference of the circle is 90°.* (∠s in semi circle **OR** diameter subtends right angle)
5.		*Angles subtended by a chord of the circle, on the same side of the chord, are equal* (∠s in the same seg)
6.		*The opposite angles of a cyclic quadrilateral are supplementary* (opp ∠s of cyclic quad)
7.		*The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.* (ext ∠ of cyclic quad)

8.		*The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.* (tan chord theorem)
9.		*The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.* (tan ⊥ radius **OR** tan ⊥ diameter)
10.		*Two tangents drawn to a circle from the same point outside the circle are equal in length* (Tans from common pt **OR** Tans from same pt)
11.		*Equal chords subtend equal angles at the centre of the circle.* (equal chords; equal ∠s)
12.		*Equal chords subtend equal angles at the circumference of the circle.* (equal chords; equal ∠s)

ACTIVITIES/ASSESSMENTS

To be done as an example by the Educator in Class

2.1.1 In the diagram below, ADF is a tangent to the circle with points E, B, C and D on the circumference of the circle. AB || DC and EC = DC.

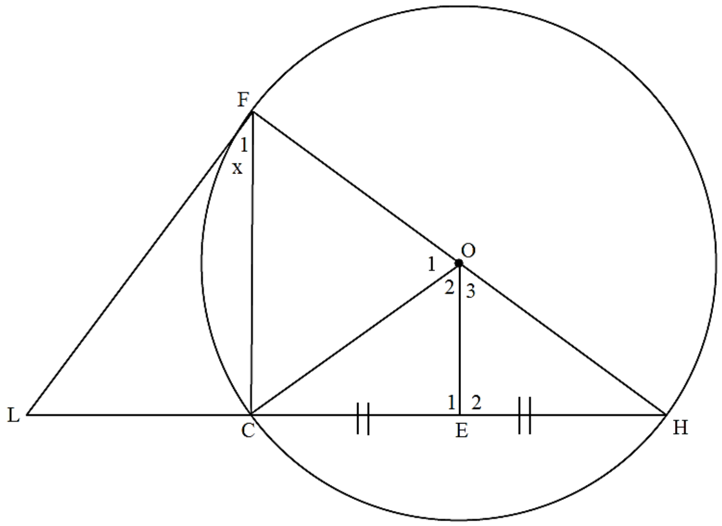


2.1.1.1. If $C\hat{D}F = x$, name with reasons, FIVE other angles equal to x.

2.1.1.2. Prove that ABCD is a parallelogram.

Day 1 – Homework Activity

2.1.2 In the diagram, FH is a diameter of the circle FCH with centre O. FC is a chord and LCH is a secant. LF is a tangent to the circle at F. E is a point on CH such that CE = HE.



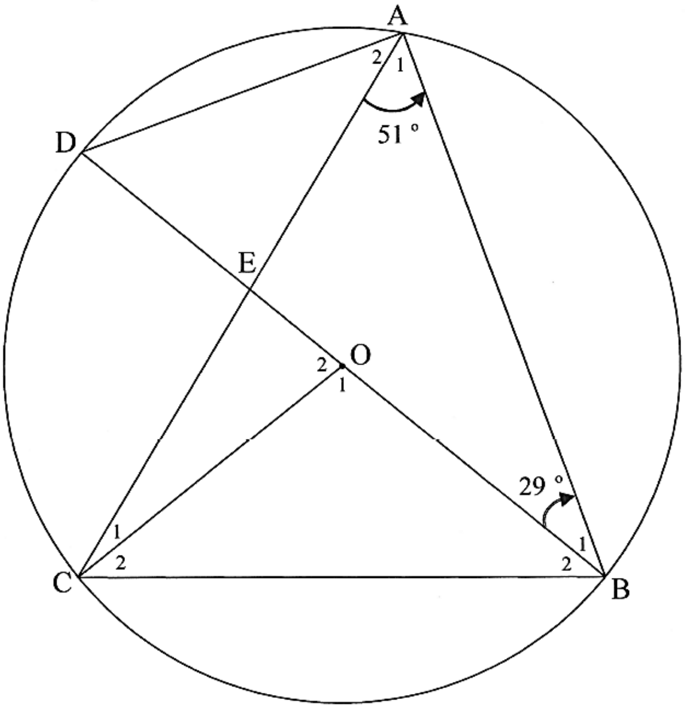
2.1.2.1. Prove that FC || OE.

2.1.2.2. Prove that OFLE is a cyclic quadrilateral.

2.1.2.3. If $\hat{F}_1 = x$, express with reasons $\hat{O}_1$ in terms of x.

Day 2 – Classwork Activity

2.1.3. In the diagram, O is the centre of the circle.
Points A, B, C and D lie on the circumference of the circle. BOD is a diameter.
AC and BD intersect at E.
$\hat{A}_1 = 51°$ and $\hat{B}_1 = 29°$.



2.1.3.1. Determine the size of $\hat{O}_1$.

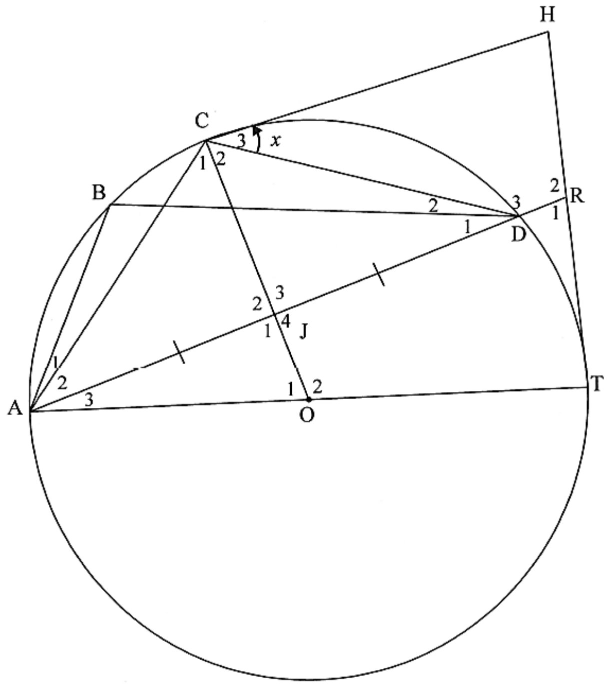
2.1.3.2. Determine the size of $\hat{A}_2$.

2.1.3.3. Determine the size of $\hat{D}$.

2.1.3.4. Determine the size of $A\hat{C}O$.

Day 2 – Classwork Activity

2.1.4. In the diagram, O is the centre of the circle through the points A, B, C, D and T.
HC and HT are tangents to the circle at C and T respectively.
AD is produced to meet HT at R.
OC bisects AD at J.
Let $\hat{C}_3 = x$.



2.1.4.1. Write down, with reasons, another angle equal to $\hat{C}_3$.

2.1.4.2. Show that CHRJ is a trapezium.

2.1.4.3. Prove that OC bisects $A\hat{C}D$

2.1.4.4. Write down, with reason, $A\hat{B}D$ in terms of x.

2.1.4.5. Determine $\hat{R}_2$ in terms of x.

TOPIC: EUCLIDEAN GEOMETRY

LESSON 2:

Term	1	**Week**	3&4	**Grade**	12
Duration	3HR	**Weighting**	27% (40/150)	**Date**	
Sub-topics	Ratio and Proportionality Theorem				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Ratio, Parallel lines, Area of triangles, Heights of triangles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

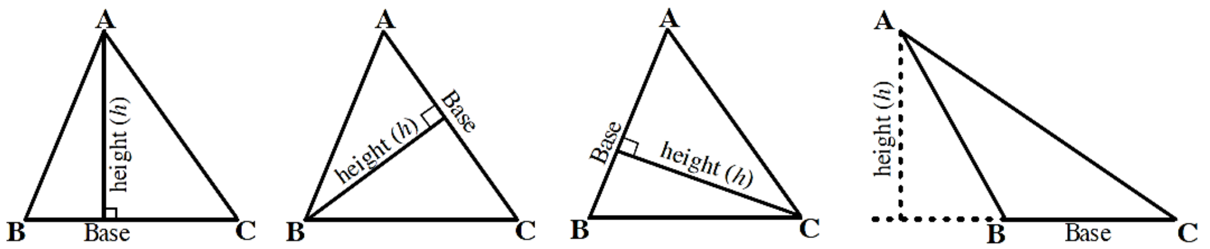
- Knowledge of all Grade 10 & 11 theorems & converses
- Grade 9 parallel lines and triangle geometry.
- Basic understanding of ratios.

RESOURCES

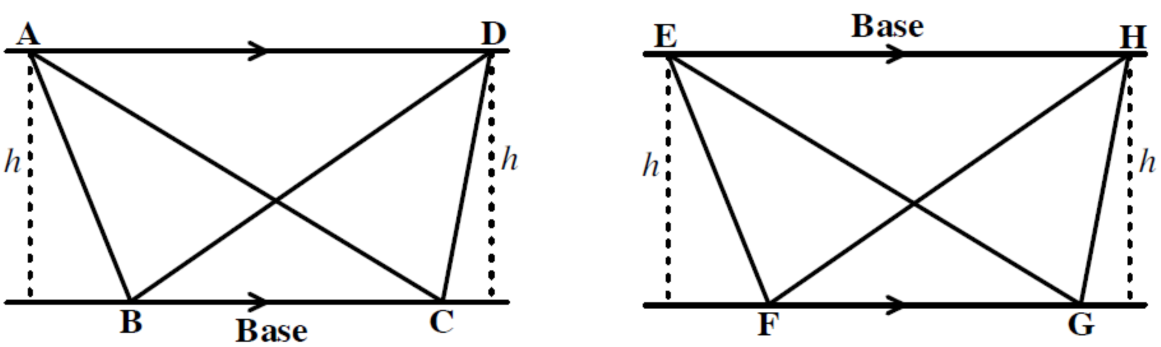
- Textbooks (Mind Action Series GR 12)
- KZN Provincial Gr 12 Investigation 2021
- Past Trial Examination Question Papers from Provinces

NOTES

- Start of by discussing the concept of ratio with learners.
- Discuss how the heights (altitudes) of triangles are obtained.



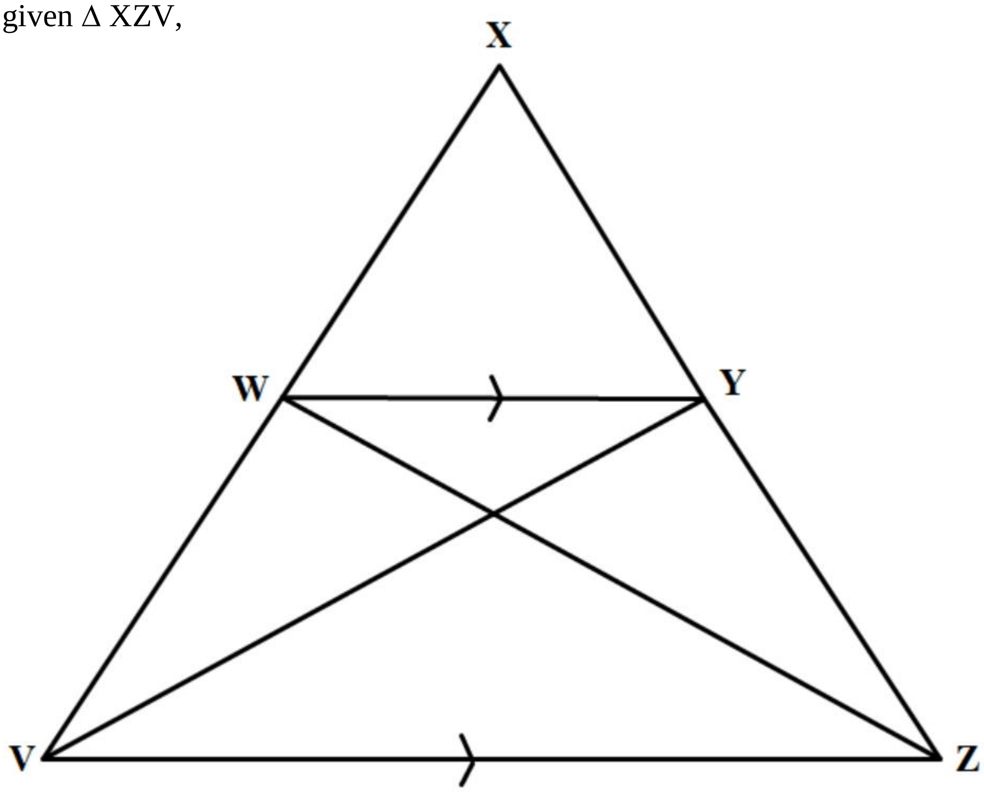
- Discuss the cases of Triangles between parallel lines which have common bases.



- Proceed to do this short investigation:

LEARNER ACTIVITY 1 | ***Day 1 – Classwork Activity***

In the diagram you are given Δ XZV,
with WY//VZ



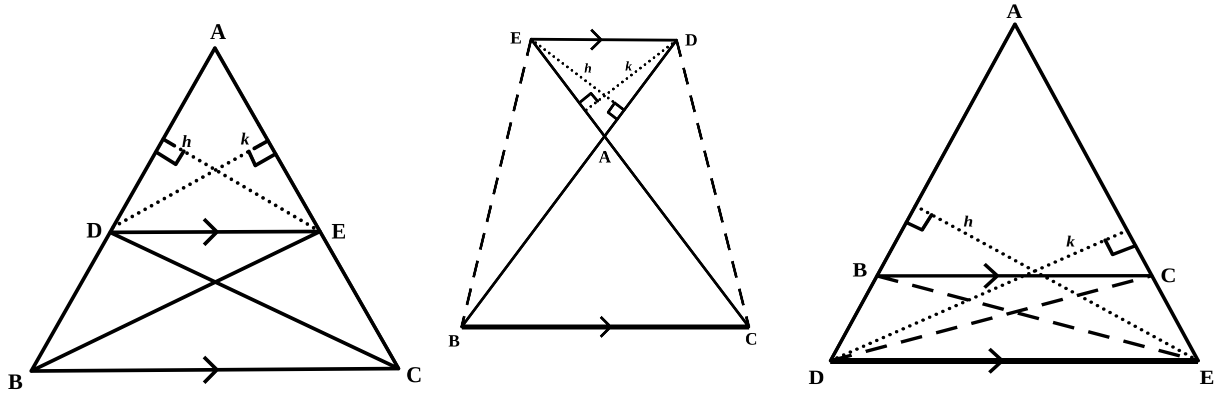
Consider the diagram above and answer the questions that follow:

1.1	Use the diagram above and construct:	
	1.1.1	Height "h" perpendicular to base XV of Δ XVY
	1.1.2	Height "k" perpendicular to base XZ of Δ XWZ
1.2	Using height "h" write down expressions for the following:	
	1.2.1	Area of Δ XWY = ……………………
	1.2.2	Area of Δ VWY = ……………………
	1.2.3	$\frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ VWY}}$ = ………………………………
	Using height "k" write down expressions for the following:	
	1.2.4	Area of Δ XWY = ……………………
	1.2.5	Area of Δ WYZ = ……………………
	1.2.6	$\frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ WYZ}}$ = ………………………………
1.3	Considering your answers in 1.2.3, 1.2.6, give a reason why the following statement can be made: $\frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ VWY}} = \frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ WYZ}}$	
1.4	Hence or otherwise, simplify $\frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ VWY}} = \frac{\text{Area of } \Delta \text{ XWY}}{\text{Area of } \Delta \text{ WYZ}}$	
1.5	Considering Δ XVZ with line WY parallel to base VZ, and your answer in 1.4, make a conjecture.	
1.6	Suggest a possible reason why the conjecture in 1.5 will not work if WY was not parallel to VZ.	

- Discuss the Formal Proof of the Ratio & Proportionality Theorem

THEOREM: The line drawn parallel to one side of a triangle divides the other two sides proportionally

Given : ΔABC, D lies on AB and E lies on AC. And DE // BC.



R.T.P : $\frac{AD}{DB} = \frac{AE}{EC}$

Proof : Join D to C and E to B. Draw altitudes *h* and *k* relative to bases AD and AE

$$\frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ BDE}} = \frac{\frac{1}{2}\text{AD}\times h}{\frac{1}{2}\text{DB}\times h} = \frac{\text{AD}}{\text{DB}} \quad \text{(same height)}$$

$$\frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ CED}} = \frac{\frac{1}{2}\text{AE}\times k}{\frac{1}{2}\text{EC}\times k} = \frac{\text{AE}}{\text{EC}} \quad \text{(same height)}$$

but Area Δ BDE = Area Δ CED (same base & between // lines)

$$\therefore \quad \frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ BDE}} = \frac{\text{Area } \Delta \text{ ADE}}{\text{Area } \Delta \text{ CED}}$$

$$\therefore \quad \frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}}$$

Discuss the converse the Ratio & Proportionality Theorem

The formal proof is not tested but the application of this theorem is!

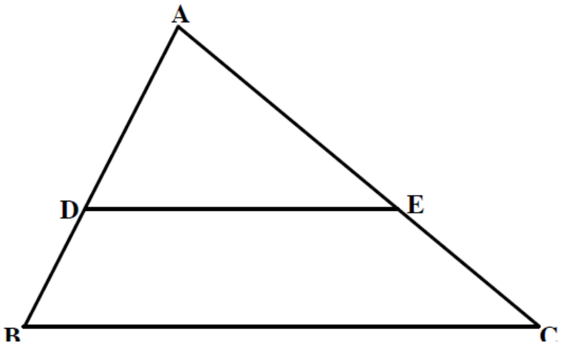
CONVERSE: If a line cuts two sides of a triangle proportionally, then that line is parallel to the third side

Whenever you use this theorem the reason you must give is:
Line divides sides ofΔ proportionally OR prop theorem; name || lines

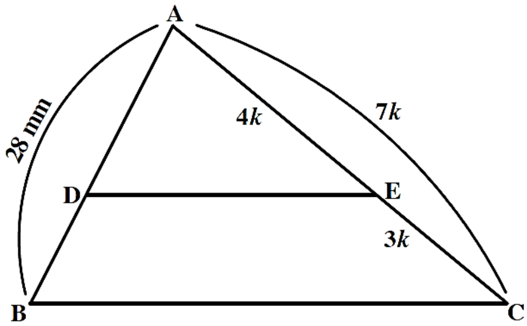
EXAMPLE 1

Question

In ΔABC, $DE \parallel BC$, $AB = 28$ mm and $AE : EC = 4 : 3$. Determine the length of BD.



Solution

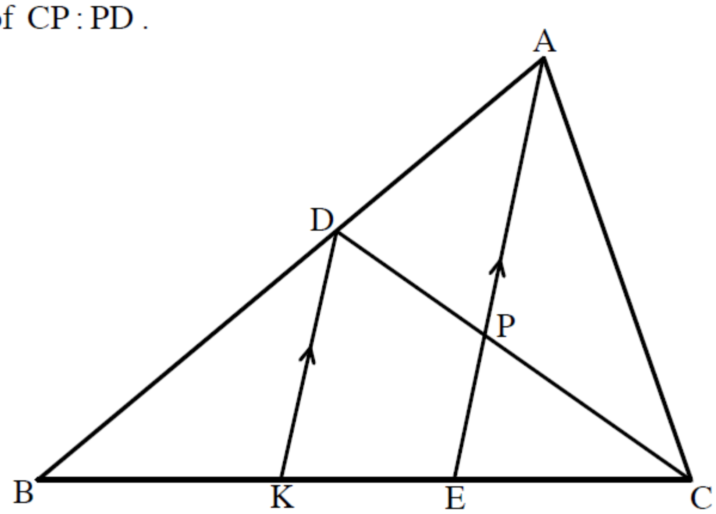


Statement	Reason
$\frac{BD}{28mm} = \frac{3k}{7k}$ $\therefore BD = \frac{3}{7} \times 28mm$ $\therefore BD = 12mm$	Line ∥ one side of ΔABC

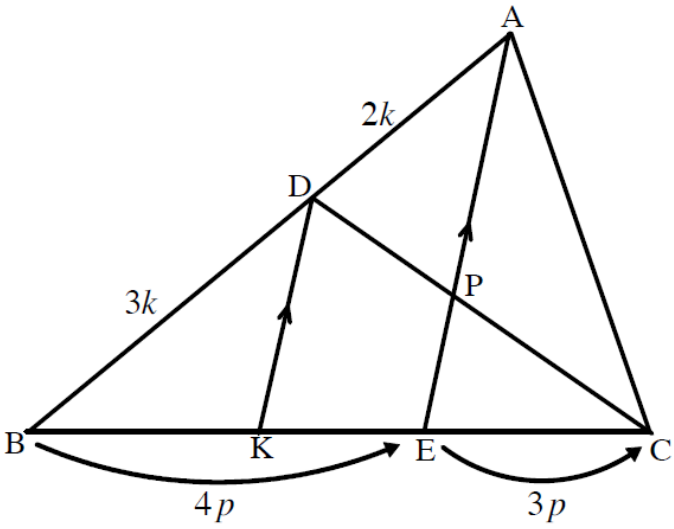
EXAMPLE 2

Question

D and E are points on sides AB and BC respectively of ΔABC such that $AD : DB = 2 : 3$ and $BE = \frac{4}{3} EC$. If $DK \parallel AE$ and AE and CD intersect at P, find the ratio of $CP : PD$.



Solution



$BE = \frac{4}{3}EC$

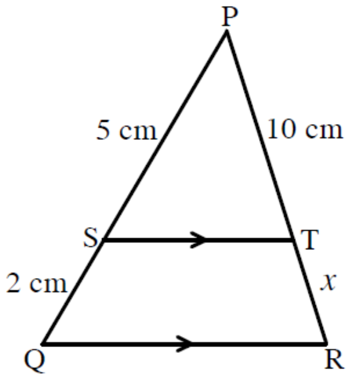
$\therefore \frac{BE}{EC} = \frac{4}{3}$

Statement	**Reason**
$\frac{CP}{PD} = \frac{3p}{EK}$	Line ∥ one side of ΔCDK
Now	
$\frac{EK}{4p} = \frac{AD}{AB}$	Line ∥ one side of ΔABE
$\therefore \frac{EK}{4p} = \frac{2k}{5k}$ $\therefore EK = \frac{2k}{5k} \times 4p$ $\therefore EK = \frac{2}{5} \times 4p$ $\therefore EK = \frac{8p}{5}$ $\therefore \frac{CP}{PD} = \frac{3p}{\frac{8p}{5}}$ $\therefore \frac{CP}{PD} = 3p \times \frac{5}{8p}$ $\therefore \frac{CP}{PD} = \frac{15}{8}$	

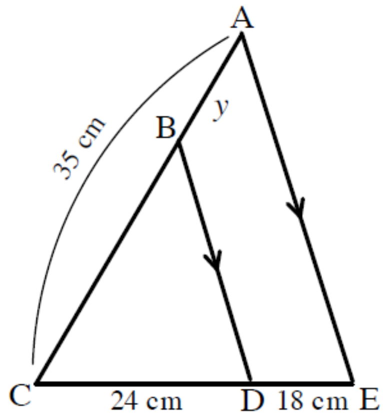
DAY 1 Activities 2.2.1 – 2.2.2
DAY 2 Activities 2.2.3 – 2.2.5
DAY 3 Activities 2.2.6 – 2.2.7

ACTIVITIES/ASSESSMENTS

2.2.1. Find, giving reasons the value of *x*



2.2.2 In ΔACE, BD||AE. Calculate the value of *y*.



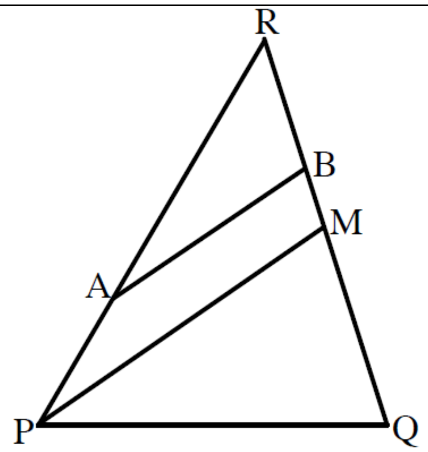
2.2.3. Consider the diagram alongside:

$\frac{RB}{RQ} = \frac{1}{3}$, PA : AR = 1 : 2 and PM//AB

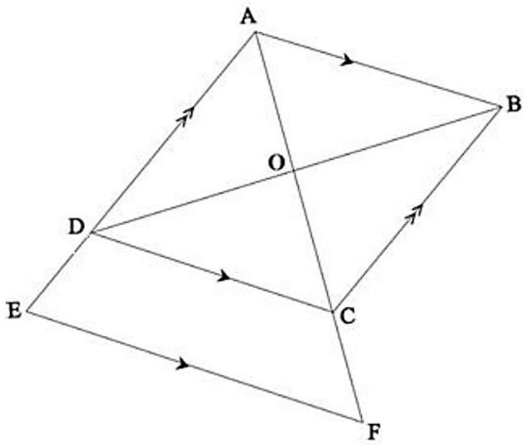
2.2.3.1 Write down the values for RA:RP and RB : BQ.

2.2.3.2 Determine BM : BR.

2.2.3.3 Prove that RM = MQ



2.2.4. In the diagram below, ABCD is a parallelogram. AD and AC are produced to E and F respectively so that EF || DC. AF and DB intersect at O.
AD = 12 units; DE = 3 units; DC = 14 units; CF = 5 units.



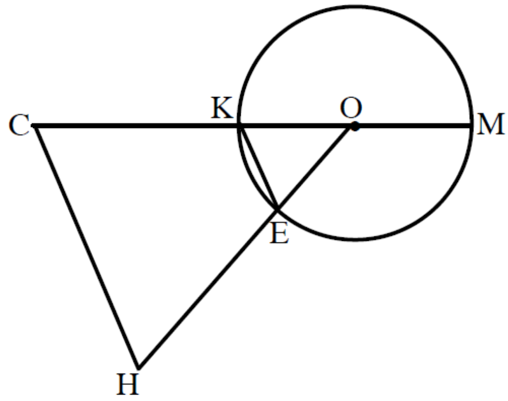
Calculate, giving reasons, the length of:

2.2.4.1 AC
2.2.4.2 AO
2.2.4.3 EF

2.2.4.4 Prove that : $\frac{\text{AREA } \Delta ADO}{\text{AREA } \Delta AEF} = \frac{8}{25}$

2.2.5. In the diagram below, KM is a diameter of the circle centre O. OK = r. OC = $4r$ and $\hat{H} = \hat{C}$.

Prove that EK // HC



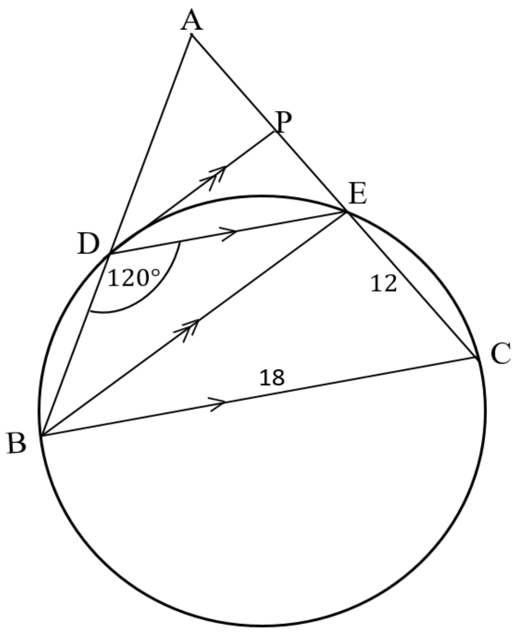
2.2. 6. In Δ ABC in the diagram alongside, D is a point on AB such that AD : DB = 5 : 4. P and E are points on AC such that DE ∥ BC and DP ∥ BE.
BC is NOT a diameter of the circle.

Given: $B\hat{D}E = 120°$, $EC = 12$ units and $BC = 18$ units.

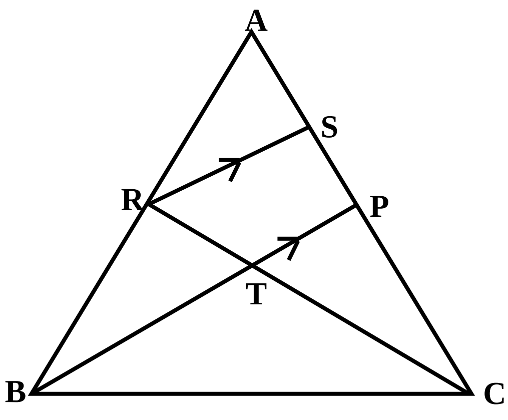
Determine, with reasons:

2.2.6.1 The length of AE

2.2.6.2 $\frac{\text{Area of } \Delta \text{ AEB}}{\text{Area of } \Delta \text{ ECB}}$



2.2.7 In ΔABC, R is a point on AB. S and P are points on AC such that RS // BP. P is the midpoint of AC. RC and BP intersects at T. $\frac{AR}{AB} = \frac{3}{5}$.



Calculate with reasons, the following ratios:

2.2.7.1 $\frac{AS}{SC}$

2.2.7.2 $\frac{RT}{TC}$

2.2.7.3 $\frac{\Delta ARS}{\Delta ABC}$

TOPIC: EUCLIDEAN GEOMETRY					
LESSON 3:					
Term	1	**Week**	4	**Grade**	12
Duration	3H	**Weighting**	27% (40/150)	**Date**	
Sub-topics	Similarity				
RELATED CONCEPTS/ TERMS/VOCABULARY					
• Similar, Equiangular, Congruent and the difference between these terms.					
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					
• Knowledge of all Grade 10 & 11 theorems & converses • Grade 9 parallel lines and triangle geometry. • Formal proofs of the 5 examinable Gr11 proofs.					
RESOURCES					
• Textbooks (Mind Action Series GR 12) • Provincial Trial Exam Papers					

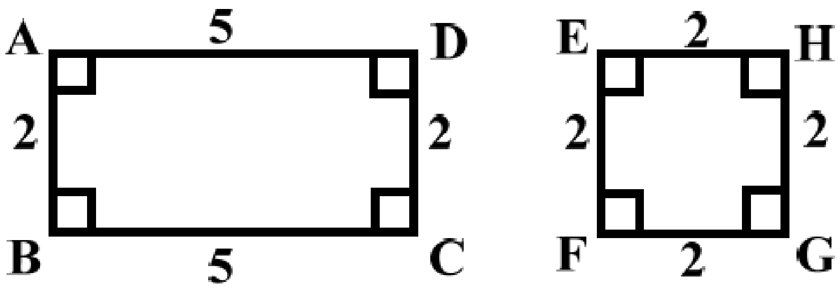
NOTES

Discuss concepts of similarity from earlier grades

Two polygons are similar if they have the same shape but not necessarily the same size.
Two conditions must **both** be satisfied for two polygons to be similar:
(a) The corresponding angles must be equal.
(b) The ratio of the corresponding sides must be in the same proportion.

Draw and discuss such examples with learners.

Have a discussion around whether the square and rectangle alongside are similar.

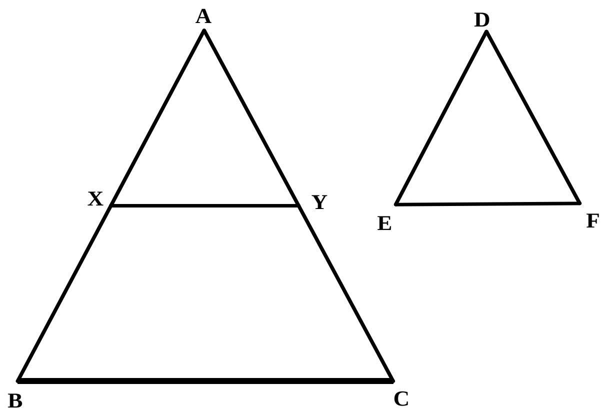


Considering Similar Triangles highlight….
With similar triangles, **only one of the two** conditions needs to be true in order for the two triangles to be similar. This is proved in the theorem below:

THEOREM : Equiangular Triangles are similar

Given Δ ABC and ΔDEF with $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$

R.T.P : $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$



Proof : Mark off X on AB and Y on AC such

That AX = DE and AY = DF. Construct XY

In Δ AXY and ΔDEF

1. $AX = DE$ (construction)

2. $AY = DF$ (construction)

3. $\hat{A} = \hat{D}$ (given)

$\therefore \Delta AXY \equiv \Delta DEF$ (SAS)

now $A\hat{X}Y = \hat{E}$ but $\hat{E} = \hat{B}$ (given)

$\therefore A\hat{X}Y = \hat{B}$

$\Rightarrow XY \parallel BC$ (correponding $\angle$'s =)

now $\frac{AB}{AX} = \frac{AC}{AY}$ (line // one side of Δ)

but $AX = DE$ and $AY = DF$ (construction)

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

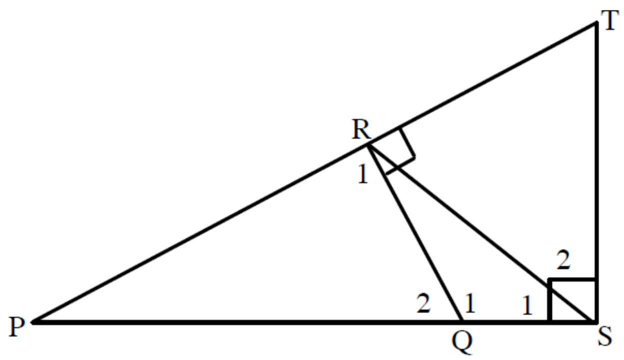
similarly by marking off equal lengths on BA and BC

it can be shown that : $\frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

CONVERSE: If the corresponding sides are in the same proportion, then the corresponding angles of the two triangles will be equal

The following Example to be discussed by the Educator in the lesson:



EXAMPLE

In ΔPST, $TS \perp PS$ and $RQ \perp PT$. Prove:

(a) $\Delta PRQ ||| \Delta PST$

(b) $RQ : PQ = ST : PT$

(c) $PR . PT = PQ . PS$

a)

In ΔPRQ and ΔPST:

(1) $\hat{P} = \hat{P}$

(2) $\hat{R}_1 = \hat{S}_1 + \hat{S}_2 = 90°$

(3) $\hat{Q}_2 = \hat{T}$

$\therefore \Delta PRQ ||| \Delta PST$

Match the corresponding angles of ΔPRQ and ΔPST as follows and then prove the pairs of angles equal.

$\hat{P}$ —— $\hat{P}$

$\hat{R}_1$ —— $\hat{S}_1 + \hat{S}_2$

$\hat{Q}_2$ ········ $\hat{T}$

Draw solid lines for each pair of corresponding angles that are equal. The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle.

b)

Since $\Delta PRQ ||| \Delta PST$:

$\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$ — corr sides of Δ's in proportion

$\therefore \frac{RQ}{ST} = \frac{PQ}{PT}$

$\therefore \frac{RQ}{PQ} = \frac{ST}{PT}$ — cross multiplication

$\therefore RQ:PQ = ST:PT$

c)

$\frac{PR}{PS} = \frac{PQ}{PT}$ — since $\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$

$\therefore PR\ .\ PT = PQ\ .\ PS$ — cross multiplication

DAY 1 Activities 2.3.1 – 2.3.3

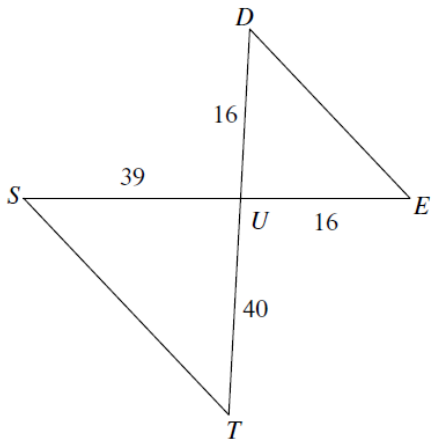
DAY 2 Activities 2.3.4 – 2.3.6

DAY 3 Activities 2.3.7 – 2.3.10

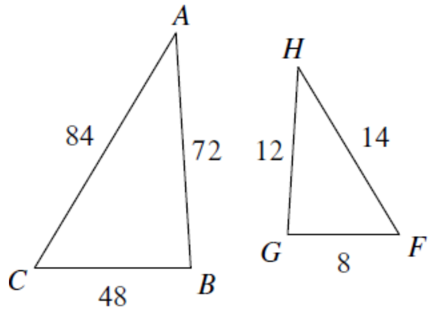
ACTIVITIES/ASSESSMENTS

2.3.1. In each of the cases below: State if the triangles are Similar OR not and write down the reason:

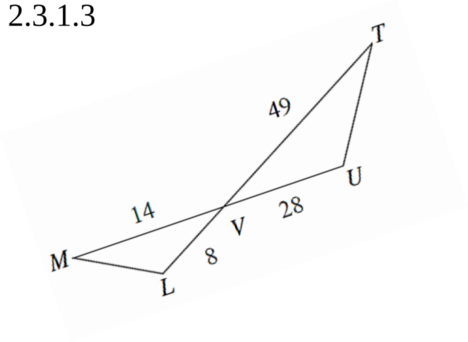
2.3.1.1



2.3.1.2

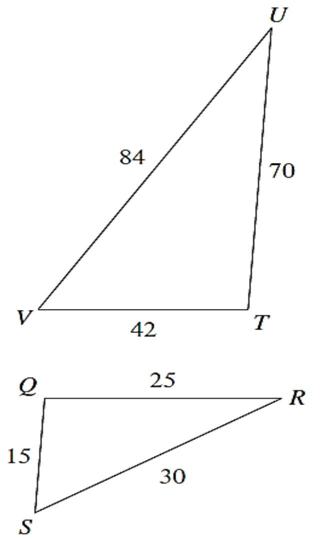


2.3.1.3

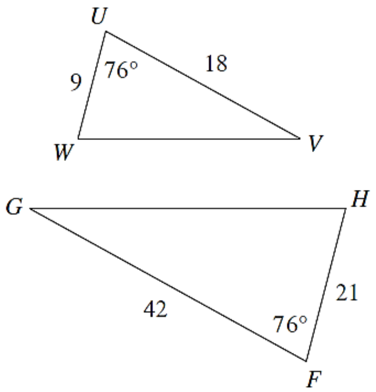


ΔMVL & ΔUVT

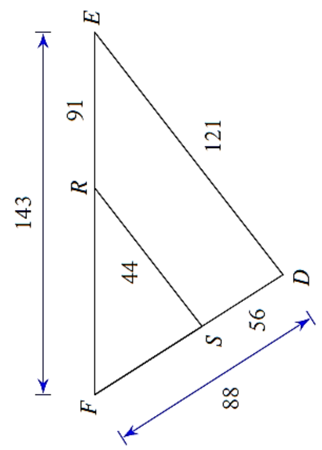
2.3.1.4



2.3.1.5



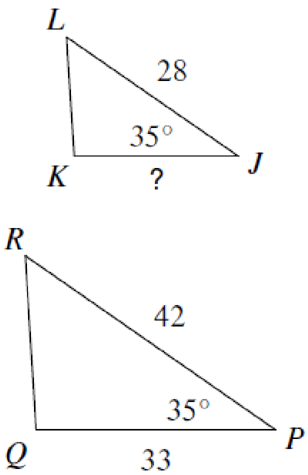
2.3.1.6



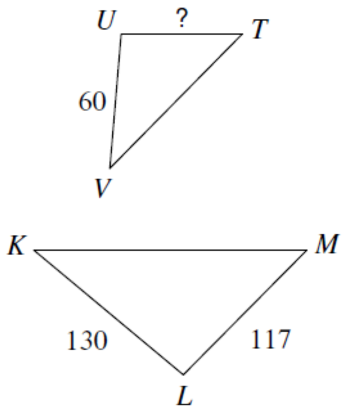
ΔFED& ΔFRS

2.3.2. Find the missing length. The triangles in each pair are similar.

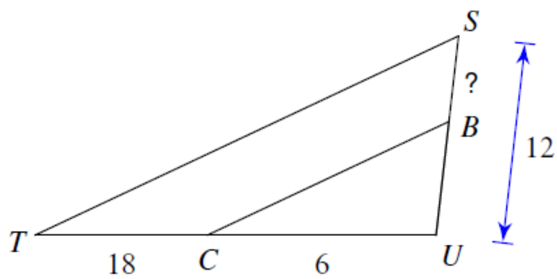
2.3.2.1



2.3.2.2

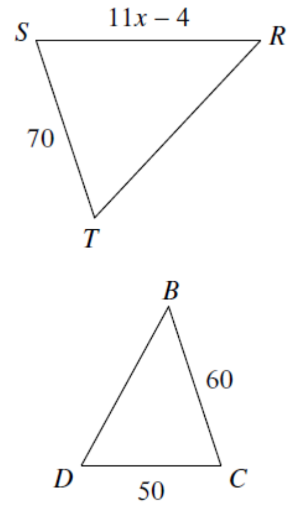


2.3.2.3

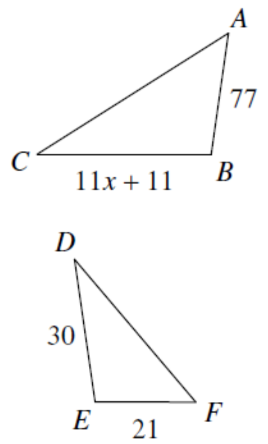


2.3.3. Solve for x. The triangles in each pair are similar.

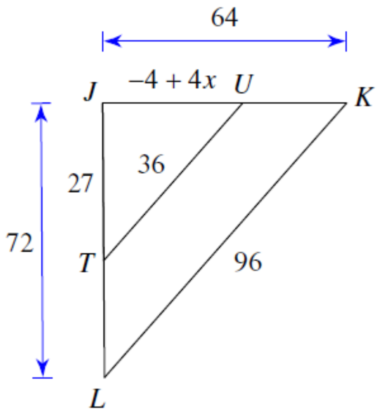
2.3.3.1



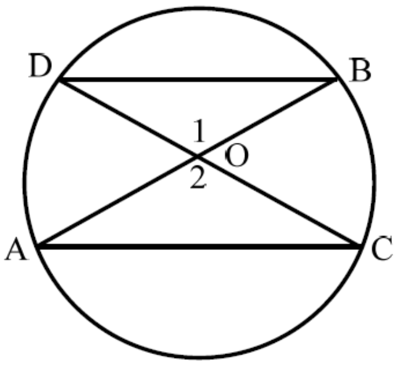
2.3.3.2



2.3.3.3



2.3.4. A, B, C and D are concyclic points. DOC and AOB are chords. DB and AC are joined.

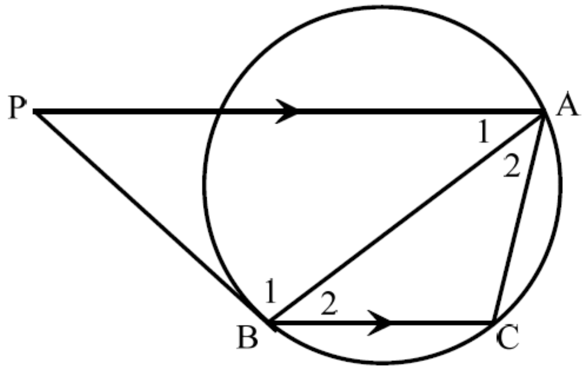


Prove that:

2.3.4.1 $\Delta AOC ||| \Delta DOB$

2.3.4.2 $\frac{OB}{OD} = \frac{OC}{OA}$

2.3.5. PB is a tangent to circle ABC. PA||BC.



Prove that:

2.3.5.1 $\Delta PAB ||| \Delta ABC$

2.3.5.2 PA:AB = AB:BC

2.3.5.3 $\frac{AP}{BP} = \frac{AB}{AC}$

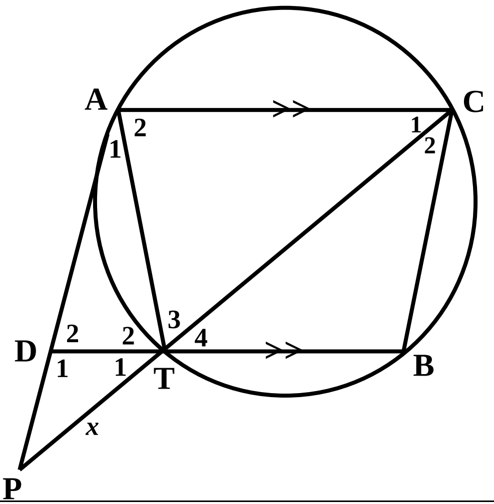
2.3.6. In the diagram alongside,
ACBT is a cyclic quadrilateral.
BT is produced to meet tangent AP on D.
CT is produced to P. AC // DB.

2.3.6.1 Prove that $PA^2 = PT \cdot PC$

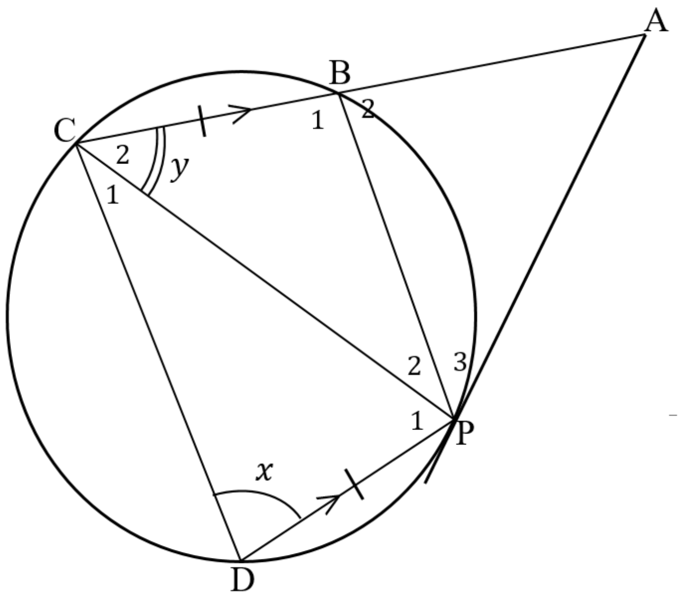
2.3.6.2 If PA = 6 units, TC = 5 units and PT = x, show that $x^2 + 5x - 36 = 0$.

2.3.6.3 Calculate the length of PT.

2.3.6.4 Calculate the length of PD.



2.3.7. AP is a tangent to the circle at P. CB ∥ DP and CB=DP. CBA is a straight line.
Let $\hat{D} = x$ and $\hat{C}_2 = y$.



Prove, with reasons that:

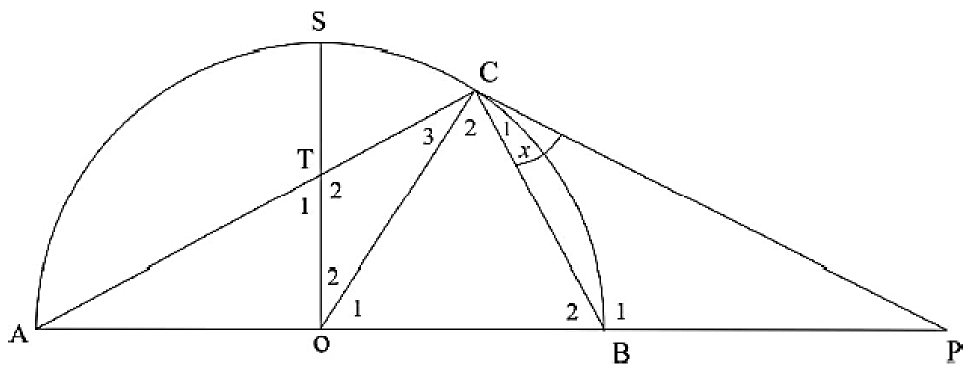
2.3.7.1 ΔAPC ||| Δ ABP

2.3.7.2 $AP^2 = AB.AC$

2.3.7.3 ΔAPC ||| Δ CDP

2.3.7.4 $AP^2 + PC^2 = AC^2$

2.3.8. In the diagram below, O is the centre of a semi-circle ACB. S is a point on the circumference and T lies on AC such that STO ⊥ AB. Diameter AB is produced to P, such that PC is a tangent to the semi-circle at C. Let $\hat{C}_1 = x$.



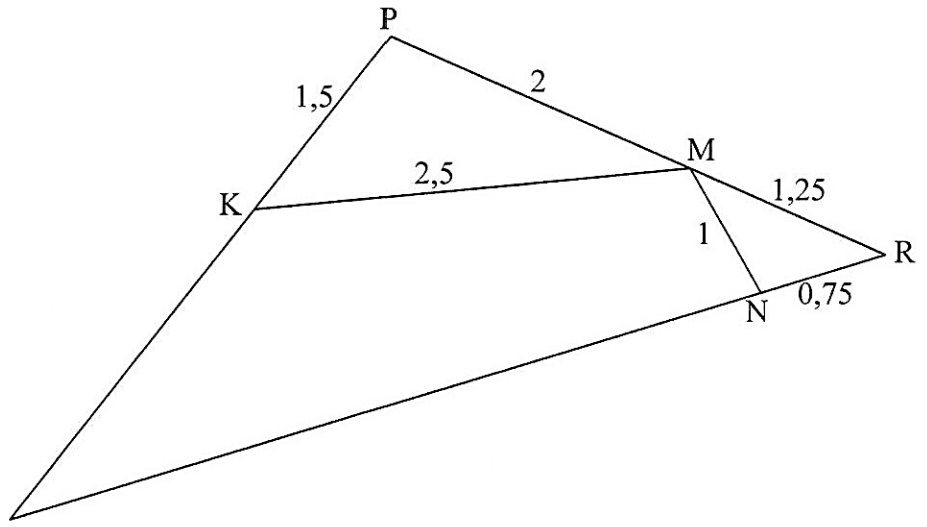
2.3.8.1 Write down, with reasons, 2 other angles equal to *x*.

2.3.8.2 Prove that ΔTOC|||ΔBPC

2.3.8.3 Prove that TO.PC = OB.BP

2.3.8.4 If BP = OB, show that $3OC^2 = PC^2$

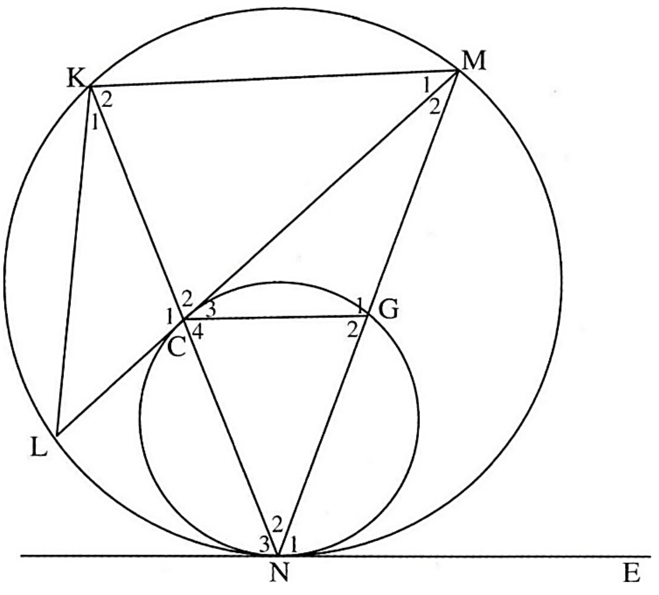
2.3.9 In the diagram below, K, M and N respectively are points on sides PQ, PR and QR of ΔPQR. 1,5•, PM = 2; KM = 2,5; MN = 1; MR = 1,25 and NR 0,75.



2.3.9.2. Prove that ΔKPM /// ΔRNM

2.3.9.3. Determine the length of NQ

2.3.10 In the diagram below NE is a common tangent to the two circles. NCK and NGM are double chords. Chord LM of the larger circle is a tangent to the smaller circle at point C. KL, KM and CG are drawn.



Prove that :

2.3.10.1. $\frac{KC}{KN} = \frac{MG}{MN}$

2.3.10.2. KMGC is a cyclic quadrilateral if CN = NG.

2.3.10.3. ΔMCG///ΔMNC

2.3.10.4. $\frac{MC^2}{MN^2} = \frac{KC}{KN}$

TOPIC: EUCLIDEAN GEOMETRY					
LESSON 4:					
Term	1	**Week**	5	**Grade**	12
Duration	1HR	**Weighting**	27% (40/150)	**Date**	
Sub-topics	Pythagoras Theorem application				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Tangents, Secants, Segments, Circles, Arcs.
- Theorems, Corollaries, Converses.
- Square, Hypotenuse, Adjacent sides

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Knowledge of all Grade 10 & 11 theorems & converses
- Grade 9 triangle geometry.

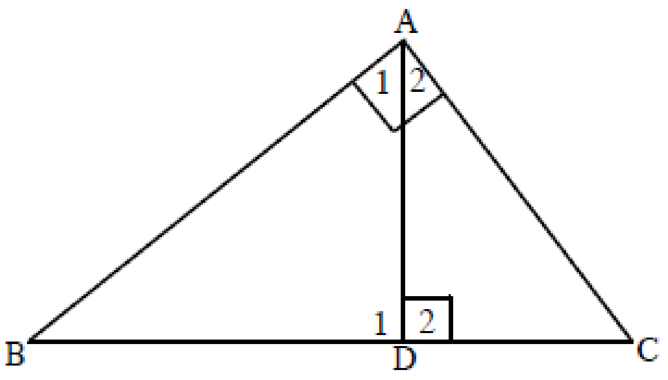
RESOURCES

- Textbooks (Mind Action Series G 12)

NOTES

- Discuss the Theorem below….The Proof is not for Examination purposes

> The perpendicular drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, divides the triangle into two triangles that are similar to each other and similar to the original triangle.



- See if learners can prove the three triangles similar by inspection and not long formal proofs.

- Develop the lesson further that since $\Delta ABC|||\Delta DBA|||\Delta DAC$ …….

<u>**Corollaries**</u>

$\Delta ABC|||\Delta DBA$	$\Delta ABC|||\Delta DAC$	$\Delta DBA|||\Delta DAC$
$\therefore \frac{AB}{DB}=\frac{BC}{BA}=\frac{AC}{DA}$	$\therefore \frac{AB}{DA}=\frac{BC}{AC}=\frac{AC}{DC}$	$\therefore \frac{DB}{DA}=\frac{BA}{AC}=\frac{DA}{DC}$
$\therefore AB^2 = BD.BC$	$\therefore AC^2 = CD.CB$	$\therefore AD^2 = BD.DC$

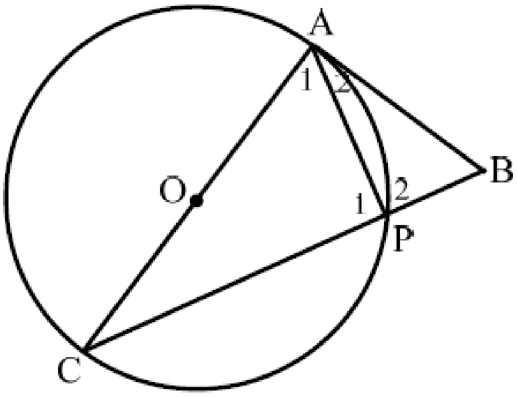
- Ultimately by using the corollaries above (and the diagram) one may prove the **Theorem of Pythagoras where: $BC^2 = AB^2 + AC^2$**

**** NOTE that the PROOF of the Theorem of Pythagoras is not for exam purposes but the APPLICATION of the Theorem is! ******

ACTIVITIES/ASSESSMENTS

2.4.1. In the figure alongside, O is the centre of the circle. AC is a diameter. Chord CP is produced to B.

Prove that: $AP^2 = PC.BP$

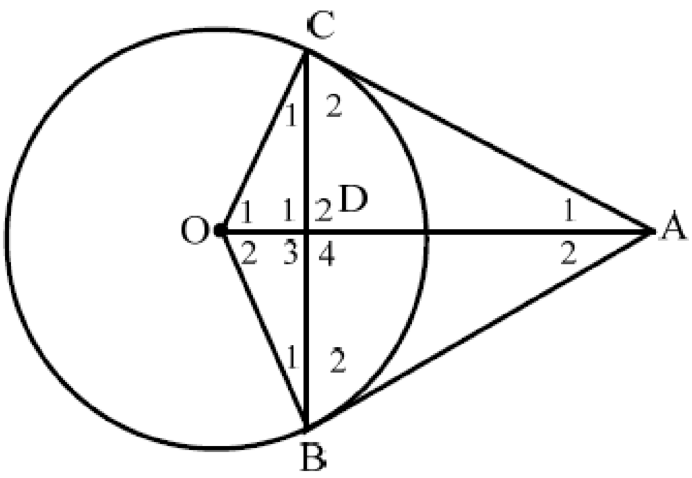


2.4.2. In the figure alongside, AB and AC are tangents to the circle with centre O. OD ⊥ BC.

Prove that:

2.1 $BD^2 = OD.DA$

2.2 $\frac{OC^2}{AC^2} = \frac{OD}{DA}$



TOPIC: EUCLIDEAN GEOMETRY					
LESSON 5:					
Term	1	**Week**	5	**Grade**	12
Duration	3HRS	**Weighting**	27% (40/150)	**Date**	
Sub-topics	Solving Euclidean Geometry Riders				

RELATED CONCEPTS/ TERMS/VOCABULARY

- All terminology related to FET Euclidean Geometry

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Knowledge of all Grade 10,11 & 12 theorems, converses, corollaries and axioms.
- Grade 9 parallel lines and triangle geometry.
- Formal proofs of all the examinable proofs.

RESOURCES

- Textbooks (Mind Action Series G 12)
- KZN Provincial Euclidean Geometry document 2015
- DBE NSC Exam Papers 2015 - 2020

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Learners attempt to solve Euclidean Riders without sound understanding of Theorems, Corollaries and Axioms.
- Learners do not adopt any strategy in answering or approaching Euclidean riders.

NOTES

Discuss the approaches below with learners.

HOW TO GO ABOUT SOLVING A GEOMETRY RIDER

1. What knowledge must you have?

- Know all terminology associated with Euclidean Geometry relevant to the School Curriculum.
- Be able to state ALL Theorems/ Converses of Theorems/ Axioms and Corollaries **AND** be able to draw a rough diagram to describe every statement. Pages 2 to 5 of this supplement indicate the important theorems and corollaries that must be learnt and illustrations that should be remembered.
- Know how to write reasons in abbreviated form for the formal writing of proofs. Approved reasons are found in the Examination Guideline

2. What approach to use?

- When you see the Diagram (involving a circle) and see the information provided use what we call the **"DOCTOR CAPE TOWN"** Method. That is look for **D**iameter/ **R**adius/ **C**yclic Quadrilaterals/ **P**arallel Lines/ **T**angents in other words **DRCPT** (Doctor Cape Town ☺) This will help you identify all the key aspects in the diagram and make problem solving easier!

- Use Colour Pencils (Maximum of 3 colours). This is particularly important when proving triangles similar.
- Always remember the order of questions is critical. Invariably what is done in a preceding question is vital to solve following questions.
- Remember correct writing of the solution is as important as solving the question itself.

3. How to Prove ....

1) That lines are Parallel :

Prove: Alternate angles equal **or**

Corresponding angles equal **or**

Co-interior angles supplementary.

2) That a quadrilateral is Cyclic :

Prove: That a pair of opposite angles are supplementary **or**

The exterior angle is equal to the interior opposite angle **or**

The angles in the same segment are equal.

3) That a chord is a diameter:

Prove: That the angle subtended by the chord on the circumference is a right angle.

The line between the chord and the tangent is a right angle.

4) That a line is a tangent :

Prove: That the angle between the line and a chord is equal to the angle subtended by the chord in the alternate segment.

or

That the line is perpendicular to the radius at point of contact on circle.

5) That two triangles are congruent:

Prove: A case of ....(Side/Side/Side) **or** (Side/Angle/Side) **or** (Angle/Side/Angle) **or** (90°/Hypotenuse/Side)

6) That two triangles are similar

Prove: A case of .... The two triangles are equiangular **or** The sides are in proportion.

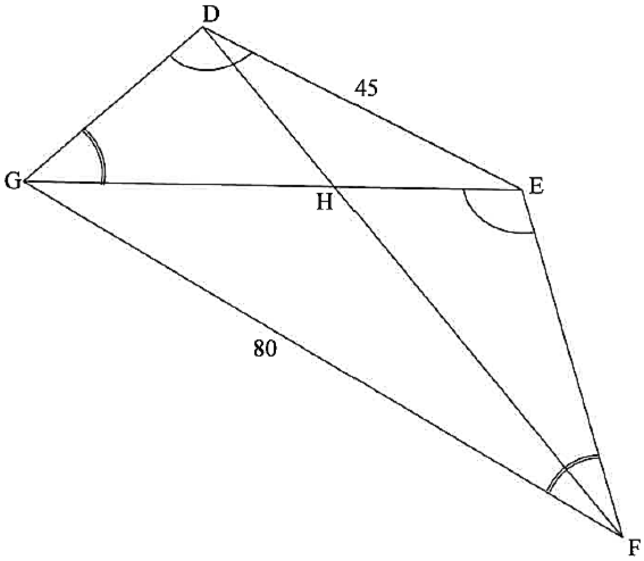
DAY 1 Activities 2.5.1 – 2.5.3

DAY 2 Activities 2.5.4 – 2.5.6

DAY 3 Activities 2.5.7 – 2.5.8

ACTIVITIES/ASSESSMENTS

2.5.1. In the diagram, DEFG is a quadrilateral with DE = 45 and GF = 80. The diagonals GE and DF meet in H. $G\hat{D}E = F\hat{E}G$ and $D\hat{G}E = E\hat{F}G$.



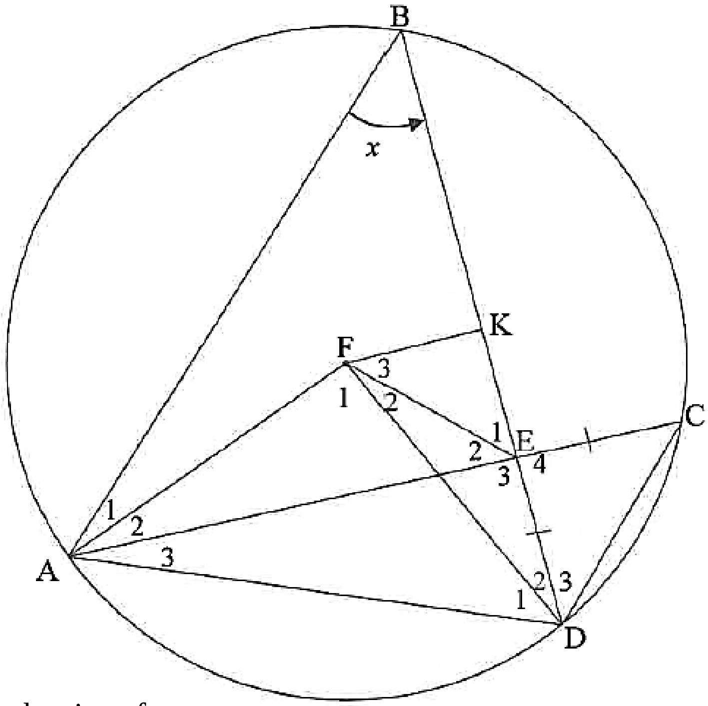
2.5.1.1 Give a reason why ΔDEG /// ΔEGF

2.5.1.2 Calculate the length of GE.

2.5.1.3 Prove that ΔDEH /// ΔFGH

2.5.1.4 Hence calculate the length of GH.

2.5.2. In the diagram, the circle with centre F is drawn. Points A, B, C and D lie on the circle. Chords AC and BD intersect at E such that EC = ED. K is the midpoint of chord BD. FK, AB, CD, AF, FE and FD are drawn. Let $\hat{B} = x$.



2.5.2.1 Determine with reasons the size of EACH of the following in terms of *x*.

a) $\hat{F}_1$

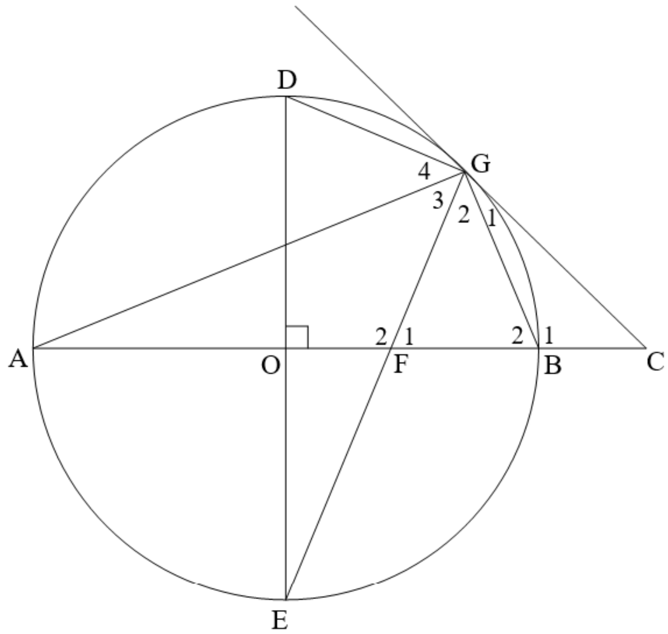
b) $\hat{C}$

2.5.2.2 Prove, with reasons, that AFED is a cyclic quadrilateral.

2.5.2.3 Prove, with reasons, that $\hat{F}_3 = x$.

2.5.2.4 If the area of ΔAEB = 6,25 × ΔDEC, calculate $\frac{AE}{ED}$

2.5.3. In the diagram, O is the centre of the circle and CG is a tangent to the circle at G. The straight line from C passing through O cuts the circle at A and B. Diameter DOE is perpendicular to CA. GE and CA intersect at F. Chords DG, BG and AG are drawn.



2.5.3.1 Prove that:

(a) DGFO is a cyclic quadrilateral

(b) GC = CF

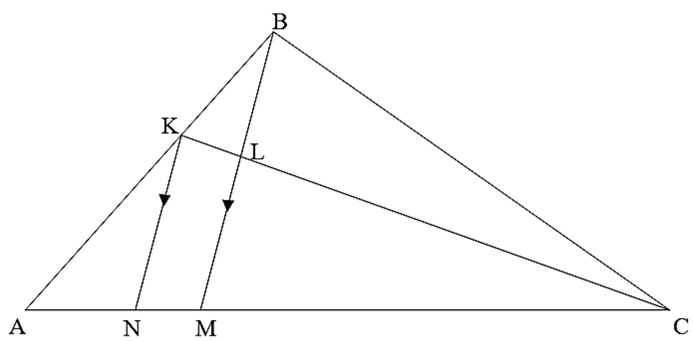
2.5.3.2 If it is further given that CO = 11 units and DE = 14 units, calculate:

(a) The length of BC

(b) The length of CG

(c) The size of $\hat{E}$.

2.5.4. In ΔABC in the diagram, K is a point on AB such that AK : KB = 3 : 2. N and M are points on AC such that KN || BM. BM intersects KC at L. AM : MC = 10 : 23.

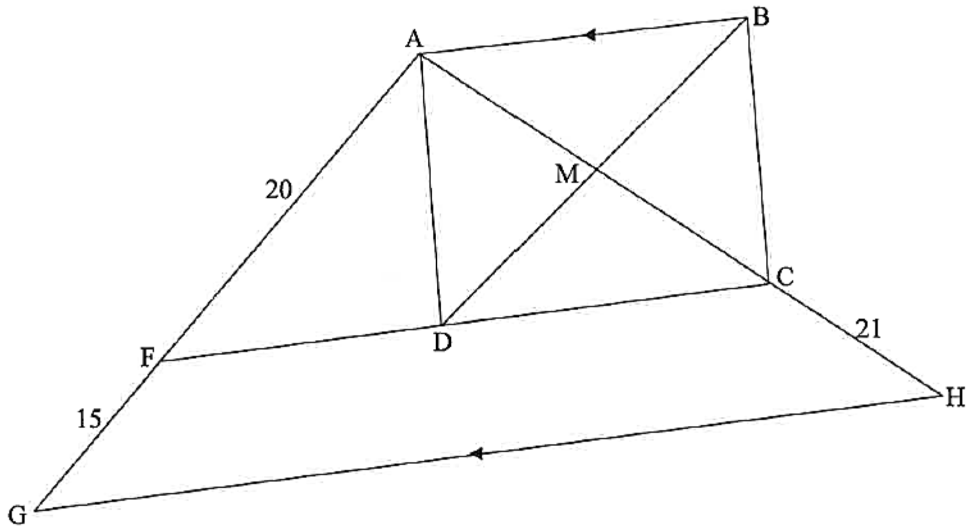


Determine, with reasons, the ratio of:

2.5.4.1 $\dfrac{AN}{AM}$

2.5.4.2 $\dfrac{CL}{LK}$

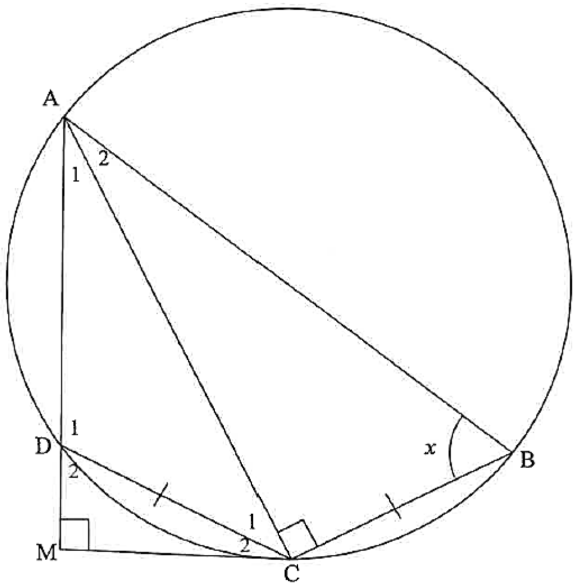
2.5.5. In the diagram below, ΔAGH is drawn. F and C are points on AG and AH respectively such that AF = 20 units, FG = 15 units and CH = 21 units. D is a point on FC such that ABCD is a rectangle with AB parallel to GH. The diagonals of ABCD intersect at M, a point on AH.



2.5.5.1 Explain why FC // GH

2.5.5.2 Calculate, with reasons, the length of DM.

2.5.6. In the diagram, ABCD is a cyclic quadrilateral such that AC ⊥ CB and DC = CB. AD is produced to M such that AM ⊥ MC. Let $\hat{B} = x$.



2.5.6.1 Prove that:

2.5.6.1.1 MC is a tangent to the circle at C

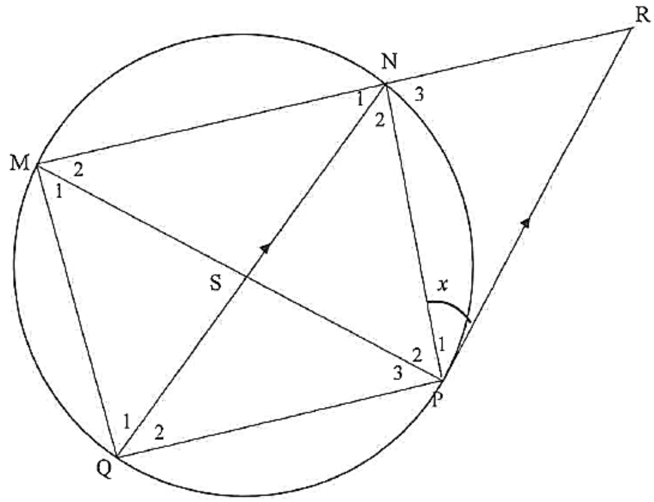
2.5.6.1.2 ΔACB /// ΔCMD

2.5.6.1 Hence, or otherwise, prove that:

2.5.6.2.1 $\frac{CM^2}{DC^2} = \frac{AM}{AB}$

2.5.6.2.2 $\frac{AM}{AB} = \sin^2 x$

2.5.7. Chord QN bisect $M\hat{N}P$ and intersects chord MP at S.The tangent at P meets MN produced at R such that QN // PR. Let $\hat{P}_1 = x$.



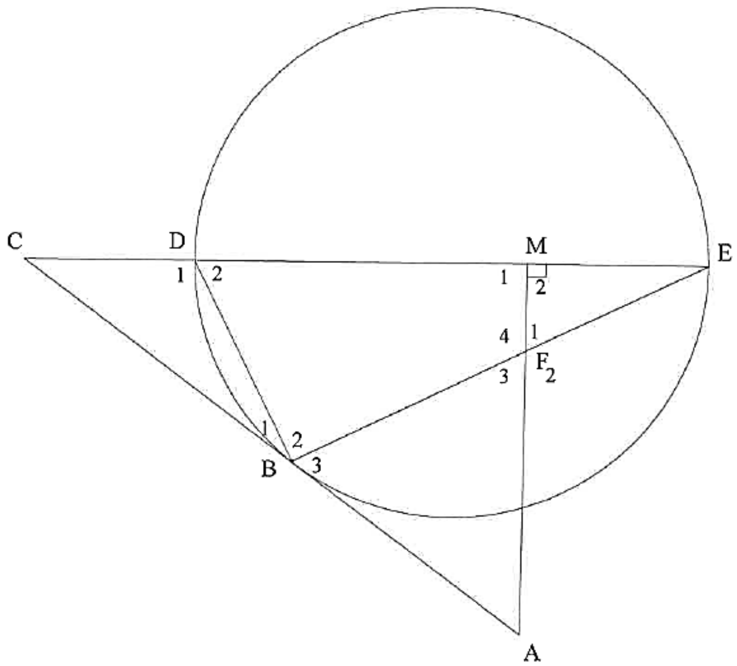
2.5.7.1 Determine the following angle in terms of x. Give reasons

a) $\hat{N}_2$

b) $\hat{Q}_2$

2.5.7.2 Prove, giving reasons, that $\frac{MN}{NR} = \frac{MS}{SQ}$

2.5.8. In the diagram, a circle passes through D,B and E. Diameter ED of the circle is produced to C and AC is a tangent to the circle at B. M is a point on DE such that AM ⊥ DE. AM and chord BE intersects at F.



2.5.8.1 Prove giving reasons that:

2.5.8.1.1 FBDM is a cyclic quadrilateral

2.5.8.1.2 $\hat{B}_3 = \hat{F}_1$

2.5.8.1.3 ΔCDB /// ΔCBE

2.5.8.2 If it further given CD = 2 units and DE = 6 units, calculate the length of:

2.5.8.2.1 BC

2.5.8.2.2 DB

TOPIC: EUCLIDEAN GEOMETRY					
LESSON 6:					
Term	1	**Week**	5	**Grade**	12
Duration	1HR	**Weighting**	27% (40/150)	**Date**	
Sub-topics	Short Test (Ratio, Proportionality)				
RELATED CONCEPTS/ TERMS/VOCABULARY					
• All terms related to Euclidean Geometry.					
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					
• Knowledge of all Grade 10 & 11 theorems & converses • Grade 9 parallel lines and triangle geometry. • Formal proofs of all examinable theorems.					
NOTES					
• Test will be written for the first 30 minutes of the lesson • Learners to interchange the task and mark during the discussion of the solutions.					

TOPIC 3 : Trigonometry					
LESSON 1:					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**Compound Angles**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Co-functions

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities

RESOURCES

Grade 12 Textbooks (Siyavula and Study &Master)

NOTES

The derivation of $\cos(\alpha+\beta)$, $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$ from $\cos(\alpha-\beta)$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

1. From the formula of $\cos(\alpha-\beta)$ derive the formula of $\cos(\alpha+\beta)$

   $\cos(\alpha+\beta)=\cos(\alpha-(-\beta)$
   $\therefore \cos[(\alpha-(-\beta)]=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$
   $\therefore \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

2. From the formula of $\cos(\alpha-\beta)$ derive the formula of $\sin(\alpha+\beta)$
   $\sin(\alpha+\beta)=\cos[90°-(\alpha+\beta)]$
   $=\cos[(90°-\alpha)-\beta]$
   $=\cos(90°-\alpha)\cos\beta+\sin(90°-\alpha)\sin\beta$
   $=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

3. From the formula of $\cos(\alpha-\beta)$ derive the formula of $\sin(\alpha-\beta)$
   $\sin(\alpha-\beta)=\cos[90°-(\alpha-\beta)]$
   $=\cos[(90°-\alpha)+\beta]$
   $=\cos[(90°-\alpha)-(-\beta)]$
   $=\cos(90°-\alpha)\cos(-\beta)+\sin(90°-\alpha)\sin(-\beta)$
   $=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

ACTIVITIES/ASSESSMENTS
3.1.1 Given that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, deduce: (a) $\sin(90°-\alpha)=\cos\alpha$ (b) $\cos(90°-\alpha)=\sin\alpha$ 3.1.2 Expand the following using the compound angle formulae, and simplify using special angles where possible: ` (a) $\cos(x-20°)$ (b) $\sin(A+45°)$ (c) $\cos 15°$

TOPIC: Trigonometry

LESSON 2:

Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	

Sub-topics	**Compound Angles**

RELATED CONCEPTS/ TERMS/VOCABULARY

- Co-functions
- Special angles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities
- Special angles

RESOURCES

- Mind action series
- Grade 12 Textbooks (Siyavula)
- Grade 12 Study &Master

NOTES

Example 1

1.1 Write down a formula for $\cos(A+B)$ in terms of trigonometric ratios of A and B.

Solution

1.1 $\cos(A+B) = \cos A \cos B - \sin A \cos B$

Example 2

1.2 Simplify the following:

$\cos 78^\circ \cos 18^\circ + \cos 72^\circ \cos 12^\circ$

Solution

1.2

$\cos 78^\circ \cos 18^\circ + \sin 18^\circ \sin 78^\circ = \cos(78^\circ - 18^\circ)$

$= \cos 60^\circ$

$= \frac{1}{2}$

ACTIVITIES/ASSESSMENTS
3.2.1Simplify the following: (a) $\sin 80^0.\sin 20^0 + \cos 20^0.\cos 80^0$. (b) $cox20° \cos 40° - \sin 20° \sin 40°$ (c) $\cos 340° \sin 80° - \sin 160° \cos 80°$ (d) $\cos 35^o.\sin 25^o - \cos(-205^o).\cos 55^o$ 3.2.2. Derive the formula for $\sin(A+B)$ if you are given that $\cos(A+B) = \cos A \cos B - \sin A \sin B$ 3.2.3. Use the compound angles to prove that: $2\sin A \cos B = \sin(B+A) - \sin(B-A)$

TOPIC: Trigonometry					
LESSON 3					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**Double angles**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Square Identities

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Identities
- Compound angles
- Double

RESOURCES

Grade 12 Textbooks (Siyavula)

NOTES

Derivation of $\sin 2\alpha$ and $\cos 2\alpha$

Derivation of sin 2α

It was shown that $\sin(\alpha + \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. If $\alpha = \beta$,

Then, $\sin 2\alpha = \sin(\alpha + \alpha)$

$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha$

Derivation of cos 2α

It was shown that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. If $\alpha = \beta$,

Then, $\cos 2\alpha = \cos(\alpha + \alpha)$

$= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\therefore \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha$ can also be written as: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= (1 - \sin^2\alpha) - \sin^2\alpha$ (Square Identity)

$\therefore \cos 2\alpha = 1 - 2\sin^2\alpha$

And, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \cos^2\alpha - (1 - \cos^2\alpha)$

$\therefore \cos 2\alpha = 2\cos^2\alpha - 1$

ACTIVITIES/ASSESSMENTS
3.3.1 Use Double angle identities to simplify each expression: (a) $\dfrac{\sin 2\theta}{2\sin^2\theta}$ (b) $\dfrac{\sin 2\theta}{2\sin\theta}$ (c) $2\sin^2\theta + \cos 2\theta$ (d) $(\sin\theta + \sin\theta)^2$ (e) $(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$ (f) $\dfrac{\sin 2\theta}{2\tan\theta}$

TOPIC: Trigonometry					
LESSON 4:					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**Compound and double angles (CAST DIAGRAM)**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- CAST diagram
- Special angles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Pythagoras theorem
- Reduction formulae
- Co-functions
- Identities
- Compound angles
- Double

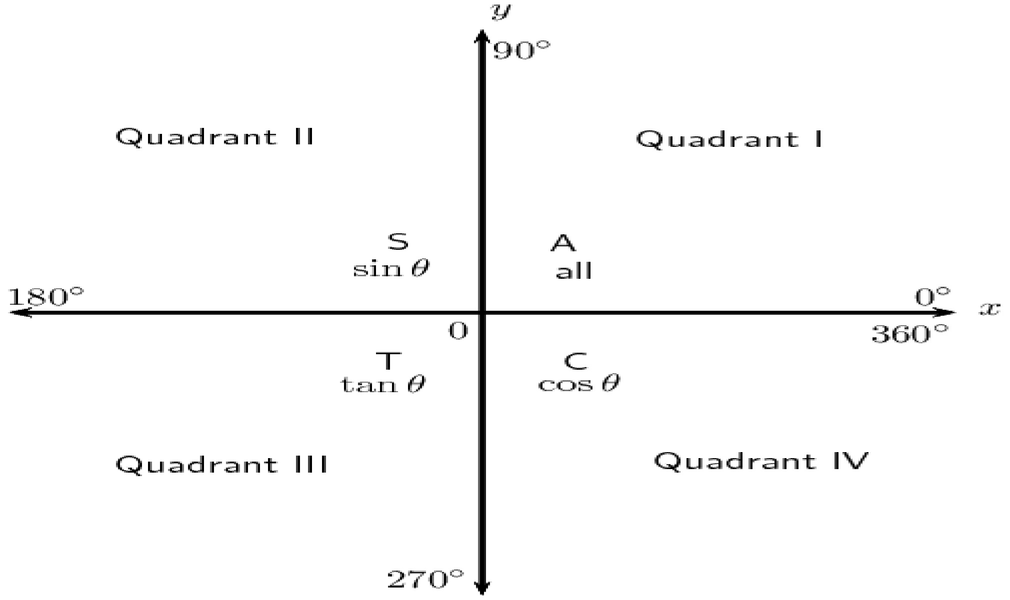
RESOURCES

Grade 12 Textbooks (Siyavula)

METHODOLOGY

- The teacher will use the CAST diagram to select the correct quadrant to draw a diagram
- Apply pythagoras theorem to determine the missing side of a right angled triangle

CAST DIAGRAM



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Examples 1:

1. If $p\sin\theta - 1 = 0$, $\cos\theta < 0$ and $p > 0$, determine the following in terms of p:

1.1 $\cos(90° - \theta)$

1.2 $\cos\theta$

1.3 $\sin(30° - \theta)$

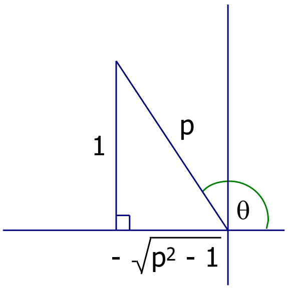
Solution:

1.

1.1 $\cos(90° - \theta)$

$= \sin\theta$

$= \dfrac{1}{p}$



1.2 $\cos\theta$

$= \dfrac{-\sqrt{p^2-1}}{p}$

1.3 $\sin(30° - \theta)$

$= \sin 30° \cos\theta - \cos 30° \sin\theta$

$= \frac{1}{2} \times \frac{-\sqrt{p^2-1}}{p} - \frac{\sqrt{3}}{2} \times \frac{1}{p}$

$= \dfrac{-\sqrt{p^2-1}-\sqrt{3}}{2p}$

Example 2:

2. If $\cos 21° = p$, determine the following in terms of p:

2.1 $\cos 201°$

2.2 $\sin 291°$

2.3 $\cos 42°$

2.4 $\tan 69°$

Solutions

2.1

$\cos 201° = \cos(180° + 21°$

$= -\cos 21°$

$= -p$

2.2

$\sin 291° = -\sin 69°$

$= -\cos 21°$

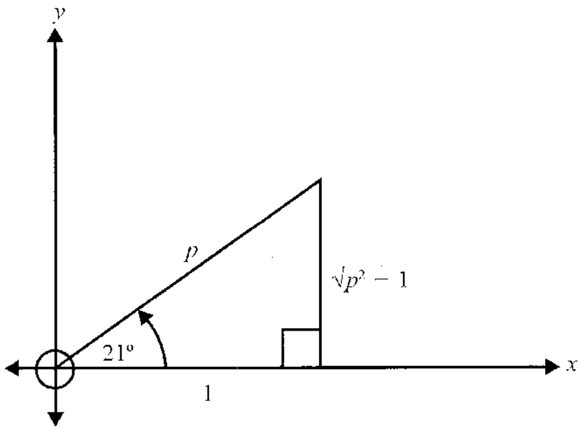
$= -p$

2.3

$\cos 42° = 2\cos^2 21° - 1$

$= 2p^2 - 1$

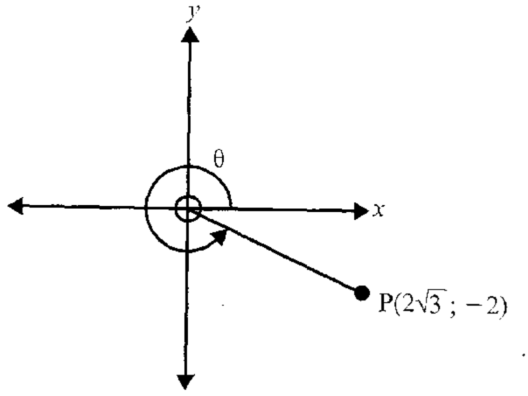
2.4



$$\tan 69° = \frac{1}{\sqrt{p^2 - 1}}$$

ACTIVITIES/ASSESSMENTS

3.4.1 If P is the point $(2\sqrt{3};-2)$ and the angle from the positive x-axis to OP is θ, without a calculator determine:



(a) the length of OP.

(b) the value of θ.

3.4.1 If $\sin 39° = p$, determine the following in terms of p:

(a) $\sin 51°$

(b) $\sin 129°$

(c) $\tan 321°$

(d) $\sin 78°$

3.4.3 Given $13\sin 2A = 12$, where $90° \leq 2A \leq 270°$. Without the use of a calculator, use a sketch in the correct quadrant to determine the following: Label the relevant angle(s).

(a) $\cos 2A$

(b) $\cos A$

3.4.4. If $(a^2+1)\sin 16° = 2a$, prove that $\frac{\cos 16°}{\sin 16°} = \frac{a}{2} - \frac{1}{2a}$.

3.4.5. If $17\cos y - 8 = 0$ and $16 + 12\tan x = 0$, determine, without a calculator, the value

$$\frac{1}{\cos(90° - y)} + \cos x \text{ if } \sin y < 0 \text{ and } \sin x > 0$$

3.4.6. Given that $\sin 15° = \frac{p}{r}$ and $p^2 + q^2 = r^2$, show, with the aid of a sketch, that $\frac{2pq}{r^2} = \frac{1}{2}$

TOPIC: Trigonometry					
LESSON : 5					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**Compound and double angles**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Co-functions
- Negative angles
- Cosine rule

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities

RESOURCES

Grade 12 Textbooks (Siyavula)
Grade 12 Platinum
Past papers

NOTES

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities
- Choose cos2A correctly to avoid time wasting

Example 1:

Simplify as far as possible:

$(\sin 15° + \cos 15°)^2$

Solution:

$= \sin^2 15 + 2\sin 15° \cos 15° + \cos^2 15°$

$= 1 + 2\sin 15° \cos 15°$

$= 1 + \sin 30°$

$\frac{3}{2}$

Example 2:
Simplify as far as possible:

$$\frac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x + 1}$$

Solution

$$= \frac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x + 1}$$

$$= \frac{2\sin x \cos -(1 - 2\sin^2 x) + 1}{2\sin x \cos +(2\cos^2 x - 1) + 1}$$

$$= \frac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x}$$

$$= \frac{2\sin x(\sin x + \cos x)}{2\cos x(\cos x + \sin x)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

ACTIVITIES/ASSESSMENTS

3.5.1Simplify the following without using a calculator

(a) $\dfrac{\cos 210°.\tan 330°}{\sin^2 225°}$

(b) $\dfrac{\sin(-x)\sin(360°-x)\sin 35°}{\cos(360°+x)\cos(90°-x)\cos 55°}$

(c) $\dfrac{\sin(x-90°).\sin 70°.\tan(x+180°)}{\cos 35°.\cos 55°.\cos(450°-x)}$

(d) $\cos 112°.\sin 22° - \dfrac{\cos 428°.\sin(-68°)}{\tan 202°}$

(e) $\sqrt[3]{\dfrac{\sin 225°.\cos 315°.\cos^2 300°.\cos(-60°)}{\sin 120°.\tan 570°}}$

(f) $\dfrac{2\sin 165°.\cos 345°}{\cos 45°\cos 15° + \sin 45°\sin 15°}$

(g) $\dfrac{\cos(-x).\tan(180° - x)}{\sin(180° - x)[\sin^2(90° + x) - \sin x.\cos(90° + x)]}$

(h) $\dfrac{\sin 2x + \sin x}{\cos 2x + \cos x + 1}$

TOPIC: Trigonometry					
LESSON : 6					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	15	**Date**	
Sub-topics	**Compound and double angles**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Co-functions
- Negative angles
- Cosine rule

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities

RESOURCES

Grade 12 Textbooks (Siyavula)
Grade 12 Platinum
Grade 12 Study & Master
Past papers

METHODOLOGY

- Verify the fundamental trigonometric identities.
- Consider the LHS or RHS as an algebraic expression
- Simplify trigonometric expressions using algebra and the identities
- Choose cos2A correctly to avoid time wasting

Example 1:

Show that: $2\sin A.\cos B = \sin(B + A) - \sin(B - A)$

Solution

RHS = $\sin A.\cos B + \sin B.\cos A - \sin B.\cos A + \sin A.\cos B = 2\sin A.\cos B$

Example 2:

Prove that: $\sin(\alpha + \beta).\sin(\alpha - \beta) = (\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta)$

Solution

$$
\begin{aligned}
LHS &= \sin(\alpha + \beta).\sin(\alpha - \beta) \\
&= [\sin\alpha\cos\beta + \cos\alpha\sin\beta][\sin\alpha\cos\beta - \cos\alpha\sin\beta] \\
&= \sin^2\alpha.\cos^2\beta - \cos^2\alpha\sin^2\beta \\
&= \sin^2\alpha\,(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta \\
&= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta \\
&= \sin^2\alpha - \sin^2\beta \\
&= (\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta) \\
&= RHS
\end{aligned}
$$

Example 3:

Prove that:

$$\frac{1}{\cos 2x} + \tan 2x = \frac{\sin x + \cos x}{\cos x - \sin x}$$

Solution:

$$LHS = \frac{1}{\cos 2x} + \tan 2x$$
$$= \frac{1}{\cos 2x} + \frac{\sin 2x}{\cos 2x}$$
$$= \frac{1}{\cos 2x} + \frac{\sin 2x}{\cos 2x}$$
$$= \frac{1 + 2\sin x \cos x}{\cos 2x}$$
$$= \frac{\sin^2 x + \cos^2 x + 2\sin x \cos x}{\cos^2 x - \sin^2 x}$$
$$= \frac{\sin^2 x + 2\sin x.\cos x + \cos^2 x}{\cos \;\; x - \sin^2 x}$$
$$= \frac{(\sin x + \cos x)(\sin x + \cos x)}{(\cos x - \sin x)(\cos x - \sin x)}$$
$$= \frac{\sin x + \cos x}{\cos x - \sin x}$$
$$\therefore LHS = RHS$$

ACTIVITIES/ASSESSMENTS

3.6.1 Prove that:

(a) $2\sin x \cos x = \dfrac{\sin 2x}{2\cos^2 x - \cos 2x}$

(b) $\dfrac{\cos 2\theta + \sin^2 \theta}{1 + \sin \theta} = 1 - \sin \theta$

(c) $\dfrac{2 - \cos^2 x - 2\sin x}{\cos^2 x} = \dfrac{1 - \sin x}{1 + \sin x}$

(d) $\dfrac{1}{\cos 2x} + \tan 2x = \dfrac{\sin x + \cos x}{\cos x - \sin x}$

(e) $\dfrac{\cos 2x.\tan x}{\sin^2 x} = \dfrac{\cos x}{\sin x} - \tan x$

3.6.2 Given the identity: $\tan 2B + \dfrac{1}{\cos 2B} = \dfrac{\cos B + \sin B}{\cos B - \sin B}$

(a) Prove the identity.

(b) Hence, determine the value of: $\dfrac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°}$

TOPIC: Trigonometry					
LESSON : 7					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**General solution**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Algebraic equations
- Compound and double angles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities

RESOURCES

Grade 12 Textbooks (Siyavula)
Grade 12 Platinum
Grade 12 Study & Master
Past papers

NOTES

- Apply the algebraic method to rearrange the equation
- Consider square identities to make trig ratios to be common
- The skill of double and compound angles is required

Example 1:

Solve for x if $3\tan^2 x - 1 = 0$ and $-180° \leq x \leq 180°$

Solution

$\tan x = \pm\frac{1}{\sqrt{3}}$

$x = 30° + n180°(n \in Z)$

$x = 45° + n180°(n \in Z)$

$x = 30° or x = 150° or x = -30° or x = -150°$

Example 2:

Determine the general solution of the following equation: $1 + \sin 2x = 4\sin^2 x$

Solution

$$1+\sin 2x = 4\sin^2 x$$
$$\sin^2 x+\cos^2 x+2\sin x\cos x-4\sin^2 x=0$$
$$\cos^2 x+2\sin x\cos x-3\sin^2 x=0$$
$$(\cos x+3\sin x)(\cos x-\sin x)=0$$
$$\cos x=-3\sin x or \cos x=\sin x$$
$$\tan x=-\frac{1}{3} or \tan x=1$$
$$x=180^\circ-18{,}43^\circ+n.180(n\in Z) or x=45^\circ+n.180^\circ(n\in Z)$$
$$x=161.57^\circ+n.180^\circ(n\in Z)$$

Example 3

Solve for x if :

$\cos 2x.\sin x-\sin 2x.\cos x=\cos(60^\circ-2x)$ without using a calculator and if $x\in[0^\circ;360^\circ]$

Solution

$$\sin(2x-x)=\cos(60^\circ-2x)$$
$$\sin x=\cos(60^\circ-2x)$$
$$\cos(90^\circ-x)=\cos(60^\circ-2x)$$
$$90^\circ-x=60^\circ-2x+n.360^\circ(n\in Z)$$
$$x=-30^\circ+n.360^\circ(n\in Z)$$
OR
$$90^\circ-x=360^\circ-(60^\circ-2x)+n.360^\circ(n\in Z)$$
$$x=-70^\circ-n.120^\circ(n\in Z)$$
$$x=50^\circ or x=170^\circ or x=290^\circ or x=330^\circ$$

ACTIVITIES/ASSESSMENTS

3.7.1 Determine the general solution of $\sin 60^\circ \cos x + \cos 60^\circ \sin x = 1$

3.7.2 Determine the general solution of $\cos 2x - 7\cos x - 3 = 0$

3.7.3 Determine the general solution of $6\sin^2 x + 7\cos x - 3 = 0$

3.7.4 Given: $1 + \tan^2 2A = 5\tan 2A - 5$,
Determine the general solution

3.7.5 Solve for $x: \sqrt{3}\sin x + \cos x = 2$

3.7.6 Determine a value for x if $\cos x; \sin x; \sqrt{3}\sin x$ is a geometric sequence.

TOPIC: Trigonometry					
LESSON : 8					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**General solution (Restrictions)**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Algebraic equations
- Compound and double angles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities

RESOURCES

Grade 12 Textbooks (Siyavula)
Grade 12 Platinum
Grade 12 Study & Master
Past papers

NOTES

- Apply the algebraic method to rearrange the equation
- Consider square identities to make trig ratios to be common
- The skill of double and compound angles is required

Example 1:

Solve for x if $3\tan^2 x - 1 = 0$ and $-180° \leq x \leq 180°$

Solution

$\tan x = \pm\frac{1}{\sqrt{3}}$

$x = 30° + n180°(n \in Z)$

$x = 45° + n180°(n \in Z)$

$x = 30° or x = 150° or x = -30° or x = -150°$

Example 2:

Determine the general solution of the following equation: $1+\sin 2x = 4\sin^2 x$

Solution

$$1+\sin 2x = 4\sin^2 x$$
$$\sin^2 x + \cos^2 x + 2\sin x\cos x - 4\sin^2 x = 0$$
$$\cos^2 x + 2\sin x\cos x - 3\sin^2 x = 0$$
$$(\cos x + 3\sin x)(\cos x - \sin x) = 0$$
$$\cos x = -3\sin x or \cos x = \sin x$$
$$\tan x = -\frac{1}{3} or \tan x = 1$$
$$x = 180° - 18{,}43° + n.180(n \in Z) or x = 45° + n.180°(n \in Z)$$
$$x = 161.57° + n.180°(n \in Z)$$

Example 3

Solve for x if :

$\cos 2x.\sin x - \sin 2x.\cos x = \cos(60° - 2x)$ without using a calculator and if $x \in [0°;360°]$

Solution

$$\sin(2x - x) = \cos(60° - 2x)$$
$$\sin x = \cos(60° - 2x)$$
$$\cos(90° - x) = \cos(60° - 2x)$$
$$90° - x = 60° - 2x + n.360°(n \in Z)$$
$$x = -30° + n.360°(n \in Z)$$
$$OR$$
$$90° - x = 360° - (60° - 2x) + n.360°(n \in Z)$$
$$x = -70° - n.120°(n \in Z)$$
$$x = 50° or x = 170° or x = 290° or x = 330°$$

ACTIVITIES/ASSESSMENTS

3.7.4 Determine the general solution of $\sin 60° \cos x + \cos 60° \sin x = 1$

3.7.5 Determine the general solution of $\cos 2x - 7\cos x - 3 = 0$

3.7.6 Determine the general solution of $6\sin^2 x + 7\cos x - 3 = 0$

3.7.4 Given: $1 + \tan^2 2A = 5\tan 2A - 5$,
Determine the general solution

3.7.5 Solve for $x : \sqrt{3}\sin x + \cos x = 2$

3.7.6 Determine a value for x if $\cos x; \sin x; \sqrt{3}\sin x$ is a geometric sequence.

ACTIVITIES/ASSESSMENTS

3.8.1 Prove that $\dfrac{1 - \cos^2 A + \sin 2A}{\sin A + 2\cos A} = \sin A$

3.8.2 For what value(s) of A is $\dfrac{1 - \cos^2 A + \sin 2A}{\sin A + 2\cos A} = \sin A$ not defined?

3.8.3(a) Prove the identity $\dfrac{\sin 2\theta + \cos\theta + 1}{\cos 2\theta} = \dfrac{2\cos\theta}{\cos\theta - \sin\theta}$

(b) Write down all the values for θ in the interval $[0°;180°]$ for which the identity is not valid.

3.8.4(a) Prove that $\dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$

(b) For which values of θ is the identity not valid?

TOPIC: Trigonometry					
LESSON : 9					
Term	1	**Week**		**Grade**	12
Duration	1 Hour	**Weighting**	50	**Date**	
Sub-topics	**Trigonometric Equations**				

RELATED CONCEPTS/ TERMS/VOCABULARY

- Algebraic equations
- Compound and double angles

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Reduction formulae
- Co-functions
- Identities
- Trig graphs

RESOURCES

Grade 12 Textbooks (Siyavula)
Grade 12 Platinum
Grade 12 Study & Master
Past papers

METHODOLOGY

- Simplify to the single ration
- Consider the amplitude
- The maximum and the minimum values are the y-values of the turning points.

Example

1.1 Given the expression: $\sin 2x.\cos 2x$

1.1.1 Calculate the maximum value of the above expression.

1.1.2 Calculate the first negative value of x for which the expression has this maximum value.

Solution

1.1.1 $\sin 2x.\cos 2x$

$= \frac{1}{2}.2\sin 2x.\cos 2x$

$= \frac{1}{2}\sin 4x$

$= \frac{1}{2}.1$

$= \frac{1}{2}$

1.1.2 $\sin 4x = 1$

$4x = 90° + k.360° \quad k \in Z$

$x = 22{,}5° + k.90° \quad k \in Z$

$x = -67{,}5°$

ACTIVITIES/ASSESSMENTS

3.9.1(a) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

(a) Hence determine the minimum value of $\frac{\sin 3A}{\sin A}$

3.9.2 Consider the expression $\sin x + \cos x$

(a) Prove that $(\sin x + \cos x)^2 = \sin 2x + 1$.

(b) Hence determine the maximum value of $\sin x + \cos x$

TOPIC: TRIGONOMETRY			
LESSSON: 10			
Term	One	**Week**	
Duration	120 minutes	**Weighting**	50
Sub-topic	2D/3D shape		
RELATED CONCEPTS/ TERMS/VOCABULARY	• Algebraic equations • Perpendicular Heights • Distance • Angle of Elevation • Angle of Depression • Types of Triangles: scalene; equilateral, isosceles etc. • Horizontal plane		

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Knowledge of polygons and their properties
- Sine rule, cosine rule and area rule
- Pythagoras theorem
- Trigonometric ratios
- Reduction formulae

RESOURCES:

- Chalkboard and other related teaching aids
- Modelling
- Grade 11 /12 textbooks (Mind Action Series and Maths Hand book and study guide)
- NSC Past exam papers

NOTES

SUMMARY

Solving Two-Dimensional Problems using the Sine, Cosine and Area Rules.

- **The sine-rule can be used when the following in known in the triangle:**
  - more than 1 angle and side
  - two sides and an angle(not included)

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{C}$$

- **The cosine rule can be used when the following is known of the triangle**
  - **three sides and one included angle**
  - $a^2 = b^2 + c^2 - 2bc\cos A$
  - $b^2 = a^2 + c^2 - 2ac\cos B$
  - $c^2 = a^2 + b^2 - 2ac\,\text{Cos}C$
- **If the lengths of the three sides are given, the formula can be written in the following form. To find $\hat{A}$,$\hat{B}$ OR $\hat{C}$ respectively:**

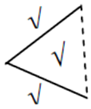
  Cos A = $\frac{b^2+c^2-a^2}{2bc}$ CosB= $\frac{a^2+c^2-b^2}{2ac}$ CosC= $\frac{a^2+b^2-c^2}{2ab}$

- **The area of a triangle can be found when at least two sides an included angle known**
  - Area of a triangle ABC $= \frac{1}{2}\ absinA$
  - Area of a triangle ABC $= \frac{1}{2}\ bcsinB$
  - Area of a triangle ABC $= \frac{1}{2}\ absinC$
- **The area rule is half the product of any two sides and an included angle**

Study Tips:

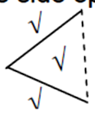
Sin Rule
In a solution of triangles question, use the sin rule to find a missing side or angle **only** if you have either two angles and one side, **or** two sides and an angle that is opposite one of the known sides. (Note: if the side opposite the given angle is the smaller of the 2 sides, there are 2 solutions)

Area Rule
To use the area rule you need to know 3 things: 2 sides and an included angle.

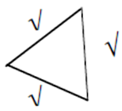
area rule  ← missing side

Cos Rule
In a solution of triangle questions use the cos rule

- To find the side opposite a given angle when we have 2 sides and an included angle

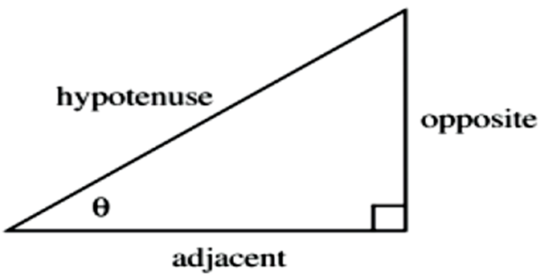
 ← missing side

- To find an angle when we have 3 sides given

cos rule 

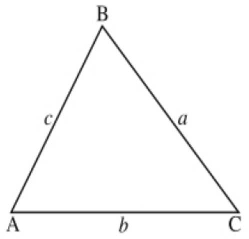
- **Overview of the triangles**

  -Right angled triangle



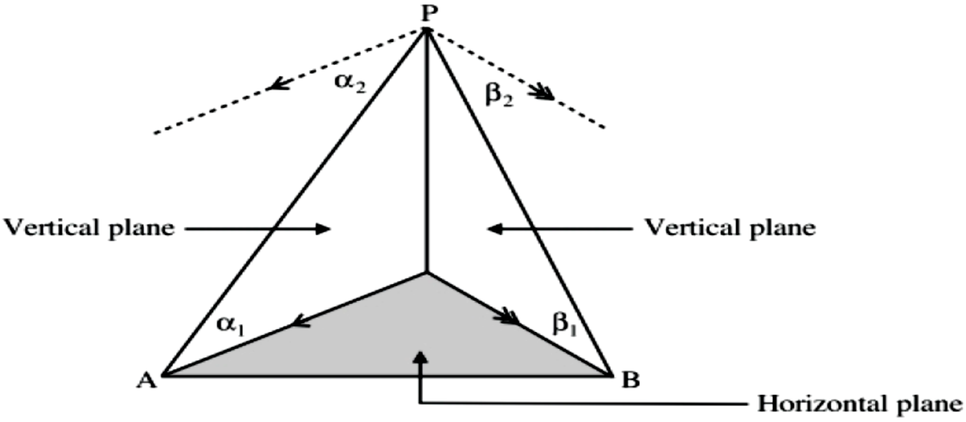
- $\sin\theta = \dfrac{opp}{hyp}$ $\cos\theta = \dfrac{adj}{hyp}$ $\tan\theta = \dfrac{opp}{adj}$

1. ALL TRIANGLES



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \textbf{or} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- **Note:**
  Angle of depression and elevation are always measured in vertical planes.

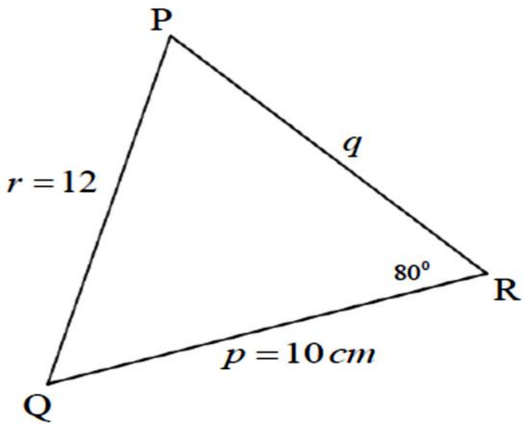


1. SINE RULE

- WORKED OUT EXAMPLE:

In ΔPQR, PQ=12m, QR=10cm and $\hat{R} = 80^0$.
Determine the:

(a) Size $\hat{P}$
(b) The length PR



Solutions:

$$\frac{\sin P}{p} = \frac{\sin R}{r}$$

$$\frac{\sin P}{10} = \frac{\sin 80^0}{12}$$

$$\sin P = \frac{10 \sin 80^0}{12}$$

$$\hat{P} = \sin^{-1}\left(\frac{10 \sin 80^0}{12}\right)$$

$$\hat{P} = 55{,}15^0$$

Note :

- For you to be able to get the length of PR you will need to know $\hat{Q}$ .Now you know two angles in ΔPQR then you can get the 3rd one by applying sum of angles in a triangle.

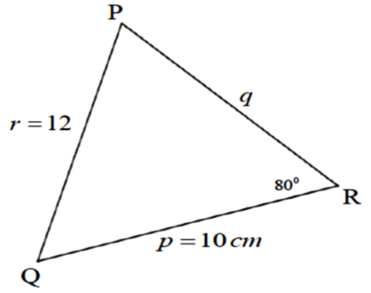
(b) $\hat{Q} = 44,85^0$

$$\frac{q}{\sin Q} = \frac{r}{\sin R}$$

$$\frac{q}{\sin 44,85^0} = \frac{12}{\sin 80^0}$$

$$q = \frac{12\sin 44,85^0}{\sin 80^0}$$

$$q = 8,59^0$$



- Sine rule is also applicable when given two angles and a side, then you will be able to use it to calculate the other sides as well as the 3rd angle.

2. Cosine rule:

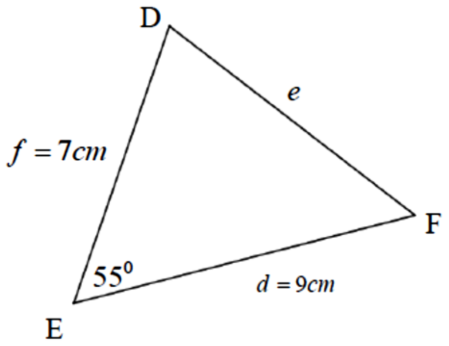
Worked out example:

In $\Delta DEF, DE = 7cm, FE = 9cm$ and $\hat{E} = 55^0$

Determine the:

(a) Length of DF

(b) Size $\hat{F}$



Solution:

$$DE^2 = EF^2 + DF^2 - 2.EF.DF\cos\hat{F}$$

$$7^2 = 9^2 + 7.60^2 - 2,9.60\cos\hat{F}$$

$$\cos\hat{F} = \frac{9^2 + (7,60)^2 - 7^2}{(2)(9)(7,60)}$$

$$\hat{F} = \cos^{-1}\left(\frac{9^2 + (7,60)^2 - 7^2}{(2)(9)(7,60)}\right)$$

$$\hat{F} = 48,99^0$$

3. Area rule:

Worked example.

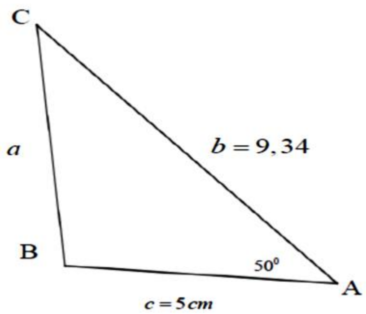
In $\Delta ABC, A = 50^0, AC = 9,34 and AB = 5cm$

(a) Determine area of ΔABC

Area of $\Delta ABC = \frac{1}{2}AC.AB\sin\hat{A}$

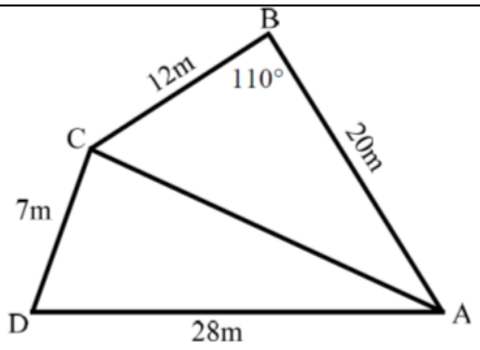
$$\Delta ABC = \frac{1}{2}9.34.5\sin 50^0$$

$$= 17,89cm^2$$



ACTIVITIES/ASSESMENT

3.10.1 A piece of land has the form of a quadrilateral ABCD with AB= $20m, BC = 12m,\ CD = 7m\ and\ AD = 28m.\ \hat{B} = 110^0$. The owner decides to divide the land into two plots by erecting a fence from A to C



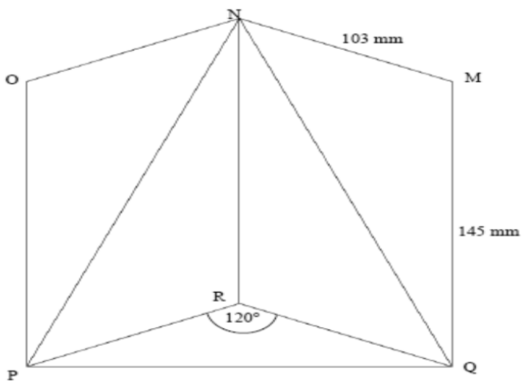
(a) Calculate the length of the fence AC correct to one decimal place.

(b) Calculate the size of $B\hat{A}C$ correct to the nearest degree.

(c) Calculate the size $\hat{D}$, correct to the nearest degree.

(d) Calculate the area of the entire piece of land ABCD, correct to one decimal place.

3.10.2 The figure shows an open birthday card. The length of the card is 145mm and the breadth is 103mm. The card is placed such that the angle formed between the two sides is 120^0



(a) Calculate the length of NP

(b) Calculate the length of PQ

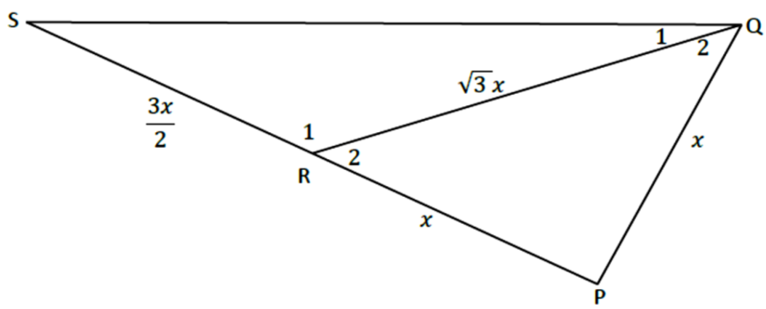
(c) Determine the size of $N\hat{P}Q$

3.10.3 Triangle PQS represents a certain area of a park. R is a point on line PS such that QR divides the area of the park into two triangular parts, as shown below. PQ= $PR = x\ units$,

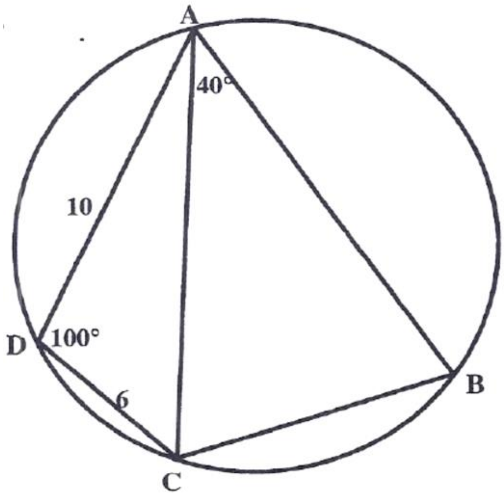
RS = $\frac{3x}{2} units$ and RQ = $\sqrt{3}x$ units

(a) Calculate the size of $\hat{P}$

(b) Determine the area of triangle QRS in terms of x



3.10.4 In the diagram below, ABCD is a cyclic quadrilateral with DC= 6 units, AD 10 units, $A\hat{D}C = 100^0$ and $C\hat{A}B = 40^0$.



Calculate the following, correct to ONE decimal place:

(a) The length of BC

(b) The area of ΔABC

TOPIC: TRIGONOMETRY			
LESSSON: 11			
Term	One	**Week**	
Duration	2HR	**Weighting**	50
Sub-topics	3D Shapes		
RELATED CONCEPTS/ TERMS/VOCABULARY	• Fractions • Heights • Distance • Elevation • Depression • Masts		

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Knowledge of polygons and their properties
- Sine rule, cosine and area rule
- Pythagoras theorem
- Trig ratios

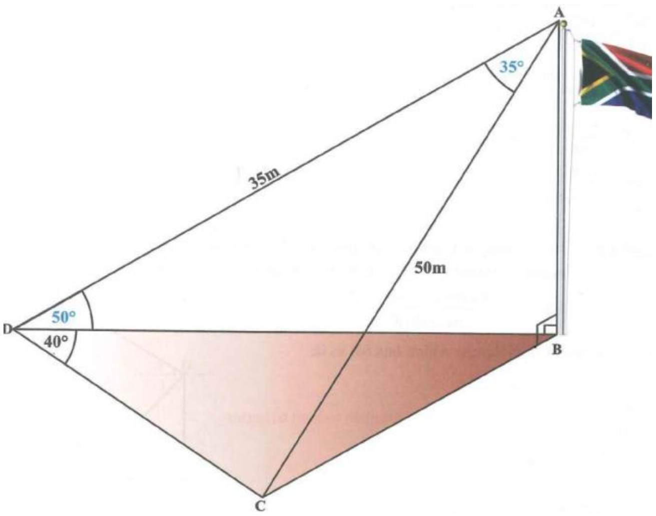
RESOURCES

- Modelling using strings and poles to demonstrate 2D shapes
- Grade 12 textbooks (Mind Action Series and Hand book study guide)
- NSC Past exam papers

NOTES

AB is a vertical flag pole with the points B, C and D in the same horizontal plane. There are two people looking at the flag points D and C. DA=35m, AC=50m. The angle of elevation of A from D is 50^0. Angle $D\hat{A}C = 35^0$ and $B\hat{D}C = 40^0$.

(a) How far is observer D standing from the flag pole?
(b) How far are the observers from each other?
(c) Calculate the area that the observers form with the flag pole.



NB: With the majority of 3D trig problems, you will have to solve a right angled triangle first (whether or not it is asked in the question.

$$\cos\theta = \frac{adj}{hyp}$$

$\therefore \cos\theta = \frac{DB}{35}$

$\therefore DB = 35\cos 50^0$
$\therefore DB = 22,50m$

(a) Using the cosine rule:

$DC^2 = AD^2 + AC^2 - 2(35)(50)\cos(D\hat{A}C)$
$\therefore DC^2 = 35^2 + 50^2 - 2(35)(50)\cos 35^0$
$\therefore DC = \sqrt{857,97}$
$\therefore DC = 29,29m$

(b) Using the area rule:

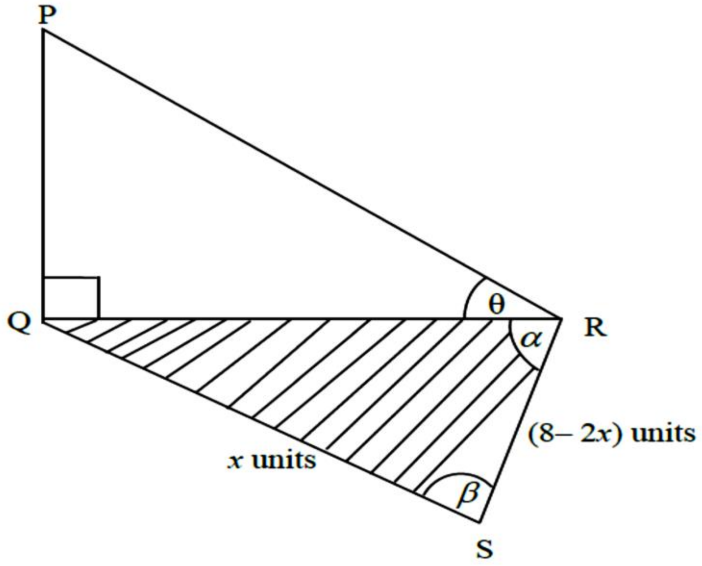
Area $\Delta BCD = \frac{1}{2}(DB)(DC)\sin(B\hat{D}C)$

Area $\Delta BCD = \frac{1}{2}(22,50)(29.29)\sin(40^0)$

Area $\Delta BCD = 211.81\, m^2$

ACTIVITIES/ASSESSMENT

1. In the diagram below, PQ is a vertical mast. R and S are two points in the same horizontal plane as Q, such that:



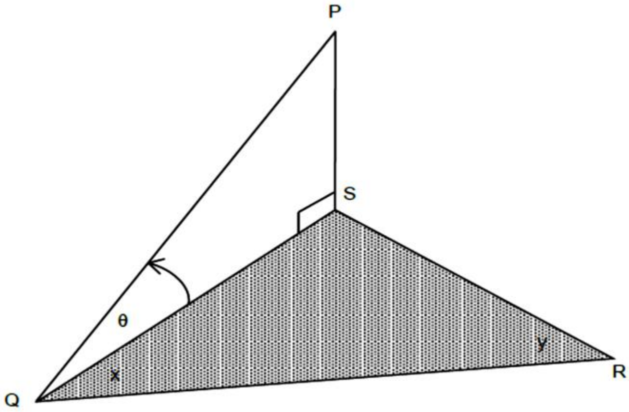
(a) Show that : $PQ = \frac{x\sin\beta.\tan\theta}{\sin(\alpha)}$

(b) If $\beta = 60^0$,show that the area of $PQ = \frac{x\sin\beta.\tan\theta}{\sin(\alpha)}$

(c) Determine the value of x for which the area of ΔQSR will maximum

(d) Calculate the length of QR if the area of ΔQSR is maximum

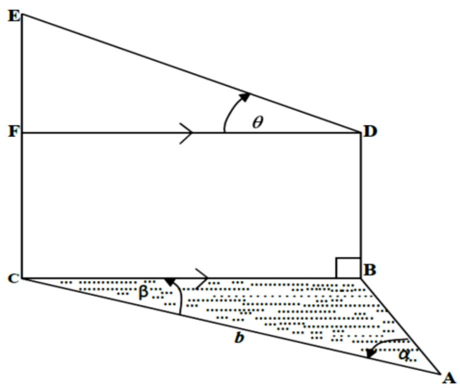
2. In the diagram below, S, Q and R are points in the same plane. PS is a vertical telephone mast. The angle of elevation of P from Q is θ. $S\hat{Q}R = x$, $S\hat{R}Q = y$, QR $= 10m$



(a) Express PS in terms of QS and θ

(b) Show that QS$= \frac{10 \sin y}{\sin(x+y)}$

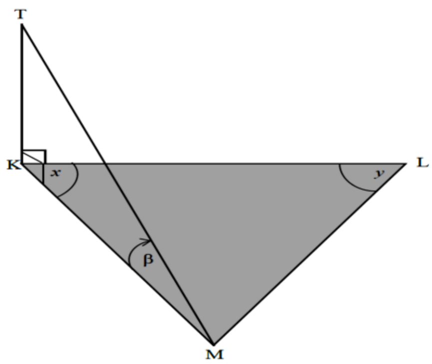
3. In the figure alongside, A, B and C are three points in the same horizontal plane D is vertically above B and E is vertically above C. The angle of elevation of E from D is θ^0. F is a point on EC such that FD$\parallel CB$. $B\hat{A}C = \alpha$, $A\hat{C}B = \beta$ and AC$= b$ metres.



(a) Express DE in terms of DF and θ

(b) Hence show that $DE = \frac{b \sin \alpha}{\sin(\alpha+\beta)\cos\theta}$

4. TK is a pole with K in the same horizontal plane as L and M. The angle of elevation of T from M is β. $L\hat{K}M = x$ and $K\hat{L}M = y$

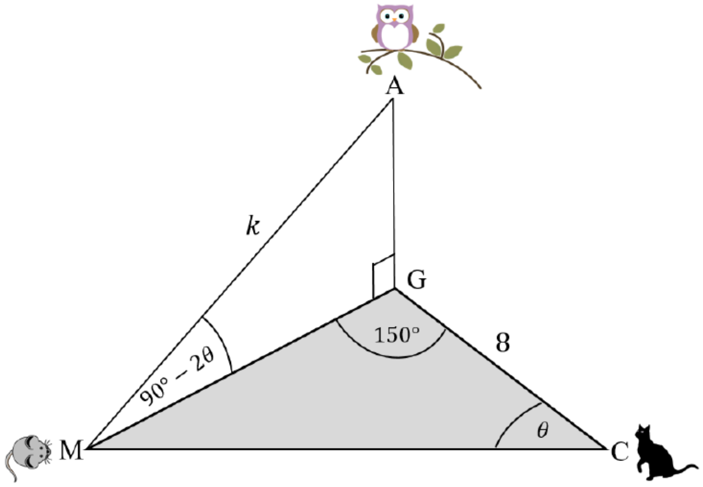


(a) Show that

$$KT = \frac{KL \sin y . \tan\beta}{\sin(x+y)}$$

5.

A mouse on the ground is looking up to an owl in a tree and a cat to his right (angle of elevation from the mouse to the owl is $(90° - 2\theta)$.
AM $= k$ *units*, GC$=$ 8 *units* , M$\hat{G}$C$=$ 150° and M$\hat{C}$G $= \theta$



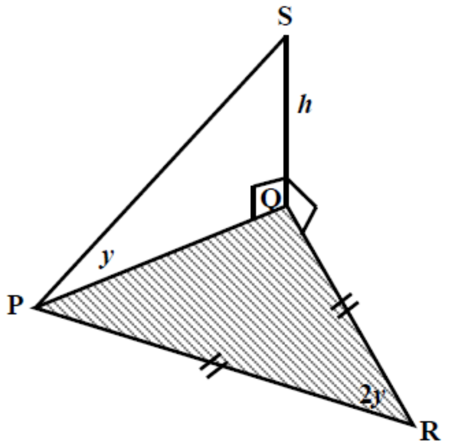
5.1 Give the size of M$\hat{A}$G in terms of θ.

5.2 Show that MG$= k \sin 2\theta$

5.3 Show that MC$= k \cos \theta$

5.4 Show that the area of ΔMGC$= 2k \sin 2\theta$

6. In the diagram QS is a vertical pole. P and R are points in the same horizontal plane as Q such that QP = QR. The angle of elevation of the top of the pole S from P is y.
Also SQ = h and P$\hat{R}$Q $= 2y$.



$$PR = \frac{h \cdot \cos^2 y}{\sin y \cdot \sin 2y}$$