KWAZULU-NATAL PROVINCE

EDUCATION

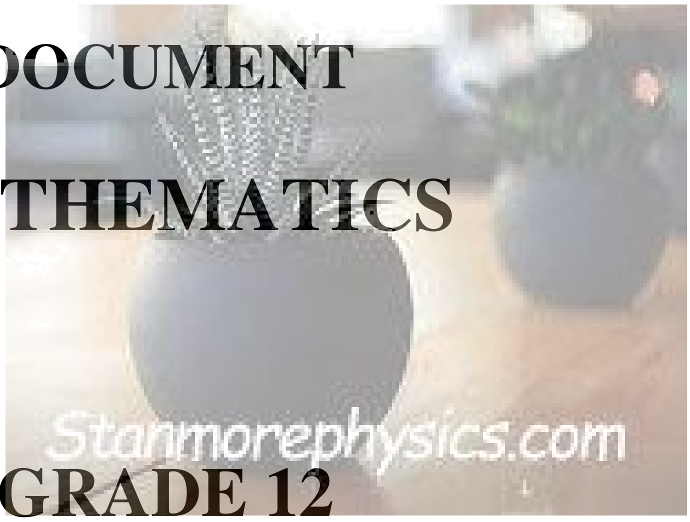
REPUBLIC OF SOUTH AFRICA

CURRICULUM GRADE 10 -12 DIRECTORATE

NCS (CAPS) SUPPORT

JUST IN TIME LEARNER REVISION DOCUMENT

MATHEMATICS



GRADE 12

2022

This document has been compiled by the FET Mathematics Subject Advisors together with Lead Teachers. It seeks to unpack the contents and to give guidance to teachers.

TABLE OF CONTENTS

TOPIC ALGEBRA

GUIDELINES, SUMMARY NOTES, & STRATEGIES

Summary	**Strategies**
Factorisation	Common factor, solve
	Transpose, factorise and solve
	Remove brackets, transpose, factorise and solve
Quadratic formula	Standard form, formula, substitution and answers correct to TWO decimal places, unless stated otherwise.
	Remove the brackets, transpose, standard form and answers correct to TWO decimal places, unless stated otherwise.
Surds	Isolate the radical sign, square both sides, solve and validate (check the validity of the answers).
Simultaneous equations	Choose the simplest equation, make x or y the subject of the formula, substitution, standard form, factorise (or use a quadratic formula), solve , substitution for the other variable.
Exponents	Laws of exponents, write as a power, factorise, equate exponents, solve
	Same base, equate exponents, solve.
	Split, factorize, simplify and solve
	Standard form, factorise, solve, laws of exponents and solve again. OR use a k -method.
Inequalities	Critical values, method, solution
	Standard form, factorise, critical values, method and solve
	Remove brackets, standard form, factorise, critical values, method and solve.
Nature of the roots	$\Delta = b^2 - 4ac$
	Standard form, discriminant, substitution, solution

Values of discriminant (Δ)	Nature of the roots
$\Delta \geq 0$	Real
$\Delta > 0$	Real, unequal, rational/ irrational
$\Delta = 0$	Real and equal
$\Delta < 0$	Non-real

REVISION QUESTIONS

1. **NW, March 2022 Question 1**

1.1 Solve for x:

1.1.1 $x(2-x)=0$ (2) **L1**

1.1.2 $2x^2-3x-7=0$ (correct to TWO decimal places) (3) **L1**

1.1.3 $2-x=\sqrt{2-7x}$ (4) **L2**

1.1.4 $(3-x)(x+1)<0$ (3) **L2**

1.2 Solve for x and y:

$2x-y=3$

$x^2+5xy+y^2=15$ (6) **L2**

1.3 Determine the value of x given that $6^x+6^x+6^x+6^x+6^x+6^x=6^{6x}$ (3) **L4**

2. **FS, March 2022 Question 1**

2.1 Solve for x:

2.1.1 $x(x+6)=0$ (2) **L1**

2.1.2 $3x^2 + 8x = -2$ (correct to TWO decimal places). (4) **L1**

2.1.3 $x^2 - 64 \leq 0$ (3) **L2**

2.1.4 $\sqrt{x+5} + 1 = x$ (5) **L2**

2.2 Solve simultaneously for x and y in the following equations:

$6x + 5xy - 5y = 8$

$x + y = 2$ (6) **L2**

3.1 Solve for x:

3.1.1 $(x-5)(x+2) = 6$ (4) **L1**

3.1.2 $(x+4)(x-3) = 3$ (correct to one decimal place) (4) **L2**

3.1.3 $2^x(x-5) \leq 0$ (3) **L3**

3.1.4 $x - 3\sqrt{x+2} = 2$ (4) **L2**

3.1.5 $3^{3x+1} = 9^{2x-4}$ (3) **L1**

3.2 Solve both x and y: (4)

$(3x-y)^2 + (x-5)^2 = 0$ **L3**

4.1 Solve for x:

4.1.1 $(x-1)^2 = 2(1-x)$ (4) **L2**

4.1.2 $-x^2 - 2x + 1 = 0$ (correct to two decimal places) (3) **L1**

4.1.3 $x - 3\sqrt{x} - 4 = 0$ (4) **L2**

4.1.4 $x = \dfrac{\sqrt{10^{1009}}}{\sqrt{10^{1011}} - \sqrt{10^{1007}}}$ (3) **L3**

4.2 Solve both x and y:

$2^{y-3x} = \dfrac{1}{16}$

$x^2 + xy = 24$ (7) **L2**

5.1 Solve for x:

5.1.1 $(x-2)(5+x) = 0$ (2) **L1**

5.1.2 $x - \dfrac{3}{x} = -2$ (4) **L1**

5.1.3 $(x-1)(x-2) \leq 6$ (4) **L2**

5.1.4 $2^{x+2} + 2^x = 40$ (3) **L2**

5.2 Solve for both x and y:

$x - 2y = 3$

$3x^2 - 5xy = 24 + 16y$ (6) **L2**

5.3 The roots of a quadratic equation are: $\dfrac{-3\pm\sqrt{4-p}}{2}$

For which values of p are the roots real? (2) **L1**

6. **Answer Series, Grade 12**

6.1 Solve for x:

6.1.1 $(x+2)^2 = 3x(x-2)$ correct to one decimal digit. (4) **L2**

6.1.2 $x^2 - 9x \geq 36$ (4) **L2**

6.1.3 $3^x - 3^{x-2} = 72$ (4) **L2**

6.1.4 $(\sqrt{x-1}-3)(\sqrt{x-1}+2) = 0$ (3) **L2**

6.2 Given: $(2m-3)(n+5) = 0$

Solve for:

6.2.1 n if $m = 1$ (1) **L1**

6.2.2 m if $n \neq -5$ (1) **L1**

6.2.3 m if $n = -5$ (2) **L1**

7. **Via Afrika, Study Guide, Grade 12**

7.1 Solve for x:

7.1.1 $x+2 = \dfrac{2}{x+1}$ (4) **L1**

7.1.2 $x - \sqrt{x} = 6$ (4) **L2**

7.1.4 $5^{x-2} + 5^{x+1} = 126$ (5) **L2**

7.2 2 is a root of $2x^2 - 3x - p = 0$. Determine the value of p and hence the other root. (4) **L1**

8. **Kevin Smith, Maths Handbook And Study Guide, Grade11**

8.1 Solve for x, rounded off to TWO decimal places where necesssary :

8.1.1 $x(x-1) = 2$ (3) **L1**

8.1.2 $2x^2 - 3x = 8$ (4) **L1**

8.1.3 $3x^2 + x - 2 \geq 0$ (4) **L2**

8.2 Given the equation $x^2 + 2xy - 8y^2 = 0$:

8.2.1 Determine the values of the ratio $\dfrac{x}{y}$. (3) **L3**

8.2.2 Hence, determine the values of x and y if $x + y = 6$ (5) **L2**

8.3 Simplify the following without the use of a calculator. Show all workings:

$$\left(\frac{\sqrt{3^{2011}} - \sqrt{3^{2009}}}{\sqrt{3^{2008}}} + \sqrt{3}\right)^2$$

(4) **L3**

9. **FEBRUARY/MARCH 2015**

9.1 Solve for x:

9.1.1 $x^2 - x - 20 = 0$ (2) **L1**

9.1.2 $2x^2 - 11x + 7 = 0$ (correct to TWO decimal places) (3) **L1**

9.1.3 $5x^2 + 4 > 21x$ (5) **L2**

9.1.4 $2^{2x} - 6.2^x = 16$ (4) **L2**

9.2 Solve for x and y simultaneously:

$y + 1 = 2x$

$x^2 - xy + y^2 = 7$ (6) **L2**

9.3 The roots of a quadratic equation are given by $x = \dfrac{-5 \pm \sqrt{20 + 8k}}{6}$, where $k \in \{-3; -2; -1; 0; 1; 2; 3\}$.

9.3.1 Write down TWO values of k for which the roots will be rational. (2) **L1**

9.3.2 Write down ONE value of k for which the roots will be no-real. (1) **L1**

9.4 Calculate a and b if $\sqrt{\dfrac{7^{2014} - 7^{2012}}{12}} = a(7^b)$ and a is not a multiple of 7. (4) **L3**

10.1 Solve for x:

10.1.1 $x^2 - x - 6 = 0$ (3) **L1**

10.1.2 $x(x+6) + 1 = 0$ (correct to TWO decimal places) (4) **L1**

10.1.3 $6x - 2x^2 \leq 0$ (3) **L2**

10.1.4 $\left(\sqrt{\sqrt{2} - x}\right)\left(\sqrt{\sqrt{x} + 2}\right) = x$ (5) **L3**

10.2 Solve for x and y simultaneously:

$x - y = 3$

$x^2 - 3y^2 = 13$ (6) **L2**

10.3 Given: $f(x) = 3(x-1)^2 + 5$ and $g(x) = 3$

10.3.1 Is it possible for $f(x) = g(x)$? Justify your answer. (2) **L3**

103.1 Determine the value(s) of k for which $f(x) = g(x) + k$ has TWO unequal roots. (2) **L4**

11.1 Solve for x:

11.1.1 $x^2 + 5x - 6 = 0$ (3) **L1**

11.1.2 $4x^2 + 3x - 5 = 0$ (correct to TWO decimal places) (3) **L1**

11.1.3 $4x^2 - 1 < 0$ (3) **L2**

11.1.4 $3^{x+1} - 3^{x-1} - 24 = 0$ (4) **L2**

11.2 Solve simultaneously for x and y:

$y + x = 12$ and $xy = 14 - 3x$ (5) **L2**

11.3 Consider the product: $1\times2\times3\times4\times...\times30$

Determine the largest value of k such that 2^k is a factor of this product. (4) **L4**

12.1 Solve for x:

12.1.1 $2x(3-x)=0$ (2) **L1**

12.1.2 $5x^2-4x=2$ (Rounded off to TWO decimal places) (4) **L1**

12.1.3 $\sqrt{7+3x}+2x=0$ (5) **L2**

12.1.4 $3^{x+2}+3^{2-x}=82$ (5) **L2**

12.2 Solve for x and y simultaneously if:

$xy=12$ and $x-4=y$ (5) **L2**

12.3 The equations $x^2+ax+b=0$ and $x^2+bx+a=0$ both have real and equal roots.

Solve for a and b, where $a>0$ and $b>0$. (7) **L4**

13.1 Solve for x:

13.1.1 $10x^2+9x-9=0$ (3) **L1**

13.1.2 $x^2-6x-5=0$ (correct to THREE decimal places) (3) **L1**

13.1.3 $(x+3)(2-x)<0$ (3) **L2**

13.1.4 $\sqrt{43-x}-x+1=0$ (5) **L2**

13.2 Given: $x+\dfrac{1}{x}=4$

13.2.1 Determine the value of $x^2+\dfrac{1}{x^2}$ (2) **L2**

13.2.2 Determine the value of $x^3+\dfrac{1}{x^3}$ (3) **L3**

13.3 Solve simultaneously for x and y:

$3x-4y=5$ and $2x^2-5xy+3y^2=4$ (6) **L2**

TOPIC SEQUENCE AND SERIES

GUIDELINES, SUMMARY NOTES, & STRATEGIES

Quadratic Pattern:
Definition: Second difference are equal where the first term forms an arithmetic sequence.
General term: $T_n = an^2 + bn + c$
To calculate the values of a, b and c:

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	$a+b+c$		$4a+2b+c$		$9a+3b+c$		$16a+4b+c$
1st difference		$3a+b$		$5a+b$		$7a+b$	
2nd difference			$2a$		$2a$		

NB: For a **MINIMUM** or **MAXIMUM** term: $n = \frac{-b}{2a}$ or $\frac{dT_n}{dn} = 0$ i.e. First derivative

Arithmetic number patterns:
Definition: All first differences are equal, i.e. you always add or subtract a constant difference
N:B $T_2 - T_1 = T_3 - T_2$

General term: $Tn = a + (n-1)d$
$d = T_2 - T_1$

Sum of n Terms:
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$S_n = \frac{n}{2}(a + l)$$
Where l is the last term or T_n

Geometric number patterns:
Definition: There exists constant ratio, i.e you multiply by the same ratio.
N:B $\frac{T_2}{T_1} = \frac{T_3}{T_2}$ (common ratio ; r)

General term: $T_n = ar^{n-1}; r = \frac{T_2}{T_1}$

Sum of n terms:
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Sum to infinity: $S_\infty = \frac{a}{1-r}$

NB: Terms of convergence: $-1 < r < -1$

NB: Given: S_n : $T_n = S_n - S_{n-1}$

REVISION QUESTIONS

DBE/May-June 2019

1. The first FOUR terms of a quadratic pattern are: 15 ; 29 ; 41 ; 51
   1.1 Write down the value of the 5th term. (1) **L1**

1.2 Determine an expression for the n^{th} term of the pattern in the form $T_n = an^2 + bn + c$. (4) **L2**

1.3 Determine the value of T_{27} (2) **L1**

IEB/November 2019

2. The table below shows the number of passengers that were on a bus after every stop.

	1st stop	**2nd stop**	**3rd stop**	**4th stop**
Number of Passengers	2	20	34	44

The number of passengers on the bus after the n^{th} bus stop can be given by $T_n = an^2 + bn + c$ where a, b and $c \in \Re$.

2.1 Write down the number of passengers on the bus after the fifth stop. (1) **L1**

2.2 Determine a, b and c. (4) **L2**

2.3 If it is given that $T_n = an^2 + bn + c$, determine the maximum number of passengers on the bus. (3) **L2**

2.4 Explain why the formula calculated in Question 2.3 does not work after the eleventh stop. (3) **L3**

BISHOPS/SEPT 2016

3. A quadratic pattern has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.
Determine the general term. (5) **L2**

WBHS/SEPT 2017

4 The p^{th} term of the first difference of a quadratic sequence is given by $T_p = 3p - 2$

4.1 Determine between which two consecutive terms of the quadratic sequence the first difference is equal to 1450 (3) **L3**

4.2 The 40^{th} term of the quadratic sequence is 2290 and $T_n = an^2 + bn + c$ is the n^{th} term of the quadratic sequence. Calculate the value of c (4) **L3**

WESTERFORD/SEPT 2016

5 A linear number pattern with a constant difference can be represented by the terms:

$$x+3; \quad 3x+2; \quad 6x-1$$

5.1 Determine the value of x (4) **L2**

5.2 Determine the value of T_5. (2) **L1**

6 Given the geometric sequence: $3 ; 2 ; k ; ...$

6.1 Write down the value of the common ratio. (1) **L1**

6.2 Calculate the value of k. (2) **L2**

6.3 Calculate the value of n if $T_n = \dfrac{128}{729}$. (4) **L2**

7 An arithmetic and geometric sequence have the same first term, 5. The common difference and common ratio have the same value. The 5^{th} term of the geometric sequence is 80. Determine the first three terms of the arithmetic sequence(s) (5) **L3**

PLATINUM MATHEMATICS/GR12(BANK QUESTIONS)

8 Given the sequence: $3 ; 6 ; 9 ; ... ; 60$

8.1 Determine the number of terms in the sequence. (3) **L2**

8.2 Determine the sum of the terms in the sequence. (3) **L1**

8.3 Determine the sum of all the natural numbers from 1 to 60 which are not multiples of 3 (4) **L3**

9 Given: A set of natural numbers up to and including 900. If the multiples of 5 are removed from the set, determine the sum of the remaining numbers. (7) **L3**

DBE/November 2021

10 Given the geometric series: $x+90+81+...$

10.1 Calculate the value of x. (2) **L2**

10.2 Show that the sum of the first n terms is $S_n = 1000\left(1-(0,9)^n\right)$ (2) **L2**

10.3 Hence, or otherwise, calculate the sum to infinity. (2) **L1**

NHHS-JUNE 2016

11 Given the sequence: $7; 1; 7; 3; 7; 5;...$

11.1 Determine the value of T_{17} (1) **L2**

11.2 Determine the sum of the sum of the sequence from T_9 up to and including T_{13} (4) **L2**

12 The sum of 16 terms of an arithmetic series is 632, and the eleventh term is 47. Determine the fifth term. (6) **L3**

DBE/November 2021

13 Consider the linear pattern: $5;7;9;...$

13.1 Determine T_{51}. (3) **L1**

13.2 Calculate the sum of the first 51 terms. (2) **L2**

13.3 Write down the expansion of $\sum_{n=1}^{5000}(2n+3)$. Show only the first 3 terms and the last term of the expansion. (1) **L1**

13.4 Hence, or otherwise, calculate $\sum_{n=1}^{5000}(2n+3)+\sum_{n=1}^{4999}(-2n-1)$.

ALL working details must be shown. (4) **L3**

NW/SEPT 2021

14 Consider the following: $\sum_{k=3}^{12} 4\left(\frac{1}{2}\right)^{k-1}$

14.1 Write down the first three terms of the series. (2) **L1**

14.2 Calculate the sum of the series. (3) **L2**

IEB/November 2019

15 Given: $\sum_{n=2}^{\infty} 4.\left(\frac{1}{2}\right)^{n+2}$

15.1 Calculate: $\sum_{n=2}^{\infty} 4.\left(\frac{1}{2}\right)^{n+2}$ (4) **L2**

15.2 Give a reason why the series converge. (1) **L1**

16 For which value(s) of t is the following infinite series convergent?

$2(t-5)+2(t-5)^2+2(t-5)^3+...$ (4) **L2**

MIND ACTION SERIES/GR12(NEW EDITION)

17 Given the geometric series: $3+2+x+\frac{8}{9}+\frac{16}{27}+...$

17.1 Calculate the value of x. (1) **L2**

17.2 Write this infinite geometric series in sigma notation. (2) **L2**

KZN/June 2021

18 Consider the geometric series where $\sum_{n=1} T_n = 27$ and $S_3 = 26$. Calculate the value of the constant ratio (r) of the series. (4) **L3**

IEB/November 2020

19 Determine the smallest value of k for which

$$\sum_{i=1}^{\infty} \frac{k}{2^i} + \sum_{i=1}^{10} 2^{2i} > 1000000 \text{for } k \in Z$$

(6) **L3**

NHHS/SEPT 2014

20 Consider the following geometric series:

$$\frac{p-1}{2p-1} + (p-1) + (p-1)(2p-1) + \ldots$$

20.1 Determine the common ratio, r, in terms of p (2) **L1**

20.2 For which value(s) of p will the series converge? (3) **L2**

MIND ACTION SERIES/GR12(NEW EDITION)

21 The sum of the first n terms of a series is given by $S_n = 2n^2 + 4n$.

21.1 Calculate the sum of the first 200 terms of the series. (2) **L1**

21.2 Calculate the value of the 100^{th} term in the series. (3) **L2**

21.3 How many terms must be added for the sum to be 4230. (3) **L2**

22 For a certain series $S_n = 64 - 64\left(\frac{1}{2}\right)^n$.

22.1 How many terms must be added for the sum to be equal to $\frac{255}{4}$? (3) **L2**

22.2 Determine the value of T_4. (3) **L2**

22.3 Show that $T_n = 2^{6-n}$. (4) **L2**

22.4 If $2^n = p$, determine the value of $S_{6-n} - S_{6+n}$ in terms of p. (4) **L3**

23 A certain quadratic pattern has the following features:

- $T_1 = p$
- $T_2 = 18$
- $T_1 = 4T_1$
- $T_3 - T_2 = 10$

Determine the value of p (5) **L3**

KZN June 2021

24 Consider the sequence: $-11;\ 2\sin 3x;\ 15;\ \ldots$

Determine the values of x in the interval $[0°;90°]$ for which the sequence will be arithmetic. (4) **L3**

25 In a geometric sequence, the third term is $5p+1$ the fifth term is 4 and the seventh term is 1. Determine the value of p. (3) **L2**

26 If the sum of the first n terms of the following geometric series is to be greater than 300, determine the smallest value of n.

$$49 + 42 + 36 + \frac{217}{7} + \ldots$$

(5) **L3**

MIND ACTION SERIES/GR12(NEW EDITION)

27 Jacob wrote 12 Maths tests. For the first test he scored 32%. However, on each successive test, his score was 1,05 times more that the preceding one. In answering the questions which follow, give all answers correct to two decimal places.

27.1 Use the information to write down the first 3 test scores as a sequence. (2) **L1**

27.2 What was Jacob's percentage for his last (twelfth) test? (3) **L2**

27.3 What was the total of all his tests? (2) **L2**

27.4 Find his average percentage for the 12 tests. (2) **L1**

28 The 1^{st}, 2^{nd} and 3^{rd} terms of a geometric progression are the 1^{st}, 9^{th} and 21^{st} terms respectively of an arithmetic progression. The 1^{st} term of each progression is 8 and the common ratio of the geometric progression is r, where $r \neq 1$. Determine the value of r (7) **L3**

RBHS/SEPT 2015

29 Consider the geometric progression: $\frac{1}{3};\frac{2}{3};\frac{4}{3};\ldots$

29.1 Determine the general term. (2) **L2**

29.2 Calculate the sum of the first 8 terms (3) **L1**

30 Calculate: $\sum_{k=1}^{21}(2-k)$ (3) **L2**

31 If $\frac{1}{b-a};\frac{1}{2b};\frac{1}{b-c};\ldots$ form an arithmetic sequence, prove that a, b and c are in geometric sequence (5) **L3**

RBHS/SEPT 2016

32 Given that a convergent geometric series, with first term $T_1 = a$; and $S_\infty = p$; where $p > 0$

32.1 Show that $a \in (0;2p)$ (5) **L2**

32.2 Determine the value of the constant ratio when $a = \frac{p}{4}$ (3) **L2**

KZN/SEPT 2021

33 $\sum_{p=1}^{5}(4y+3p)+\sum_{k=4}^{7}3.(2)^{k-1}=\sum_{j=1}^{\infty}(\frac{1}{3})^{j-1}$ (7) **L3**

WESTERFORD/SEPT 2014

34 Parents of a new born baby decide they will save for the child's future. They decide to save one cent this month when the baby has just been born, then two cents next month and so on, doubling the amount every month. How old will the child be when they have saved a total of R1 000 000, if they keep saving in this way? Give you answer correct to the nearest whole number. (4) **L2**

NW/SEPT 2021

35 Consider the series: $\cos\theta + \sin 2\theta + 4\sin^2\theta.\cos\theta + \ldots$ where θ is an acute angle.

37.1 Prove that it is a geometric series. (4) **L2**

37.2 Calculate for which values of θ it will be a converging series. (3) **L2**

BISHOPS/SEPT 2014

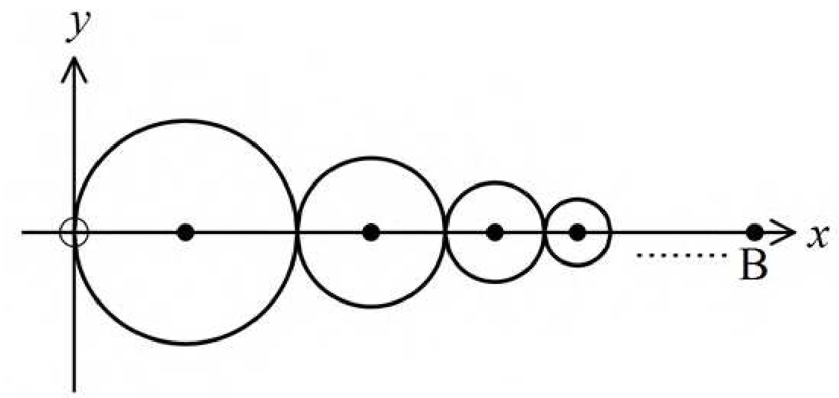
36 An arithmetic series is given such that

$T_1 + T_3 + T_5 + T_7 + \ldots + T_{2013} = 3029$ and $T_2 + T_4 + T_6 + T_8 + \ldots + T_{2014} = 6050$.

Determine the common difference between each term. (3) **L4**

37 An infinite number of circles, all touching one another, are drawn with their centres on the *x*-axis as shown. The first circle with radius 9 units, touches the *y*-axis at the origin. The radius of the second circle is one-third the radius of the radius of the first circle, the radius of the third circle is one-third the radius of second circle, and so on.

Show that the circles will never pass the point B $(27;0)$.



(4) **L3**

TOPIC FUNCTIONS

GUIDELINES, SUMMARY NOTES, & STRATEGIES

FUNCTIONS

- Intercepts (x and $y-$intercepts)
- Domain and the range

1. **STRAIGHT LINE:** $y = mx + c$

- Given two points
- Given gradient (***m***) and one point
- Given line parallel or perpendicular to the line to be determined and one point
- Domain : $x \in R$ Range : $y \in R$

2. **QUADRATIC FUNCTION (PARABOLA)**

$y = a(x+p)^2 + q$	$y = ax^2 + bx + c$
• Turning point: $(-p;q)$ • Axis of symmetry: $x = -p$ • Maximum/minimum value: $y = q$	• Axis of symmetry: $x = \frac{-b}{2a}$ • Derivative: $y' = 2ax + b$

3. **HYPERBOLA**: $y = \dfrac{a}{x+p} + q$

- Equations of asymptotes: $x = -p$ **and** $y = q$
- Line of symmetry: $\left.\begin{cases} y = x + c \\ y = -x + c \end{cases}\right\}$ **substitute point of intersection of asymptotes**

  OR $\left.\begin{cases} y = (x-p) + q \\ y = -(x-p) + q \end{cases}\right\}$
- **Domain:** $x \in R$**,** $x \neq -p$
- Range : $y \in R;\ y \neq q$

4. **EXPONENTIAL FUNCTION**: $y = a.b^{x+p} + q$**,** $b > 0$ **and** $b \neq 1$

- One asymptote: $y = q$
- Domain : $x \in R$ **Range:** $y > q$ if $a > 0$

INVERSE FUNCTION

Indicated as f^{-1}

- **Swop x and y in the given function**
- **Make y subject of the formula in the new function**
- **The graph of the given function and the graph of its inverse are reflected about the line** $y = x$

1. STRAIGHT LINE: $y = mx + c$

- **Inverse:** $x = my + c$ **and** $y = \dfrac{x}{m} - \dfrac{c}{m}$ **… is a function**
- Domain : $x \in R$ **Range:** $y \in R$

2. **PARABOLA:** $y = ax^2$

- **Inverse**: $x = ay^2$ **and** $y = \pm\sqrt{\dfrac{x}{a}}$ **… not a function**

- Restricting domain: $\begin{Bmatrix} x \geq 0 \\ x \leq 0 \end{Bmatrix}$ **… inverse will be a function**
- **Domain:** $x \geq 0$ or $x \leq 0$ **Range:** $y > 0$ **if** $a > 0$ $y < 0$ if $a < 0$

3. EXPONENTIAL: $y = b^x$

- **Inverse:** $x = b^y$ **and** $y = \log_b x$ **… is a function**
- Domain : $x > 0$ Range : $y \in R$

Point of intersection of two graphs: Graph 1 = Graph2
A line between two graphs: Above graph –graph Below

REVISION QUESTIONS

DBE /MAY-JUNE2017

1 1 Given: $f(x) = x^2 - 5x - 14$ and $g(x) = 2x - 14$

1.1 On the same set, sketch the graph of f and g .Clearly indicate all intercepts with the axes and turning points. (6) **L2**

1.2 Determine the equation of the tangent to f at $x = 2\frac{1}{2}$. (2) **L2**

1.3 Determine the values of k for which $f(x) = k$ will have two unequal positive real roots. (2) **L3**

1.4 A new graph h is obtained by first reflecting g in the x-axis and then translating it 7 units to the left. Write down the equation of h in the form $h(x) = mx + c$. (2) **L3**

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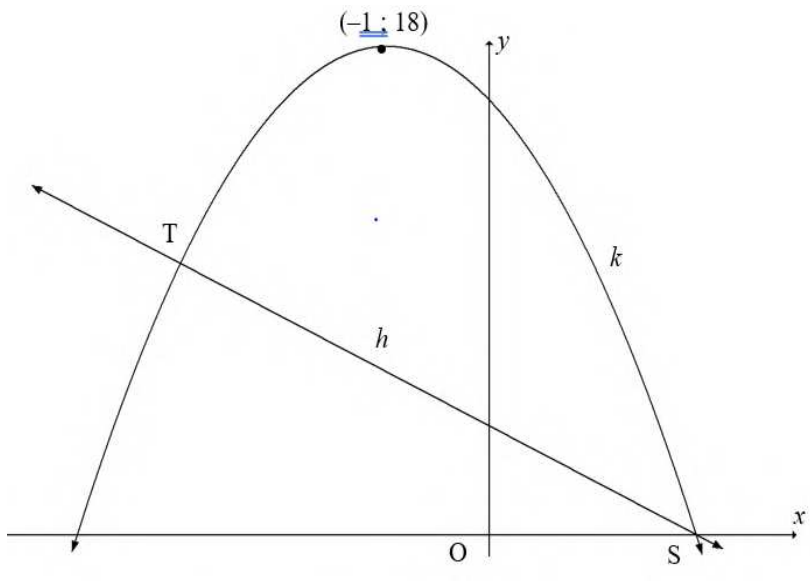
2 Sketched below are the graphs of $k(x) = ax^2 + bx + c$ and $h(x) = -2x + 4$.
Graph k has a turning point at (-1;18). S is the x-intercept of h and k .
Graphs h and k also intersect at T.



2.1 Calculate the coordinates of S. (2) **L1**

2.2 Determine the equation of k in the form $y = a(x+p)^2 + q$ (3) **L2**

2.3 If $k(x) = -2x^2 - 4x + 16$, determine the coordinates of T. (5) **L3**

2.4 Determine the value(s) of x for which $k(x) < h(x)$. (2) **L3**

2.5 It is further given that k is the graph of $g'(x)$.

2.5.1 For which values of x will the graph of g be concave up? (2) **L2**

2.5.2 Sketch the graph of g, showing clearly the x-values of the turning points and the point of inflection. (3) **L3**

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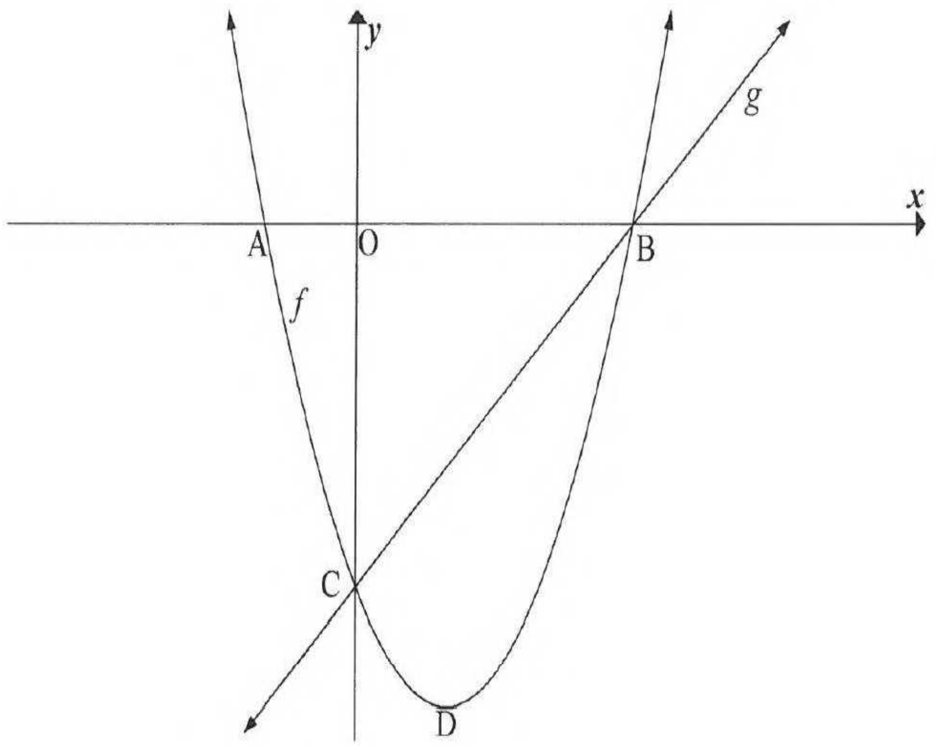
3 The sketch below shows the graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = x - 3$.

- A and B are the x-intercepts of f.
- The graphs of f and g intersect at C and B.

D is the turning point of f.



3.1 Determine the coordinates of C. (1) **L1**

3.2 Calculate the length of AB. (4) **L2**

3.3 Determine the coordinates of D. (2) **L2**

3.4 Calculate the average gradient of f between C and D. (2) **L2**

3.5 Calculate the size of $O\hat{C}B$. (2) **L3**

3.6 Determine the values of k for which $f(x) = k$ will have two unequal positive real roots. (3) **L3**

3.7 For which values of x will $f'(x).f''(x) > 0$? (3) **L4**

4 The graph of a parabola f has x-intercepts at $x = 1$ and $x = 5$. $g(x) = 4$ is a tangent to f at P, the turning point of f. Sketch the graph of f, clearly showing the intercepts with the axes and the coordinates of the turning point. (5) **L3**

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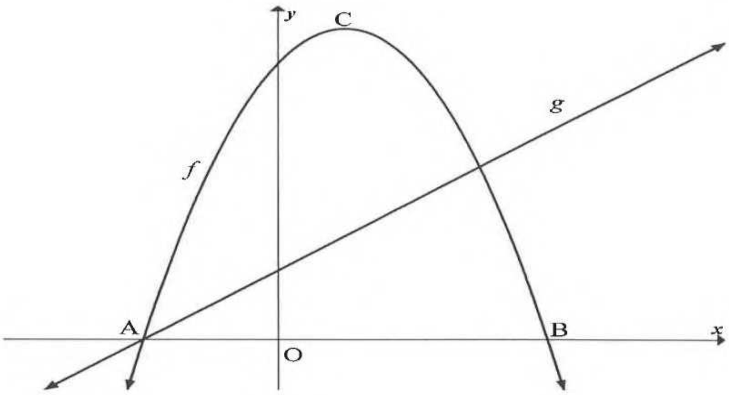
5 Sketched below are the graphs of $f(x)=-2x^2+4x+16$ and $g(x)=2x+4$. A and B are the x-intercepts of f .C is the point on f .



5.1	Calculate the coordinates of A and B.	(3)	**L2**
5.2	Determine the coordinates of C the turning point of f .	(2)	**L2**
5.3	Write down the range of f .	(1)	**L2**
5.4	The graph of $h(x)=f(x+p)+q$ has a maximum value of 15 at $x=2$. Determine the values of p and q .	(3)	**L3**
5.5	Determine the equation of g^{-1}, the inverse of g , in the form $y=...$	(2)	**L2**
5.6	For which value(s) of x will $g^{-1}(x).g(x)=0$?	(2)	**L3**
5.7	If $p(x)=f(x)+k$, determine the values of k for which p and g will not intersect.	(5)	**L4**

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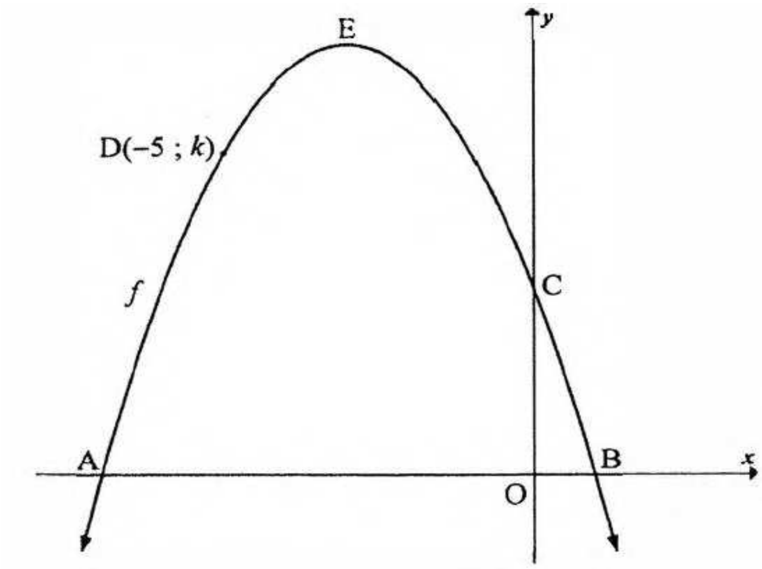
6. The sketch below shows the graph of $f(x)=-x^2-6x+7$.
C is the $y-$intercept of f . A and B are the $x-$intercepts of f . D(-5; k) is a point on f .



6.1	Calculate the coordinates of E, the turning point of f .	(3)	**L2**
6.2	Write down the value of k	(1)	**L1**
6.3	Determine the equation of the straight line passing through C and D.	(4)	**L2**
6.4	A tangent parallel to CD, touches f at P. Determine the coordinates of P.	(4)	**L3**
6.5	For which values of x will $f(x)-12>0$?	(2)	**L3**

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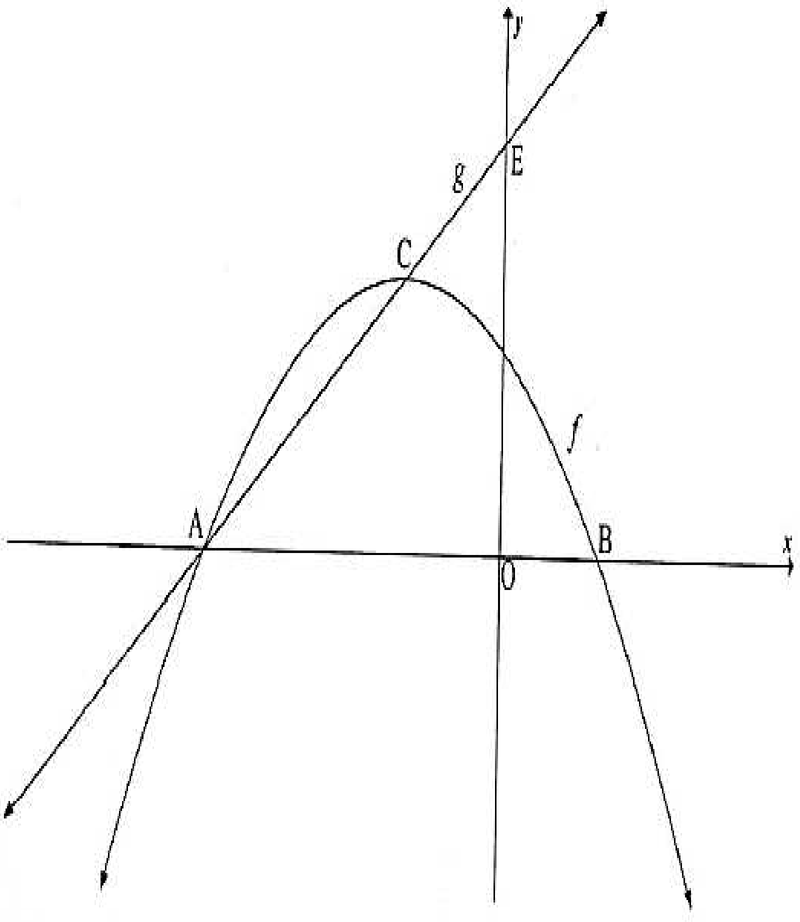
7 The sketch below shows the graphs of $f(x) = -x^2 - 2x + 3$ and $g(x) = mx + q$.
The graph f has x-intercept at A and B(1;0) and a turning point at C.
The straight line g, passing through A and C, cuts the y-axis at E.



7.1	Write down the coordinates of the y-intercept of f.	(1)	**L1**
7.2	Show that the coordinates of C are $(-1;4)$.	(3)	**L2**
7.3	Write down the coordinates of A.	(1)	**L2**
7.4	Calculate the length of CE.	(6)	**L3**
7.5	Determine the values of k if $h(x) = 2x + k$ is a tangent to the graph of f.	(5)	**L3**
7.6	Determine the equation of g^{-1}, the inverse of g, in the form $y = ...$	(2)	**L2**
7.7	For which values of x is $g(x) \geq g^{-1}(x)$?	(3)	**L3**

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8. Given: $f(x) = \dfrac{1}{x+2} + 3$

8.1	Determine the equation of the asymptotes of f.	(2)	**L1**
8.2	Write down the y-intercept of f.	(1)	**L1**
8.3	Calculate the x-intercept of f.	(2)	**L2**
8.4	Sketch the graph of f. Clearly label ALL intercepts with the axes and any asymptotes	(3)	**L2**

9 Determine the equation of asymptotes of : $f(x) = \dfrac{7-x}{x-1}$ (3) **L2**

10 Given: $f(x)=\dfrac{4}{2-x}+1$

10.1 Write down the equation of the asymptotes. (2) **L1**

10.2 Sketch the graph of $g(x)$ if $g(x)=-f(x)$ (4) **L3**

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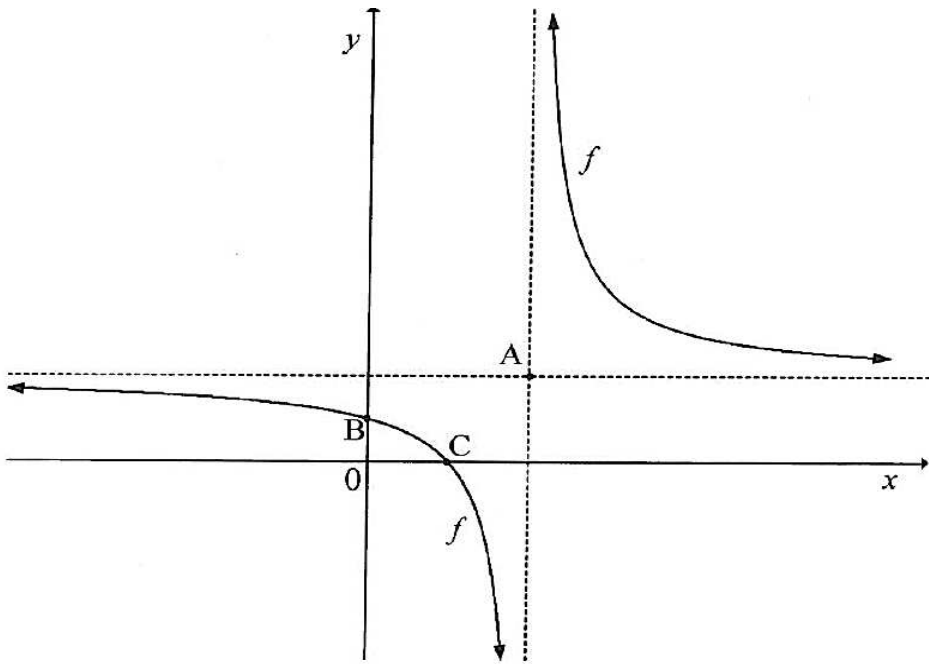
11. The sketch below shows the graph of $f(x)=\dfrac{6}{x-4}+3$. The asymptotes of f intersect at A. The graph f intersects the x-axis and y- axis at C and B respectively.



11.1 Write down the coordinates of A. (1) **L1**

11.2 Calculate the coordinates of B. (2) **L1**

11.3 Calculate the coordinates of C. (2) **L2**

11.4 Calculate the average gradient of f between B and C. (2) **L1**

11.5 Determine the equation of a line of symmetry of f which has a positive y- intercept. (2) **L2**

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12 The function f, defined by $f(x)=\dfrac{a}{x+p}+q$, has the following properties:

- The range of f is $y\in R, y\neq 1$.
- The graph f passes through the origin.
- $P\left(\sqrt{2}+2;\sqrt{2}+1\right)$ lies on the graph of f.

12.1 Write down the value of q. (1) **L1**

12.2 Calculate the values of a and p. (5) **L3**

12.3 Sketch a neat graph of this function. Your graph must include the asymptotes, if any. (4) **L2**

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13 Consider: $g(x)=\dfrac{a}{x+p}+q$

The following information of g is given:

- Domain: $x \in R\,;\, x \neq -2$
- x-intercept at K(1; 0)
- y-intercept at $N\left(0;-\dfrac{1}{2}\right)$

13.1	Show that the equation of g is given by: $g(x)=\dfrac{-3}{x+2}+1$	(6)	**L3**
13.2	Write down the range of g.	(1)	**L1**
13.3	Determine the equation of h, the axis of symmetry of g, in the form $y=mx+c$, where $m>0$	(3)	**L2**
13.4	Write down the coordinates of K', the image of K reflected over h.	(2)	**L3**

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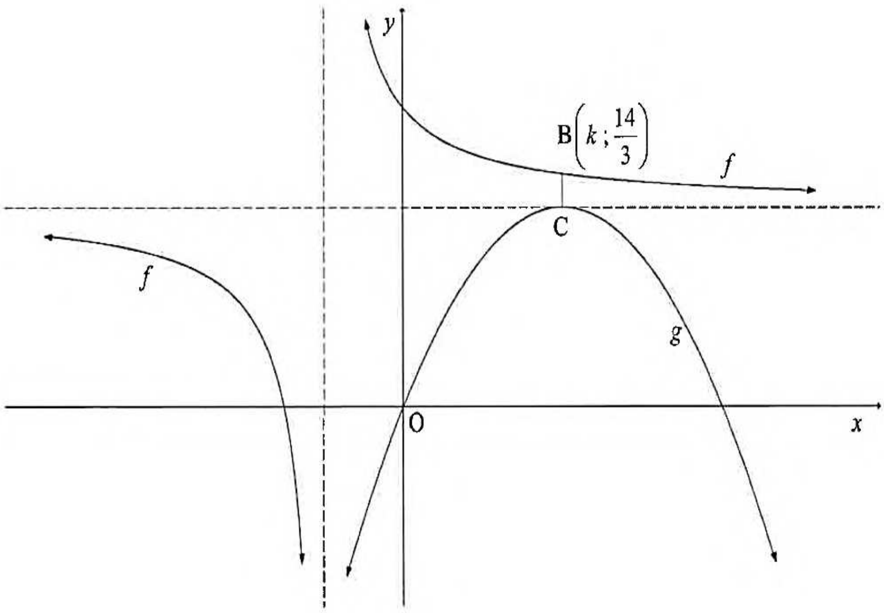
14 The graphs of $f(x)=\dfrac{2}{x+1}+4$ and parabola g are drawn below.

- C, the turning point of g, lies on the horizontal asymptotes of f.
- The graph of g passes through the origin.
- $B\left(k;\dfrac{14}{3}\right)$ is a point on f such that BC is parallel to the y-axis.



14.1	Write down the domain of f.	(2)	**L1**
14.2	Determine the x-intercept of f.	(2)	**L2**
14.3	Calculate the value of k.	(3)	**L2**
14.4	Write down the coordinates of C.	(2)	**L1**
14.5	Determine the equation of g in the form $y=a(x+p)^2+q$.	(3)	**L2**
14.6	For which value(s) of x will $\dfrac{f(x)}{g(x)} \leq 0$?	(4)	**L3**

14.7 Use the graphs of f and g to determine the number of real roots of $\frac{2}{x}-5=-(x-3)^2-5$. Give reasons for your answer (4) **L4**

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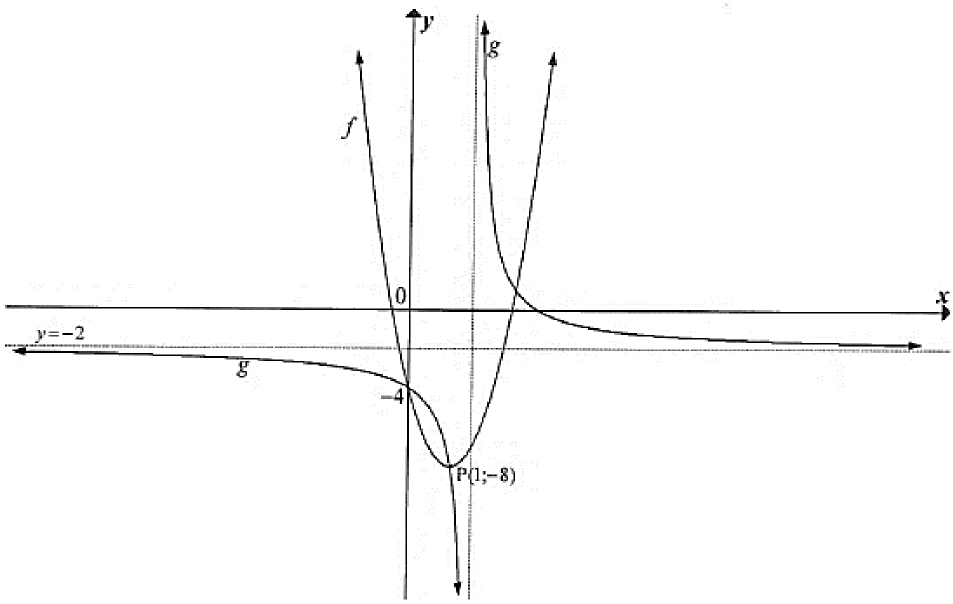
15 The graphs of the functions $f(x)=a(x+p)^2+q$ and $g(x)=\frac{k}{x+r}+d$ are sketched below. Both graphs cut the $y-$axis at -4. One of the points of intersection of the graphs is $\text{P}(1;-8)$, which is also the turning point of f. The horizontal asymptote of g is $y=-2$.



15.1	Calculate the values of a, p and q.	(4)	**L2**
15.2	Calculate the values of k, r and d.	(6)	**L3**
15.3	Determine the value(s) of x in the interval $x \le 1$ for which $g(x) \ge f(x)$.	(2)	**L2**
15.4	Determine the value(s) of k for which $f(x)=k$ has two, unequal positive roots.	(2)	**L2**
15.5	Write down the equation for the axis of symmetry of g that has a negative gradient.	(3)	**L2**
15.6	The point P is reflected in the line determined in QUESTION 15.5 to give the point Q. Write down the coordinates of Q.	(2)	**L3**

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16. Given: $f(x)=2^x+1$

16.1	Determine the coordinates of the $y-$intercept of f.	(1)	**L1**
16.2	Sketch the graph of f, clearly indicating ALL intercepts with the axes as well as any asymptotes.	(3)	**L2**
16.3	Calculate the average gradient of f between the points on the graph where $x=-2$ and $x=1$.	(3)	**L2**
16.4	If $h(x)=3f(x)$, write down an equation of the asymptote of h.	(1)	**L3**

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17. Given the exponential function: $g(x)=\left(\frac{1}{2}\right)^x$

17.1	Write down the range of g.	(1)	**L1**
17.2	Determine the equation of g^{-1} in the form $y=....$	(2)	**L2**
17.3	Is g^{-1} a function? Justify your answer.	(2)	**L1**

17.4 The point M(a; 2) lies on g^{-1}.

17.4.1 Calculate the value of a (2) **L1**

17.4.2 M′, the image of M, lies on g. Write down the coordinates of M′ (1) **L2**

17.5 If $h(x) = g(x+3)+2$, write down the coordinates of the image of M′ on h. (3) **L3**

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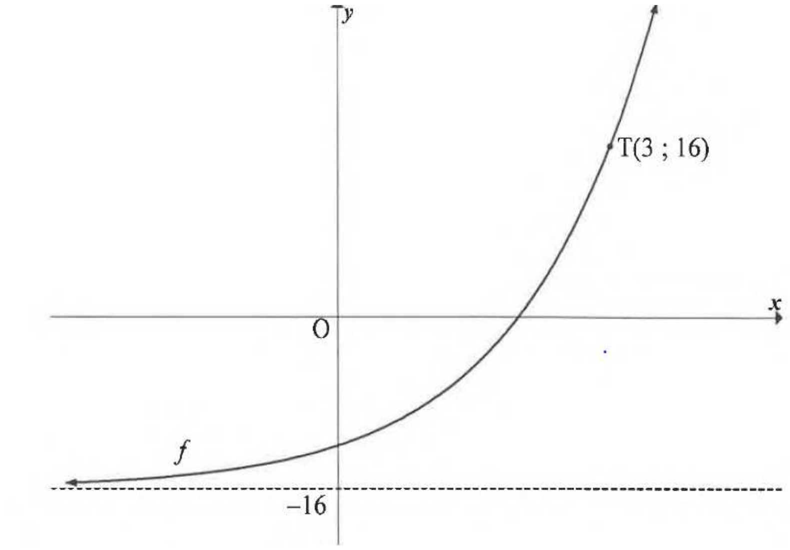
18 18.1 Given: $g(x) = 3^x$

18.1.1 Write down the equation g^{-1} in the form $y = ...$ (2) **L2**

18.1.2 Point P(6;11) lies on $h(x) = 3^{x-4} + 2$. The graph of h is translated to form g. Write down the coordinate of the image of P on g. (2) **L2**

18.2 Sketched is the graph of $f(x) = 2^{x+p} + q$. T(3;16) is a point on f and the asymptote of f is $y = -16$.



Determine the values of p and q (4) **L2**

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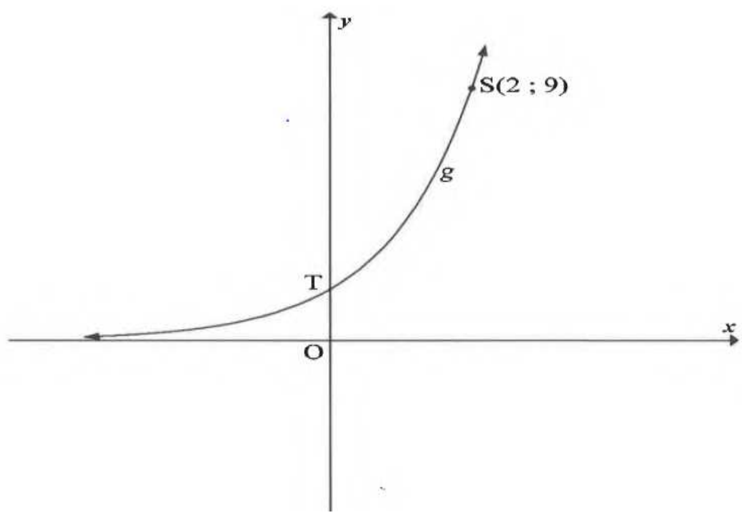
19 The graph of $g(x) = a^x$ is drawn in the sketch below. The point S(2;9) lies on g. T is the y-intercept of g.



19.1 Write down the coordinates of T. (2) **L1**

19.2 Calculate the value of a. (2) **L2**

19.3 The graph h is obtained by reflecting g in the y-axis. Write down the equation of h. (2) **L2**

19.4 Write down the values of x for which $0 < \log_3 x < 1$. (2) **L3**

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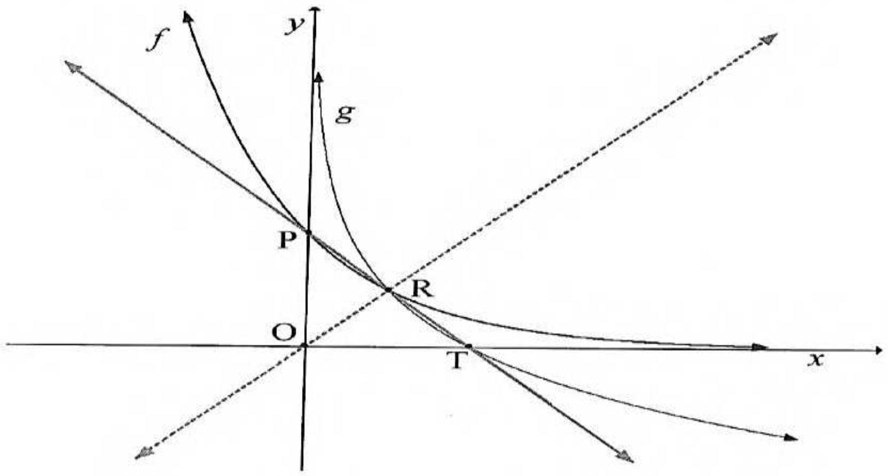
20 In the sketch below, P is the y-intercept of the graph of $f(x) = b^x$. T is the x-intercept of graph g, the inverse of f. R is the point of intersection of f and g. Straight lines are drawn through O and R and through P and T.



20.1 Determine the equation of g (in terms of b) in the form $y = ...$ (2) **L2**

20.2 Write down the equation of the line passing through O and R. (1) **L1**

20.3 Write down the coordinates of point P. (1) **L1**

20.4 Determine the equation of the line passing through P and T. (2) **L2**

20.5 Calculate the value of b. (5) **L3**

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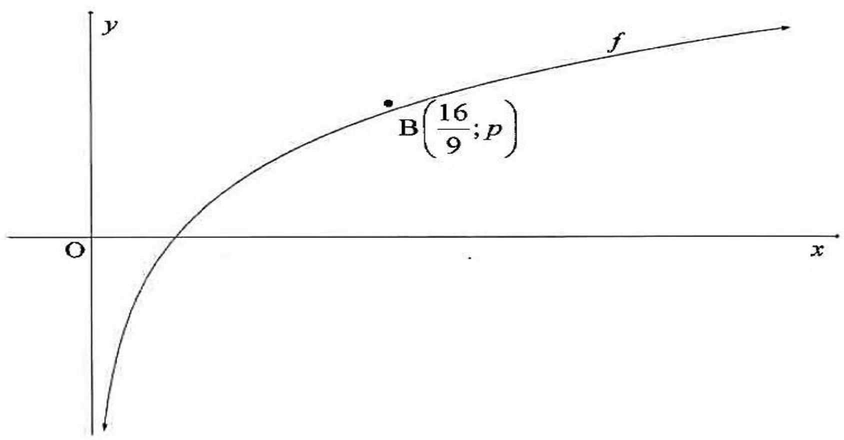
21 The graph of $f(x) = \log_{\frac{4}{3}} x$ is drawn below. $B\left(\frac{16}{9}; p\right)$ is a point on f.



21.1 For which value(s) of x is $\log_{\frac{4}{3}} x < 0$? (2) **L2**

21.2 Determine the value of p ,without using a calculator (3) **L2**

21.3 Write down the equation of the inverse of f in the form $y = ...$ (2) **L2**

21.4 Write down the range of $y = f^{-1}(x)$ (2) **L1**

21.5 The function $h(x)=\left(\frac{3}{4}\right)^{x}$ is obtained after applying two reflections on f. Write down the coordinates of B'', the image of B an h. (2) **L3**

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22 The graph of $g(x)=a\left(\frac{1}{3}\right)^{x}+7$ passes through point E(-2; 10)

22.1 Calculate the value of a. (3) **L2**

22.2 Calculate the coordinates of the y-intercept of g. (2) **L2**

22.3 Consider: $h(x)=\left(\frac{1}{3}\right)^{x}$

22.3.1 Describe the translation from g to h. (2) **L3**

22.3.2 Determine the equation of the inverse of h, in the form $y=....$ (2) **L2**

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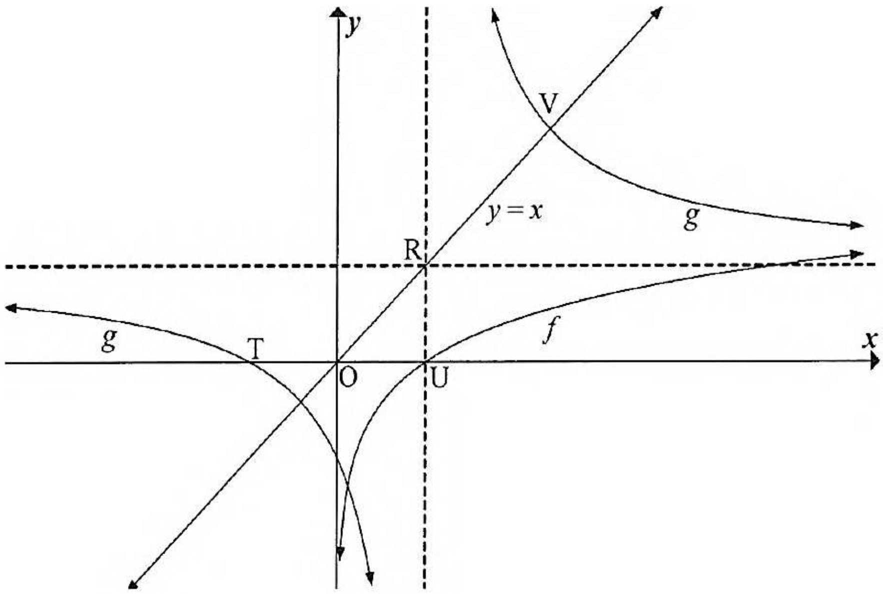
23 The sketch below shows the graphs of $f(x)=\log_5 x$ and $g(x)=\frac{2}{x-1}+1$.

- T and U are the x-intercepts of g and f respectively.
- The line $y=x$ intersects the asymptotes of g at R, and the graph of g at V.



23.1 Write down the coordinates of U. (1) **L1**

23.2 Write down the equation of the asymptotes of g. (2) **L2**

23.3 Determine the coordinates of T. (2) **L2**

23.4 Write down the equation of h, the reflection of f in the line $y=x$, in the form $y=...$ (2) **L2**

23.5 Write down the equation of the asymptotes of $h(x-3)$. (1) **L3**

23.6 Calculate the coordinates of V. (4) **L2**

23.7 Determine the coordinates of T′ the point which is symmetrical to T about the point R. (2) **L2**

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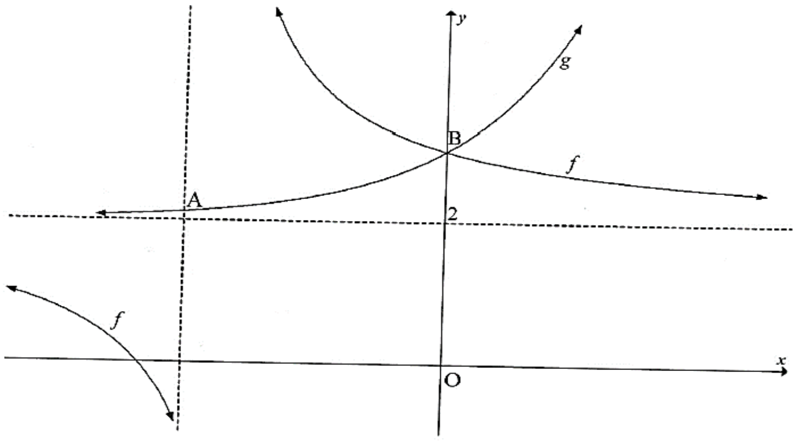
24 The sketch below shows the graphs of $f(x)=\dfrac{3}{x-p}+q$ and $g(x)=2^{x}+r$

- g intersects the vertical asymptotes of f at A.
- B is the common y-intercept of f and g.
- $y=2$ is the common horizontal asymptote of f and g



24.1	Write down the value of r.	(1)	**L1**
24.2	Determine the value of p.	(4)	**L2**
24.3	Determine the coordinates of A.	(3)	**L2**
24.4	For which value(s) of x is $f(x)-g(x)\geq 0$?	(2)	**L2**
24.5	If $h(x)=f(x-2)$, write down the equation of h.	(2)	**L2**

TOPIC: FINANCIAL MATHEMATICS

GUIDELINES, SUMMARY NOTES, & STRATEGIES

SIMPLE INTEREST AND COMPOUND INTEREST (A>P)

- When simple interest is used, interest is calculated as a percentage of the original amount invested or borrowed.

  $A = P(1+in)$

- When compound interest is used, interest is added after every period and the interest for the next period is calculated as a percentage of the new total.

  $A = P(1+i)^n$

DIFFERENT COMPOUNDING PERIODS

- Interest is not always calculated annually (yearly), but often based on a different period such as:
- **Monthly** (Divide interest rate by 12 and multiply years by 12)
- **Quarterly** (Divide the interest by 4 and multiply years by 4)
- **Half-year or Semi-annually or bi-annually** (Divide interest rate by 2 and multiply years by 2)

Example 1

At what interest rate, compounded quarterly, must R25 000 be invested in order to grow to R40 000 in 5 years' time?

Solution to example 1

n = 5 x 4 = 20 quarters

$A = P(1+i)^n$

$40000 = 25000(1+i)^{20}$

$(1+i)^{20} = \frac{40}{25}$

$1+i = \sqrt[20]{\frac{40}{25}}$

$i = 0,023778486$

Annual rate = 9,51%

Example 2

How long will it take for the value of an investment to treble, if interests is calculated at 22% p.a. compounded semi-annually?

Solution to example 1

$A = P(1+i)^n$

$3x = x\left(1+\frac{0,22}{2}\right)^n$

$3 = (1,11)^n$

$n = \log_{1,11}(3)$

$n = 10,527138$

n = 5 years and 3 months

EFFECTIVE AND NOMINAL INTEREST RATES

- To determine the annual effective rate, we use the formula:

$$1+i_{eff} = \left(1+\frac{i_{nom}}{m}\right)^m$$

Example 3

Calculate the effective annual interest rate corresponding to each of the following nominal rates:

8,5 p.a. compounded monthly

Solution to example 3

$$1+i_{eff} = \left(1+\frac{i_{nom}}{m}\right)^m$$

$$1+i_{eff} = \left(1+\frac{0,085}{12}\right)^{12}$$

$$1+i_{eff} = 1,0884$$

$$i_{eff} = 0,0884$$

$\therefore$ *Effective rate is* 8,84%

Example 4

Convert an effective annual interest rate of 12% per annum to a nominal rate p.a. compounded quarterly.

Solution to example 4

$$1+i_{eff} = \left(1+\frac{i_{nom}}{m}\right)^m$$

$$1+0,12 = \left(1+\frac{i_{nom}}{4}\right)^4$$

$$1+\frac{i_{nom}}{4} = \sqrt[4]{1,12}$$

$$i_{nom} = \left(\sqrt[4]{1,12}-1\right)\times 4$$

$$i_{nom} = 0,1149$$

$\therefore$ Nominal rate is 11,49% p.a. compounded quarterly.

Example 5
Convert an interest rate of 13% p.a. compounded monthly to an interest rate compounded quarterly.

Solution to example 5

$$(1+i_{new})^m = (1+i_{nom})^n$$

$$\left(1+\frac{i_{new}}{4}\right)^4 = \left(1+\frac{0,12}{12}\right)^{12}$$

$$1+\frac{i_{new}}{4} = \sqrt[4]{\left(1+\frac{0,12}{12}\right)^{12}}$$

$$i_{new} = 0,1212$$

$$= 12,12\%$$

- When working with different compounding periods use the formula $(1+i_{new})^m = (1+i_{nom})^n$ where:

DEPRECIATION (A<P)

- The formulae that are used to do depreciation calculations are:
  $A = P(1-in)$ Straight line depreciation
  $A = P(1-i)^n$ Reducing balance depreciation

ANNUITIES

- An annuity is a series of equal payments made at regular time intervals
- The annuity formulae are used under the following conditions:
  - ✓ All payments are equal
  - ✓ The payments are made at regular intervals
  - ✓ The interest rate remains fixed and
  - ✓ the compounding period for interest is the same as the payment intervals

THE FUTURE VALUE FORMULA

- We can use the following formula to calculate the future value of an annuity:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

F is the future value.
x is the payment.
i is the interest rate per interval.
n is the number of payments.

THE PRESENT VALUE FORMULA

- We can use the following formula to calculate the present value of an annuity:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

P is the present value.
x is the payment.
i is the interest rate per interval.
n is the number of payments.

Example 6
On her 25th birthday, Jeanine decides to start saving for retirement. One month after her 25th birthday, she takes out a retirement annuity and immediately pays R1 000 into the account. She continues making monthly payments of R1 000 until she retires on her 65th birthday. Interest is calculated at 7,3% p.a. compounded monthly. How much money will be in her annuity when she retires?

Example 7
Amahle wants to save up R100 000 to pay a deposit on a house in 4 years' time. She makes equal quarterly payments into a savings account, starting immediately. She makes the last payment at the end of the 4 years. How much must she save each quarter, if interest is calculated at 12% p.a. compounded quarterly?

Solution to example 6

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$F=\frac{1000\left[\left(1+\frac{0,073}{12}\right)^{480}-1\right]}{\frac{0,073}{12}}$$

$F=R2856657,21$

Solution to example 7

$$=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$100000=\frac{x\left[\left(1+\frac{0,12}{4}\right)^{17}-1\right]}{\frac{0,12}{4}}$$

$x=R4595,25$

Example 8

Jaco wants to save up R25 000 to redo his swimming pool. He can afford to save R1 000 per month and the interest rate on his savings account is 6,9% p.a. compounded monthly. How long will Jaco have to save before he will have at least R25 000 in his savings account?

Solution to example 8

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$25000=\frac{1000\left[\left(1+\frac{0,069}{12}\right)^{n}-1\right]}{\frac{0,069}{12}}$$

$1,14375=(1,00575)^n$

$n=\log_{(1,00575)}(1,14375)$

$n=23,4257594$ months

$n=1,952$ years

$\therefore$ He must save for 2 years.

SINKING FUND

i) $A=P(1-i)^n$ (scrap value of old asset)

ii) $A=P(1+i)^n$ (cost of new asset)

iii) Sinking fund = new – old

iv) Calculate x

v) Withdrawals (calculate x_{new}) – treat it separately and add it back

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$796685,71=\frac{x\left[\left(1+\frac{0,1375}{12}\right)^{120}-1\right]}{\frac{0,1375}{12}}$$

$x=R3121,69$

Example 9

Jan's Jam has to replace their canning in 10 years' time. Their current canning machine is valued at R270 000 and depreciates at 17% p.a. on a reducing balance. The price of a replacement canning machine increases by 12% p.a. The old machine will be sold at scrap value and the proceeds used towards purchasing the new machine. The company decides to set up a sinking fund to cover the replacement cost of the machine. Payments are made into the sinking fund on a monthly basis. The first payment is made one month after the purchase of the original canning machine and the last payment at the end of the 10 years. Calculate the monthly payment into the sinking fund if the interest rate is 13,75% p.a. compounded monthly.

Solution to example 9

Depreciation:

$A=P(1-i)^n$

$A=270000(1-0,17)^{10}$

$A=R41893,31$

Inflation:

$A=P(1+i)^n$

$A=270000(1+0,12)^{10}$

$A=838579,02$

Difference:

$=R838579,02-R41893,31$

$=R796685,71$

THE PRESENT VALUE FORMULA

- We can use the following formula to calculate the present value of an annuity:

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

Solution to example 10

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$1500000=\frac{x\left[1-\left(1+\frac{0,105}{12}\right)^{-20\times 12}\right]}{\frac{0,105}{12}}$$

$$x=\frac{1500000\times\frac{0,105}{12}}{\left[1-\left(1+\frac{0,105}{12}\right)^{-240}\right]}$$

$$x=R14975,70$$

THE OUTSTANDING BALANCE ON A LOAN

Outstanding Balance = Loan with interest to date – Repayments with interest to date

$$OB=P(1+i)^n-\frac{x\left[(1+i)^n-1\right]}{i}$$ **Or**

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

Note:

- When using the P formula, use the remaining number of payments.
- When using OB=A-F, use n as number of payments made.

Solution to example 12

a) After eleven months, Lethiwe will owe:

$$A=P(1+i)^n$$

$$A=82000\left(1+\frac{0,15}{12}\right)^{11}$$

$$=R94006,79$$

b)

Example 10 (November 2018)

Tshepo takes out a home loan over 20 years ago to buy a house that costs R1 500 000.

Calculate the monthly instalment if the interest is charged at 10,5% p.a., compounded monthly.

Example 11

Refer to example 14 above and calculate the outstanding balance immediately after the 144th payment was made.

Solution to example 11

$$Balance=1500000\left(1+\frac{0,105}{12}\right)^{144}-\frac{14975,70\left[\left(1+\frac{0,105}{12}\right)^{144}-1\right]}{\frac{0,105}{12}}$$

$$=R5259229,61-R4289302,47$$

$$=R969927,14$$

THE LAST/FINAL PAYMENT

Last payment = Outstanding balance after the last full payment multiplied by $(1+i)^1$

Example 12 (Feb/March 2018)

On 1 February 2018, Lethiwe took a loan of R82 000 from the bank to pay for her studies. She will make her first repayment of R3 200 in 1 February 2019 and continue to make payments of R3 200 on the first of each month thereafter until she settles the loan. The bank charges interest at 15% p.a., compounded monthly.

a) Calculate how much Lethiwe will owe the bank on 1 January 2019.

b) How many instalments of R3 200 must she pay?

c) Calculate the final payment, to the nearest rand, Lethiwe has to pay to settle the loan.

c)

$$\left(1+\frac{0,15}{12}\right)^{-n}=1-0,3672147...$$

$$-n=\log_{\left(1+\frac{0,15}{12}\right)}(1-0,3672147...)$$

$$-n=-36,8382$$

$$n=36,84$$

Lethiwe will have to pay 36 instalments of R3 200

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$94006,79=\frac{3200\left[1-\left(1+\frac{0,12}{12}\right)^{-n}\right]}{\frac{0,15}{12}}$$

Final Payment will be:

$$A=P(1+i)^{n}$$

$$A=2651,72\left(1+\frac{0,15}{12}\right)^{1}$$

$$A=R2685,00$$

DELAYED/ DEFERRED ANNUITIES

- When the first payment of a loan is made more than one period after the loan was received, this payment is referred to as a ***deferred annuity*.**
- Apply the compound interest to the loan to move it to the same point on the timeline as the present value of the annuity.

Monthly payment:

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$268018,43=\frac{x\left[1-\left(1+\frac{0,14}{12}\right)^{-42}\right]}{\frac{0,14}{12}}$$

$$x=R8108,43$$

Example 14 (September 2019)

Andile takes a loan of R950 000 to buy a house. The interest is 14,25% p.a. compounded monthly. His first instalment will commence one month after taking out the loan.

a) Calculate the monthly repayments over a period of 20 years.

b) Determine the balance on the loan after the 100th instalment.

c) If Andile failed to pay the 101st, 102nd, 103rd and 104th instalments, calculate the value of the new instalment that will settle the loan in the same time period.

d)

$$Balance=P(1+i)^{n}-\frac{x\left[(1+i)^{n}-1\right]}{n}$$

$$94006,79\left(1+\frac{0,15}{12}\right)^{36}-\frac{3200\left[\left(1+\frac{0,15}{12}\right)^{36}-1\right]}{\frac{0,15}{12}}$$

$$=R2651,72$$

Example 13

On 1 January 2020, Amahle takes out a loan of R250 000 to pay for her wedding. She will repay the loan by means of equal monthly payments, starting on 31 July 2020 and ending 31 December 2023. The interest rate on the loan is 14% p.a. compounded monthly.

Calculate her monthly payment.

Solution to example 13

Grow the loan for 6 months

$$A=P(1+i)^{n}$$

$$A=250000\left(1+\frac{0,14}{12}\right)^{6}$$

$$A=R268018,43$$

MISSED PAYMENTS

To calculate the new payment:

- We calculate the outstanding balance immediately after the last payment made.

We then apply the compound interest to this outstanding balance, till one period before payments resume. The result is the present value of the new annuity consisting of all the remaining payments.

Solution to example 14

a)

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$950000=\frac{x\left[1-\left(1+\frac{0,1425}{12}\right)^{-240}\right]}{\frac{0,1425}{12}}$$

$$x=\frac{950000\times\frac{0,1425}{12}}{\left[1-\left(1+\frac{0,1425}{12}\right)^{-240}\right]}$$

$$x=R11986,67$$

Outstanding Balance on the loan

$$= P(1+i)^n - \frac{x\left[(1+i)^n - 1\right]}{n}$$

b) $$=950000\left(1+\frac{0,1425}{12}\right)^{100} - \frac{11986,33\left[\left(1+\frac{0,1425}{12}\right)^{100}-1\right]}{\frac{0,1425}{12}}$$

$$= R816048,67$$

c)

$$A = P(1+i)^n$$

$$A = 816048,67\left(1+\frac{0,1425}{12}\right)^4$$

$$A = R855506,92$$

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$855506,92 = \frac{x\left[1-\left(1+\frac{0,1425}{12}\right)^{-136}\right]}{\frac{0,1425}{12}}$$

$$x = R12711,51$$

Example 15 (November 2017)

Noxolo takes a loan from the bank to buy a car for R235 000. She agrees to repay the loan over a period of 54 months. The first instalment will be paid one month after the loan is granted. The bank charges interest at 11% p.a., compounded monthly.

a) Calculate Noxolo's monthly instalment.

b) Calculate the total amount of interest that Noxolo will pay during the first year of the repayment of the loan.

Solution to example 15

a)

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$235000 = \frac{x\left[1-\left(1+\frac{0,11}{12}\right)^{-54}\right]}{\frac{0,11}{12}}$$

$$x = R5536,95$$

b)

Amount paid = $(5536,95\times 12) = R66443,40$

$$Balance = 235000\left(1+\frac{0,11}{12}\right)^{12} - \frac{5536,95\left[\left(1+\frac{0,11}{12}\right)^{12}-1\right]}{\frac{0,11}{12}}$$

$$= R192296,17$$

$$Interest = 66443,40 + 192296,17 - 235000$$

$$= R23739,57$$

REVISION QUESTION

1 A new cellphone was purchased for R7 200. Determine the depreciation value after 3 years if the cellphone depreciates at 25% p.a. on a reducing balance method. (3) **L2**

2 Sipho negotiates a loan of R300 000 with a bank which has to be repaid by means of monthly payments of R5 000 and a final payment which is less than R5 000. The repayments start one month after the granting of the loan. Interest is fixed at 18% p.a., compounded monthly.

2.1 Determine the number of payments required to settle the loan. (6) **L3**

2.2 Calculate the balance outstanding after Sipho has paid the last R5 000. (5) **L2**

2.3 Calculate the value of the final payment made by Sipho to settle the loan. (2) **L2**

2.4 Calculate the total amount that Sipho paid to the bank. (2) **L2**

3 James buys a house and takes out a loan of R2 million. He repays the loan over fifteen years. The interest charged on the outstanding balance of the loan is 8,5% p.a., compounded monthly.

3.1 Calculate his monthly payment of the loan. (4) **L2**

3.2 What is the outstanding balance on the loan at the end of five years. (3) **L2**

3.3 Determine the amount of money paid on the loan at the end of the first five years. (2) **L1**

3.4 What is the interest paid on the loan during the first five years. (4) **L3**

4 R1 430,77 was invested in a fund paying i% p.a. compounded monthly. After 18 months the fund had a value of R1 711,41. Calculate i. (4) **L3**

5 A father decided to buy a house for his family for R800 000. He agreed to pay monthly instalments of R10 000 on a loan which incurred interest at a rate of 14% p.a. compounded monthly. The first payment was made at the end of the first month.

5.1 Show that the loan would be paid off in 234 months. (4) **L3**

5.2 Suppose the father encountered unexpected expenses and was unable to pay any instalments at the end of the 120th, 121st, 122nd and 123rd months. At the end of the 124th month he increased his payment so as to still pay off the loan in 234 months by 111 equal monthly payments. Calculate the value of this new instalment. (7) **L3**

6 A farmer buys a tractor for R2,2 million.

6.1 Determine the book value of a tractor at the end of 5 years if the depreciation is calculated at 14% p.a. on a reducing balance method. (3) **L1**

6.2 Determine the expected cost of buying a new tractor in five years' time if the average rate of inflation is expected to be 6% p.a. (3) **L1**

6.3 The farmer decides to replace the old tractor in five years' time. He will trade in the old tractor. Calculate the sinking fund. (3) **L1**

6.4 Calculate the monthly payment into the sinking fund if the payments commenced one month after he bought the tractor if the interest rate is 7% per annum compounded monthly. (4) **L2**

6.5 Suppose that at the end of each year he withdraws R5 000 from his account to pay for the maintenance of the tractor. Determine the new monthly deposit. (4) **L3**

7 A business buys a machine that costs R120 000. The value of the machine depreciates at 9% per annum according to the reducing balance method.

7.1 Determine the scrap value of the machine at the end of 5 years. (3) **L1**

7.2 After 5 years the machine needs to be replaced. During this time the inflation remained constant at 7% per annum. Determine the cost of the new machine at the end of 5 years. (3) **L1**

7.3 The business estimates that it will need R90 000 by the end of five years. A sinking fund for R90 000 into which equal monthly instalments must be paid, is set up. Interest on this is 8,5% p.a. compounded monthly. The first payment will be made immediately, and the last payment will be made at the end of the 5-year period. Calculate the monthly deposits. (5) **L2**

8 At the beginning of October 2018, Lungile opened a savings account with a single deposit of R10 000. She then made 24 monthly deposits of R1600 at the end of every month starting at the end of October 2018. She earns 15% p.a. interest compounded monthly in her account.
Calculate the amount that should be in her savings account immediately after she makes the last deposit. (5) **L3**

9 How many years will it take for an investment to double in value, if it earns interest at a rate of 8,5% p.a., compounded quarterly? (5) **L3**

May/ June 2022 Question 7

10 A company purchased machinery for R500 000. After 5 years, the machinery was sold for R180 000 and new machinery was bought.

10.1 Calculate the rate of depreciation of the old machinery over the 5 years, using the reducing balance method. (4) **L2**

10.2 The rate of inflation for the cost of the new machinery is 6,3% p.a. over the 5 years. What will the new machinery cost at the end of 5 years. (2) **L1**

10.3 The company set up a sinking fund and made the first payment into this fund on the day the old machinery was bought. The last payment was made three months before the new machinery was purchased at the at the end of the 5 years. (5) **L3**

The interest earned on the sinking fund was 10,25% p.a., compounded monthly. The money from the sinking fund and the R180 000 from the sale of the old machinery was used to pay for the new machinery.
Calculate the monthly payment into the sinking fund.

March 2010 Question 9

11 Lindiwe receives a bursary of R80 000 for her studies at university. She invests the money at a rate of 13,75% p.a. compounded yearly. She decides to withdraw R25 000 at the end of each year for her studies, starting at the end of the first year.
Determine for how many full years will this investment finance her studies. (4) **L3**

November 2012 Question 7

12 Lorraine receives an amount of R900 000 upon her retirement. She invests this amount immediately at an interest rate of 10,5% per annum, compounded monthly. She needs an amount of R18 000 per month to maintain her current lifestyle. She plans to withdraw the first amount at the end of the first month.
For how many months will she be able to live from her investment? (6) **L3**

March 2015 Question 7

13 For each of the three years from 2010 to 2012 the population of town X decreased by 8% per year and the population of town Y increased by 12% per year. At the end of 2012 the populations of these two towns were equal.
Determine the ratio of the population of town X (call it P_x) to the population of town Y (call it P_Y) at the beginning of 2010. (4) **L4**

March 2017 Question 6

14 Lerato wishes to apply for a home loan. The bank charges interest at 11% per annum, compounded monthly. She can afford a monthly instalment of R9 000 and wants to repay the loan over a period of 15 years. She will make the first monthly repayment one month after the loan is granted. Calculate, to the nearest thousand rand, the maximum amount that Lerato can borrow from the bank. (5) **L2**

DGC September 2021 Question 5

15 Khwezi is planning to buy her first home. The bank will allow her to use a maximum of 30% of her monthly salary to repay the bond. She earns R18 480 per month. Suppose, at the end of each month, Khwezi repays the maximum amount allowed by the bank. The first instalmet is made one month after the loan is granted.

15.1 How much money does Khwezi borrow if she takes 25 years to repay the loan at a rate of 8% p.a. compounded monthly. (4) **L2**

15.2 Calculate the outstanding balance after 20 years of paying back the loan. (3) **L2**

Rustenburg Girls' High School Question3

16 Lynne purchases a new car for R350 000. They take out a 6-year loan on 1 January 2019. The monthly instalments are paid at the end of every month. Interest is fixed at 18% p.a. compounded monthly.

16.1 Calculate the monthly repayment. (4) **L2**

16.2 Due to financial difficulty, Lynne misses the 40^{th}, 41^{st} and 42^{nd} payments. Determine the balance outstanding at the end of the 42^{nd} month. (4) **L3**

16.3 If Lynne's monthly repayment is R10 000. How many month will it take her to pay back the rest of the loan. (4) **L3**

TOPIC DIFFERENTIAL CALCULUS

GUIDELINES, SUMMARY NOTES, & STRATEGIES
TEACHING APPROACHES (CALCULUS)

1. FIRST PRINCIPLES:

The learners:

- ✓ Need to understand what is meant by determining the gradient from first principles and know the first principles formula.
- ✓ must be able to copy the first principle formula from the formula sheet correctly.
- ✓ Be able to simplify the first principles expression (It seems as if learners handled this question better when they determine $f(x+h)$ separately and then bring it back to the formula).
- ✓ Need to be mindful of the notation and apply it correctly when they simplify the first principle expression.
- ✓ At this stage, learners can also determine the equation of the tangent at a point.

2. RULES FOR DIFFERENTIATION

- ✓ The learners:
  - i. need to revise how to simplify surds, rational, irrational exponents.
  - ii. Must know how to simplify expressions before differentiation.
  - iii. Must know how to tell which variable they are required to differentiating with respect to.
- ✓ Must expose themselves to variety of questions having different notations including where a variable is given as constant.
- ✓ Following instructions is once more important, on how the answer should be provided whether with a + ve or - ve.
- ✓ Must always use of correct notation.

3. CUBIC FUNCTIONS $f(x) = ax^3 + bx^2 + cx + d$

The learners need to know and follow these steps when sketching a cubic function:

- ✓ Before learners can sketch a cubic function, they at least need to know the shape of their graph as guided by value of a where a could be $a > 0$ and $a < 0$.
- ✓ The learners must be able to Factories a third-degree polynomial using any other method to determine the ***x*-intercepts** (the x-intercepts are known as the: zero, roots, $f(x) = 0$. It would be an advantage if they can be able factories using a calculator.
- ✓ They must also be able to find the ***y*- intercept**, which is when $x = 0$, or given by the value of d.
- ✓ Learners must be able to use the first derivative to find the coordinates of the turning points, which are also known as the Stationery points or local minima and local maxima. In simple terms, this is finding $f'(x) = 0$, solve for x, and then find the corresponding y-values to give the coordinate of the turning point.
- ✓ Examiners often require learners to write the intercepts with the axes, stationary points and

points of inflection in coordinate form $(a ; b)$. Make sure that the learners are aware of this.

4. INTERPRETATION OF A CUBIC FUNCTION:

The learners must be able to:

- ✓ Tell what the domain is, that $x \in R$
- ✓ Understand the relationship between the graph of a function and the graph of its derivative is important in that it explains to the learners why the second derivative is zero at a point of inflection. (REFER TO EXAMPLE BELOW)
- ✓ Understand that the point of inflection is determined by equating the second derivative to zero and solving for *x*. An alternative method is to add up the *x*-coordinates of the turning points and divide by 2 (i.e. determining the midpoint of the two turning points).
- ✓ Tell for which values of x will $f(x)$ be concave up: $f''(x) > 0$ & Concave down:
- ✓ $f''(x) < 0$
- ✓ Tell where f is increasing or decreasing ($f'(x) > 0$, decrease $f'(x) < 0$).
- ✓ Determine the values of *x*, for which: $x.f(x) > 0$, $f'(x) > 0$, $f'(x).f(x) < 0$
- ✓ when will f have three real roots, one real root?

5. OPTIMIZATION

The learners need to develop the conceptual understanding on Optimization

- **Calculus of motion**
- ✓ In this regard, the equation will be given.
- ✓ The learners need to know that, Velocity is the derivative of displacement, and
- ✓ Acceleration (2^{nd} derivative) is the derivative of velocity
- **Rates of change**
- ✓ Knowledge of formulae for the surface area and volume of right prisms is required from learners.
- ✓ A list of relevant formulae will only be provided for the surface area and volume of cones, spheres and pyramids. Learners must select the correct one to use.

REVISION QUESTIONS

1.

1.1	Determine the derivative of f, using the first principle, if $f(x) = 2 - 5x^2$	(5)	**L2**
1.2	Given. $f(x) = x^2 + 2x$ Determine $f'(x)$ from first principles	(5)	**L2**
1.3	Given $f(x) = -\dfrac{2}{x}$, determine $f'(x)$ from first principles.	(5)	**L2**
1.4	Determine the derivative of f, using the first principle, if $f(x) = -2x^3$	(5)	**L2**
1.5	Determine $f'(x)$ from first principles if it is given that $f(x) = -x^2$.	(5)	**L2**
1.6	Determine $f'(x)$ from first principles if it is given that $f(x) = ax^2 + b$.	(5)	**L2**
1.7	Determine $f'(x)$ from first principles if it is given that $f(x) = 4 - 7x$	(4)	**L2**

2 **Rules: Determine:**

2.1 $g'(x)$ **if** $g(x)=(7x-3)^2$ (3) **L1**

2.2 $D_x\left[\dfrac{x^3-4x^2-5}{\sqrt{x}}\right]$ (4) **L3**

2.3 $\dfrac{dy}{dx}$ **if** $y=\left[\dfrac{x^3-125}{5-x}\right]$ (4) **L2**

2.4 the derivative of f if **;** $f(x)=\sum_{r=0}^{3} r.x^{3-r}$ (4) **L3**

2.5 $\dfrac{dy}{dx}$ **if** $y=-\dfrac{\sqrt{x}}{2}-\dfrac{1}{x^2}$ (4) **L2**

2.6 $f'(x)$ **if** $f(x)=\dfrac{3x^2-7x-6}{x-3}$ (3) **L2**

2.7 $D_x\left(\dfrac{(x^2-2)^2}{x}\right)$ (4) **L2**

2.8 $\dfrac{dy}{dx}$ **if** $y=3x^3a^4-a^5x$ (2) **L2**

2.9 $D_x\left[\dfrac{-6\sqrt[3]{x}+2}{x^4}\right]$ (4) **L3**

2.10 Given $y=ax^2+a$

2.10.1 $\dfrac{dy}{dx}$ (1) **L1**

2.10.2 $\dfrac{dy}{da}$ (2) **L1**

2.11 Determine $\dfrac{dy}{dx}$ if

2.11.1 $y=3x^3+6x^2+x-4$ (3) **L1**

2.11.2 $yx-y=2x^2-2x$; $x\neq 1$ (4) **L3**

3 3.1 Determine the equation of the tangent to the curve $t(x)=\sqrt{x^3}$ at $x=4$ (4) **L2**

3.2 If $f(x)=3x^2-2$, calculate the gradient of the tangent to the curve of f at the point where $x=-1$ (3) **L2**

3.3 Determine the point on the curve of $y=4x^2+3x$ where the gradient is -1 (3) **L2**

3.4 Prove that $x+y=0$ is a tangent to the curve $y=x^3-10x^2+24x$ (5) **L4**

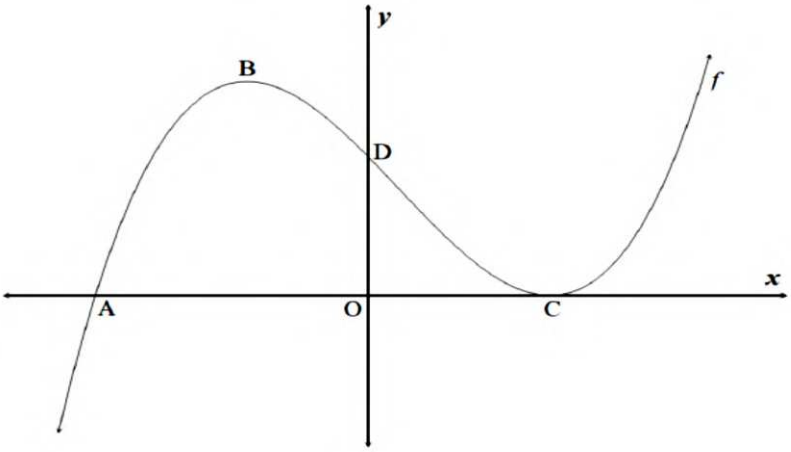
3.5 The line $y=2x+3$ is a tangent to the curve , $y=x^2+ax+b$ at the point $(2;7)$. Calculate the values of a and b. (5) **L3**

3.6 If g is a linear function with $g(1)=5$ and $g'(3)=2$, determine the equation of g in the form $y=$ ….. (3) **L2**

3.7 The function $f(x) = x^3 + bx^2 + cx - 4$ has a point of inflection at (2 ; 4). Calculate the values of b and c. (7) **L3**

3.8 The curve with equation $y = x + \frac{12}{x}$ passes through the point A (2;b) . Determine the equation of the line perpendicular to the tangent to the curve at A . (4) **L3**

4 The graph of $f(x) = x^3 - x^2 - 8x + 12$ is sketched below. B and C are turning points, A and C are the x-intercepts, and D is the y-intercept.



4.1 Write down the coordinates of D **(2) L1**

4.2 Determine the coordinates of the turning points of f **(5) L2**

4.3 Show that $f(x)$ has a point of inflection at $x = \frac{1}{3}$. **(4) L2**

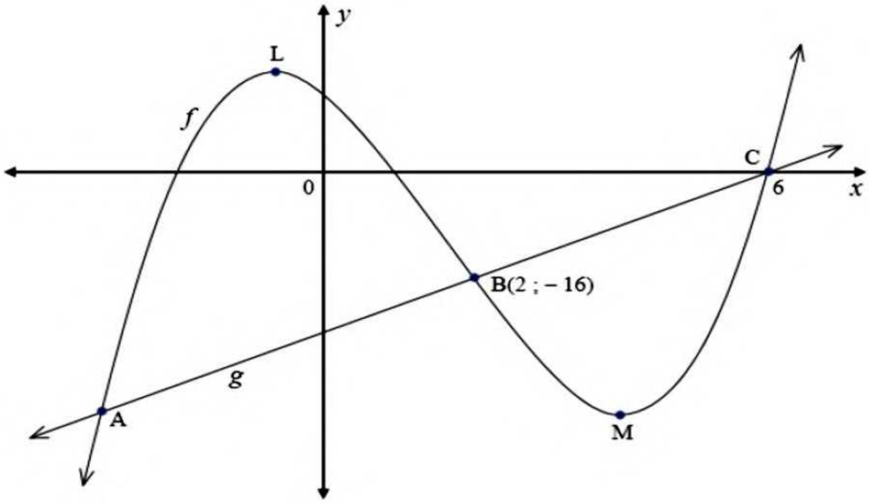
4.4 If $g(x) = f(-x) + 1$ write down the coordinates of C', the image of C. **(2) L3**

4.5 Write down the value(s) of k for which $f(x) = k$ will have

4.5.1 two unequal real roots **(2) L2**

4.5.2 one of the roots equal to 0 **(2) L2**

5 The sketch below represents the functions $f(x) = x^3 + bx^2 + cx + d$ and $g(x) = ax + q$. The points A, B(2;-16) and C, are points where the two graphs intersect. C(6;0) is an x-intercept of f, while L and M are the turning points of f.



Show that $b = -5$, $c = -8$ *and* $d = 12$ if it is given that

5.1 $f'(x) = 3x^2 - 10x - 8$ **(4) L3**

5.2 Determine the coordinates of the turning points, L and M, of f. **(5) L2**

5.3 Determine the equation of g **(3) L2**

If it is further given that the coordinates of point A are $(x;-36)$, determine the length of

5.4 AM. **(3)** **L2**

5.5 For which value(s) of x…..

5.5.1 is the graph of f increasing? **(2)** **L1**

5.5.2 is the graph of f concave down? **(2)** **L2**

6 The following information is about a cubic polynomial $y = f(x)$

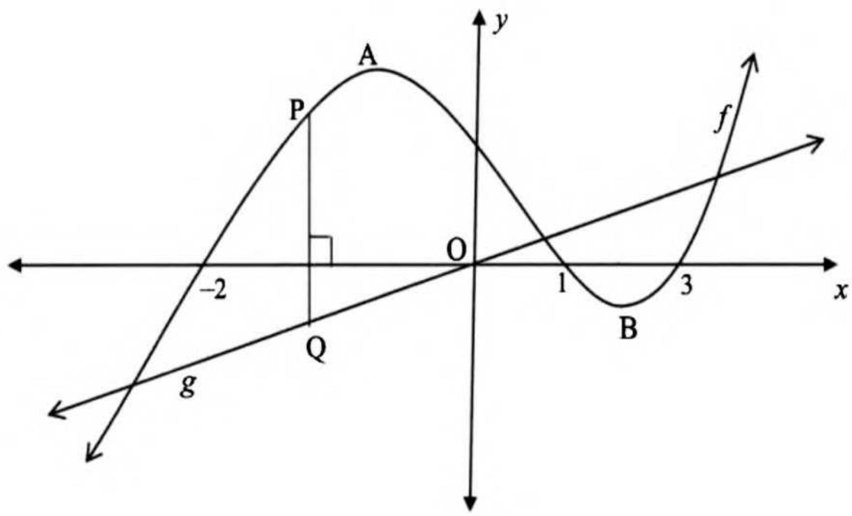
- $f(-1) = 0$
- $f(5) = 0$
- $f(0) = -2$
- $f'(-1) = f'(3) = 0$
- $f(3) = 6$
- If $x < -1$ then $f'(x) > 0$
- If $x > 3$ then $f'(x) > 0$

6.1 Sketch a neat graph of $f(x)$ showing all intercepts and turning points. **(5)** **L3**

6.2 Use the graph to find the x-value of the point of inflection **(2)** **L2**

6.3 For which values of x is the graph decreasing? **(2)** **L1**

7 In the diagram, the graphs of $f(x) = x^3 - bx^2 - cx + d$ and $g(x) = 2x$ are drawn. The graph passes through the x-axis at $x = -2, x = 1$ and $x = 3$. A and B are the turning points of f. P is a point on f and Q is a point on g such that PQ is perpendicular to the x-axis. $x_p < 0$



7.1 Show that $f(x) = x^3 - 2x^2 - 5x + 6$ **(2)** **L2**

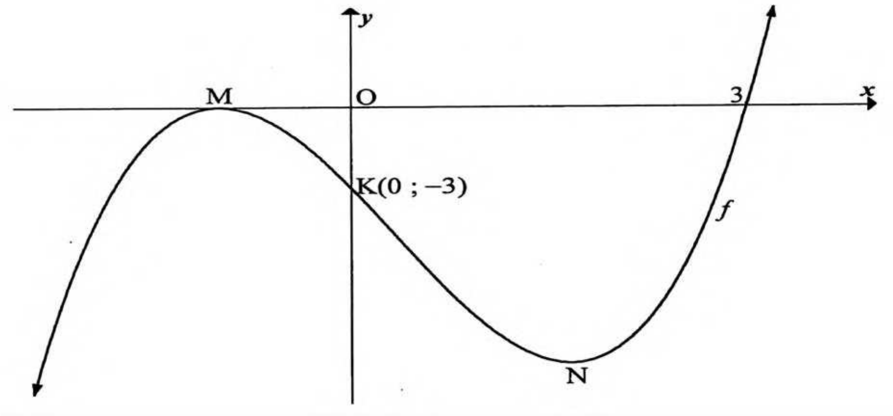
7.2 Calculate the x-coordinate of B. **(3)** **L2**

7.3 A tangent to f has gradient of -1. Explain why the point of contact of the tangent and the graph of f lies between A and B. **(1)** **L1**

7.4 For which values of x will f be concave up? **(2)** **L2**

7.5 Determine the maximum length of the line PQ. **(5)** **L3**

8 Sketched below is the graph of $f(x)=x^3+ax^2+bx+c$. The x-intercepts of f are at $(3;0)$ and M, where M lies on the negative x-axis.
$K(0;-3)$ is the y-intercept of f.
M and N are the turning point of f



8.1 Show that the equation of f $f(x)=x^3-x^2-5x-3$ (5) **L3**

8.2 Calculate the coordinates on N. (5) **L2**

8.3 For which values of x will:

8.3.1 $f(x)\leq 0$ (2) **L1**

8.3.2 f is increasing (2) **L1**

8.3.3 f is concave up (3) **L2**

8.4 Determine the maximum vertical distance between the graphs of f and f' in the interval $-1<x<0$. (5) **L4**

9 A cubic function $h(x)=-2x^3+bx^2+cx+d$ cuts the x-axis at (–3 ; 0); $\left(-\frac{3}{2}\,;0\right)$ and (1 ; 0).

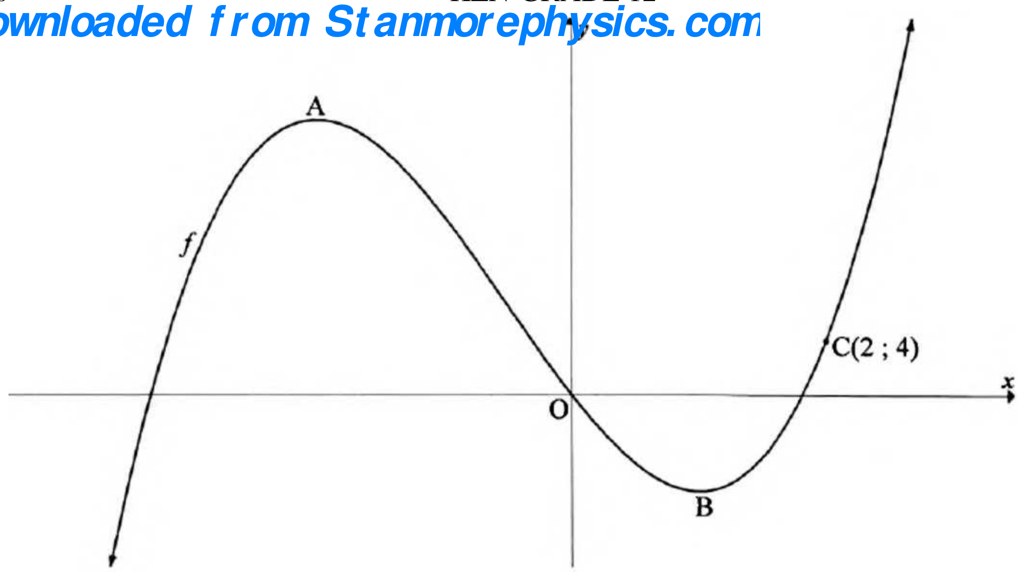
9.1 Show that $h(x)=-2x^3-7x^2+9$. (3) **L2**

9.2 Calculate the x-coordinates of the turning points of h. (3) **L2**

9.3 Determine the value(s) of x for which h will be decreasing. (3) **L1**

9.4 For which value(s) of x will there be a tangent to the curve of h that is parallel to the line $y-4x=7$. (4) **L3**

10 The graph of $f(x)=2x^3+3x^2-12x$ is sketched below. A and B are the turning points of f. C(2 ; 4)is a point on f



10.1 Determine the coordinates of A and B. (5) **L2**

10.2 For which values of x will f be concave up? (3) **L2**

10.3 Determine the equation of the tangent to f at C(2 ; 4) (3) **L2**

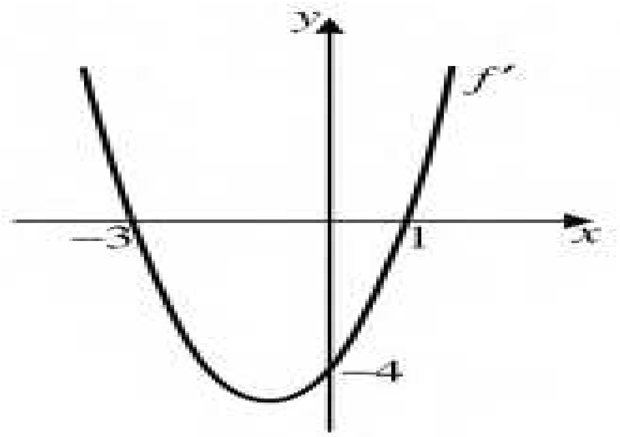
11 **If** g is a cubic function with

- $g(3)=g'(3)=0$
- $g(0)=27$
- $g''(x)>0$ when $x<3$ and $g''(x)<0$ when $x>3$,

Draw a sketch of g indicating ALL relevant points. (3) **L3**

12

In the sketch below , the gragh of $y = f'(x)$ is shown



12.1 What is the gradient of the tangent to f at $x = 0$? (1) **L1**

12.2 Write down the x-coordinates of the stationary points of f. (2) **L1**

12.3 What is the x-coordinate of the point of inflection of f (2) **L1**

12.4 For which values of x is f

12.4 1 Increasing (2) **L2**

12.4.2 decreasing (2) **L2**

13 Given $f(x) = 3x^3$

13.1 Solve $f(x) = f'(x)$ (3) **L2**

13.2 The graphs f, f' and f'' all pass through the point $(0;0)$

13.2.1 For which of the graphs will $(0;0)$ be a stationary point? (1) **L1**

13.2.2 Explain the difference, if any, in the stationary points referred to in QUESTION13.2.1 (2) **L2**

13.3 Determine the vertical distance between the graphs of f' and f'' at $x=1$ (2) **L2**

13.4 For which value(s) of x is $f(x)-f'(x)<0$ (4) **L3**

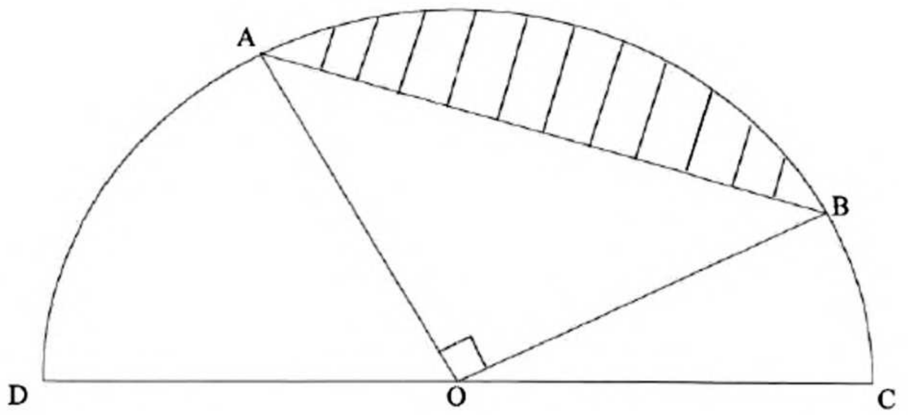
14 The graph of $f(x)=ax^3+bx^2+cx+d$ has two turning points.

The following information about f is also given:

- $f(2)=0$
- The x-axis is a tangent to the graph of f at $x=-1$
- $f'(1)=0$
- $f'\left(\frac{1}{2}\right)>0$

Without calculating the equation of f, use this information to draw a sketch graph of f, only indicating the x-coordinates of the x-intercepts and turning points. (4) **L3**

15 **O** is the centre of a semicircle passing through A, B, C and D. The radius of the semicircle is $(x-x^2)$ units for $0<x<1$. ΔAOB is right angled at O.



15.1 Show that area of the shaded part is given by:

$$Area=\left(\frac{\pi-2}{4}\right)\left(x^4-2x^3+x^2\right)$$
(5) **L3**

15.2 Determine the value of x for which the shaded area will be a maximum. (4) **L3**

16 Given: $f(x)=x^3-x^2-x+1$

16.1 Write down the coordinates of the y-intercepts of f (1) **L1**

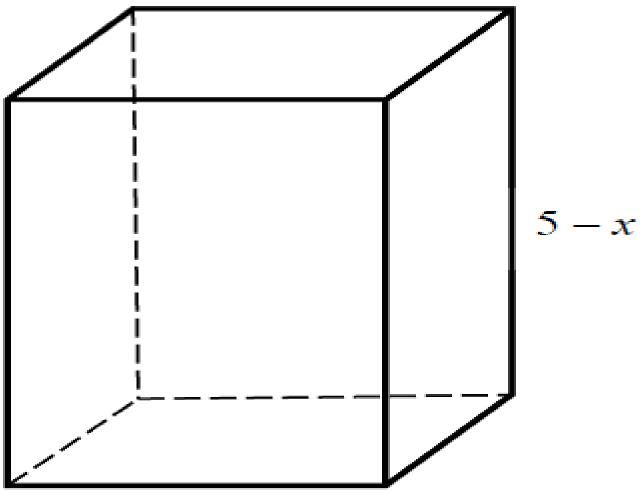
16.2 Calculate the coordinates of the x-intercepts of f (5) **L2**

16.3 Calculate the coordinates of the turning points of f (6) **L2**

16.4 Sketch the graph of f. Clearly indicate all intercepts with the axes and the turning points. (3) **L2**

16.5 Write down the values of x for which $f'(x)<0$ (2) **L2**

17 A rectangular box of cereal has a height of $(5-x)$ units.
The expression for the volume (V) of the box is given by
$V(x)=x^3-8x^2+5x+50$

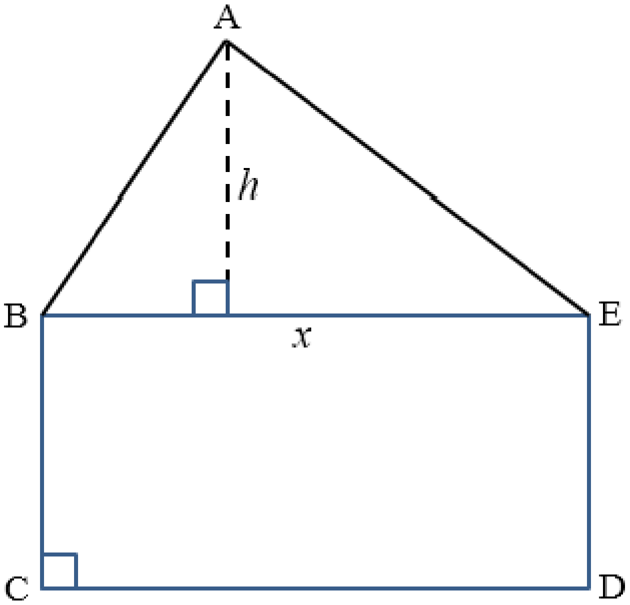


If the height of the cereal box is $(5-x)$ units, determine the area of the base of the box

17.1 in terms of x. (3) **L3**

17.2 Calculate the value of x for which the volume of the box will be at maximum. (5) **L2**

18 It the diagram below, Triangle ABC has a base of x metres. The base and the perpendicular height off the triangle add up to 10 metres. The triangle is mounted on a rectangle BCDE which has a perimeter of 32 metres.

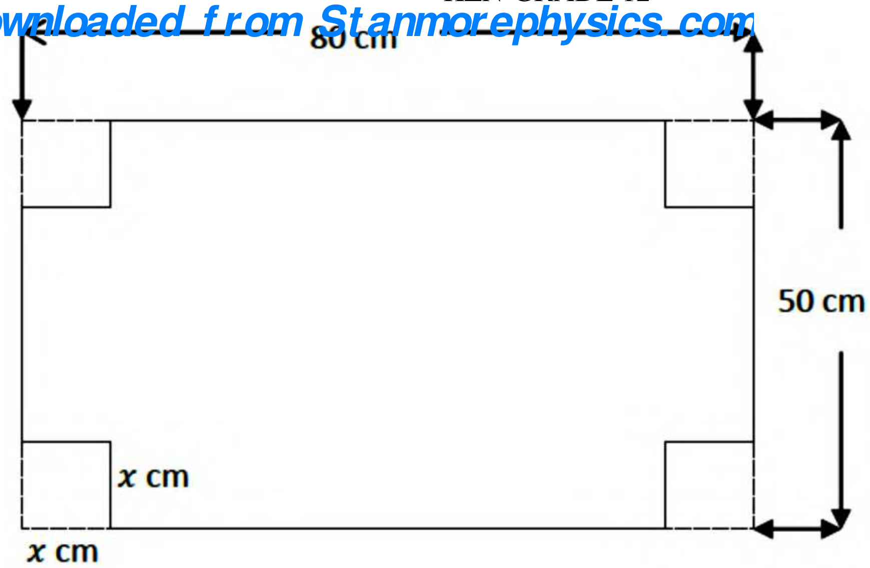


18.1 Show that the new area of the figure ABCDE is equal to $-\frac{3}{2}x^2+21x$ (5) **L2**

18.2 Determine the value of x for which ABCDE has a maximum area. (3) **L2**

18.3 Hence, determine the maximum area of ABCDE (2) **L2**

19 A piece of metal sheet, 80cm long and 50cm wide, is used to make a rectangular container without a lid. Squares of $x\ cm$ long are cut from the corners of the sheet for proper folding to make a height of $x\ cm$. The folded parts are then welded together to close the corners properly. The outside surfaces of the container are painted to decorate it.



Prove that the volume of the container is given by

19.1 $V(x)=4x^3-260x^2+4000x$ (3) **L4**

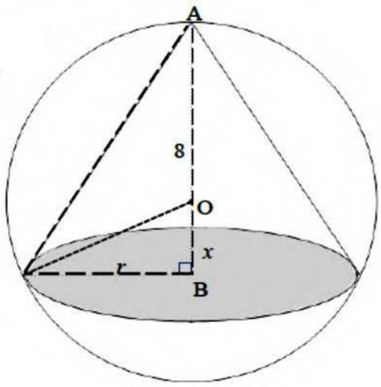
19.2 For which values of x will the volume be a maximum. (2) **L2**

19.3 Calculate the surface area that is to be painted. (3) **L2**

20 A cone with radius r cm and height AB is inscribed in a sphere with centre O and a radius of 8 cm. OB = x.

Volume of sphere = $\frac{4}{3}\pi r^3$

Volume of cone = $\frac{1}{3}\pi r^2 h$



20.1 Calculate the volume of the sphere. (1) **L1**

20.2 Show that $r^2=64-x^2$. (1) **L1**

20.3 Determine the ratio between the largest volume of this cone and the volume of the sphere. (7) **L3**

21 A stone is thrown upwards. Its height (in metres) above the ground at t seconds is given by $h(t)=-t^2+6t+16$

21.1 Determine the initial height of the stone above the ground. (1) **L1**

21.2 Determine the time taken to reach the maximum height (3) **L2**

21.3 How fast was the stone travelling when it hit the ground (4) **L2**

22 After flying a short distance an insect came to rest on a wall. Thereafter the insect started crawling on the wall. The path that the insect crawled **can be** described by

$h(t) = (t - 6)(-2t^2 + 3t - 6)$, where h is the height (in cm) above the floor and t is the time (in minute) since the insect started crawling .

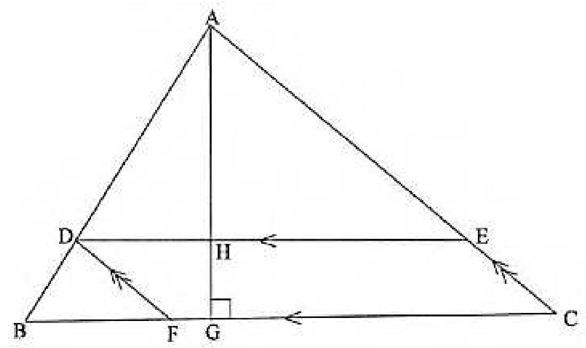
22.1 At what height above the floor did the insect start to crawl? (1) **L1**

22.2 How many times did the insect reach the floor? (3) **L2**

22.3 Determine the maximum height that the insect reached above the floor (4) **L2**

23 In ΔABC:

- D is a point on AB , E is a point on AC and F is a point on BC such that DECF is a parallelogram.
- BF:FC = 2:3
- The perpendicular height AG is drawn intersecting DE at H.
- AG =t units
- $BC = (5 - t)$ units.



23.1 Write down AH: HG (1) **L1**

23.2 Calculate t if the area of the parallelogram is a maximum.
Note: (Area of a parallelogram =base x perpendicular height (5) **L3**

TOPIC	PROBABILITY

GUIDELINES, SUMMARY NOTES, & STRATEGIES

- The probability scale: $0 \le P \le 1$. If P(an event) = 0, the event is impossible; If P(an event) = 1, the event is certain to happen.
- The **definition of probability**: $P(E) = \dfrac{n(E)}{n(S)}$
- **Addition Rule** for any 2 events A and B: **P(A or B) = P(A) + P(B)−P(A and B)**
- **Mutually exclusive events** A and B: **P(A or B) = P(A) + P(B)**
  NOTE: Since P(A and B) = 0
- **Independent events** A and B: **P(A and B) = P(A) × P(B)**
- **The complementary rule**: **P(not A) = 1−P(A)**
- **The fundamental counting principle**:
  If one operation can be done in m ways and a second operation can be done in n ways then the total possible number of different ways in which both operations can be done is $m \times n$.

REVISION QUESTIONS

1. **November 2008 Question 4**

A smoke detector system in a large warehouse uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0,95. The probability that it will be detected by device B is 0,98 and the probability that it will be detected by both devices simultaneously is 0,94.

1.1 If smoke is present, what is the probability that it will be detected by device A or device B or both devices? (3) **L2**

1.2 What is the probability that the smoke will not be detected? (1) **L1**

2. **March 2009 Question 4**

In a company there are three vacancies. The company had identified candidates to fill each post.

POST	CANDIDATES
Clerk	Craig, Luke and Tom
Sales representative	Ann, Sandile, Sizwe and Devon
Sales manager	John and Debby

2.1 In how many different ways can these three posts be filled? (3) **L1**

2.2 If it is certain that Craig will get the job as clerk, in how many different ways can the three posts be filled? (2) **L2**

3. **March 2012 Question 7**

Three items from four different departments of a major chain store will be featured in a one-page newspaper advertisement. The page layout for the advertisement is shown in the diagram below where one item will be placed in each block.

A	B	C
D	E	F
G	H	I
J	K	L

3.1 In how many different ways can all these items be arranged in the advertisement? (2) **L1**

3.2 In how many different ways can these items be arranged if specific items are to be placed in blocks A, F and J? (2) **L2**

3.3 In how many different ways can these items be arranged in the advertisement if items from the same department are grouped together in the same row? (3) **L2**

4. **March 2013 Question 5**

Consider the word: PRODUCT

4.1 How many different arrangements are possible if all the letters are used? (2) **L1**

4.2 How many different arrangements can be made if the first letter is T and the fifth letter is C? (2) **L1**

4.3 How many different arrangements can be made if the letters R, O and D must follow each other, in any order? (3) **L2**

5. **November 2014 Question 11**

A survey concerning their holiday preferences was done with 180 staff members. The options they could choose from were to:

- Go to the coast
- Visit a game park
- Stay at home

The results were recorded in the table below:

	Coast	**Game Park**	**Home**	**Total**
Male	46	24	13	83
Female	52	38	7	97
Total	98	62	20	180

5.1 Determine the probability that a randomly selected staff member:

5.1.1 Is male (1) **L1**

5.1.2 Does not prefer visiting a game park (2) **L1**

5.2 Are the events 'being a male' and 'staying at home' independent events. Motivate your answer with relevant calculations. (4) **L2**

6. **CAPS Exemplar 2014 Question 12**

Consider the word M A T H S.

6.1 How many different 5-letter arrangements can be made using all the above letters? (2) **L1**

6.2 Determine the probability that the letters S and T will always be the first two letters of the arrangements in question 6.1. (3) **L2**

7. **November 2015 Question 11.2**

The letters of the word DECIMAL are randomly arranged into a new 'word', also consisting of seven letters. How many different arrangements are possible if:

7.1 Letters may be repeated (2) **L1**

7.2 Letters may not be repeated (2) **L1**

7.3 The arrangements must start with a vowel and end in a consonant and no repetition of letters is allowed. (4) **L2**

November 2015 Question 11.3

8. There are t orange balls and 2 yellow balls in the bag. Craig randomly selects one ball from the bag, records his choice and returns the ball to the bag. He then randomly selects a second ball from the bag, records his choice and returns it to bag. It is known that the probability that Craig will select two balls of the same colour from the bag is 52%.

Calculate how many orange balls are in the bag. (6) **L3**

9. **November 2016 Question 12**

The digits 1 to 7 are used to create a four-digit code to enter a locked room. How many different codes are possible if the digits may not be repeated and the code must be an even number bigger than 5000? (5) **L3**

10. **March 2016 Question 10**

Each passenger on a certain Banana Airways flight chose exactly one beverage from tea, coffee or fruit juice. The results are shown in the table below.

	MALE	FEMALE	TOTAL
Tea	20	40	60
Coffee	***b***	***c***	80
Fruit juice	***d***	***e***	20
TOTAL	60	100	***a***

10.1 Write down the value of $\boldsymbol{a}$. (1) **L1**

10.2 What is the probability that a randomly selected passenger is male? (2) **L1**

10.3 Given that the event of a passenger choosing coffee is independent of being a male, calculate the value of $\boldsymbol{b}$. (4) **L2**

11. **June 2016 Question 11**

Five boys and four girls go to the movies. They are all seated next to each other in the same row.

11.1 One boy and girl are a couple and want to sit next to each other at any end of the row of friends. In how many different ways can the entire group be seated? (3) **L2**

11.2 If all the friends are seated randomly, calculate the probability that all the girls are seated next to each other. (3) **L3**

12. **November 2016 Question 11**

A survey was conducted among 100 boys and 60 girls to determine how many of them watched TV in the period during which examinations were written. Their responses are shown in the partially completed table below.

	WATCHED TV DURING EXAMINATIONS	DID NOT WATCH TV DURINGEXAMINATIONS	TOTALS
Male	**80**	***a***	
Female	**48**	**12**	
Totals	***b***	**32**	**160**

12.1 Calculate the values of a and b. (2) **L1**

12.2 Are the events 'being a male' and 'did not watch TV during examinations' mutually exclusive? Give a reason for your answer. (2) **L2**

12.3 If a learner who participated in this survey is chosen at random, what is the probability that the learner:

12.3.1 Watched TV in the period during which the examinations were written? (2) **L2**

12.3.2 Is not a male and did not watch TV in the period during which examinations were written? (2) **L2**

13. **June 2018 Question 10**

Ben, Nhlanhla, Owen, Derick and 6 other athletes take part in a 100 m race. Each athlete will be allocated a lane in which to run. The athletic track has 10 lanes.

13.1 In how many different ways can all the athletes be allocated a lane? (2) **L1**

13.2 Four athletes taking part in the event insist on being placed in lanes next to each other. In how many different ways can the lanes be allocated to the athletes now? (3) **L2**

13.3 If lanes are randomly allocated to athletes, determine the probability that Ben will be placed in lane 1, Nhlanhla in lane 3, Owen in lane 5 and Derick in lane 7. (2) **L2**

14. **November 2018 Question 12**

Given: $P(A) = 0{,}45$; $P(B) = y$ and $P(A \text{ or } B) = 0{,}74$.

Determine the value(s) of y if A and B are mutually exclusive. (3) **L2**

15. **June 2017 Question 11.3**

Grade 12 learners in a certain town may choose to attend any one of three high schools. The table below shows the number of Grade 12 learners (as a percentage) attending the different schools in 2016 and the matric pass rate in that school (as a percentage) in 2016.

SCHOOLS	NUMBER OF LEARNERS ATTENDING (%)	MATRIC PASS RATE (%)
A	20	35
B	30	65
C	50	90

If a learner from this town, who was in Grade 12 in 2016, is selected at random, determine the probability that the learner:

15.1 Did not attend School A (2) **L1**

15.2 Attended School B and failed Grade 12 in 2016 (3) **L2**

15.3 Passed Grade 12 in 2016 (4) **L2**

16. March 2018 Question 11

Veli and Bongi are learners at the same school. Some days they arrive late at school. The probability that neither Veli nor Bongi will arrive late on any day is 0,7.

16.1 Calculate the probability that at least one of the two learners will arrive late on a randomly selected day. (1) **L2**

16.2 The probability that Veli arrives late for school on a randomly selected day is 0,25, while the probability that both of them arrive late for school on that day is 0,15. Calculate the probability that Bongi will arrive late for school on that day. (3) **L2**

16.3 The principal suspects that the latecoming of the two learners is linked. The principal asks you to determine whether the events of Veli arriving late for school and Bongi arriving late for school are statistically independent or not. What will be your response to him? Show ALL calculations. (3) **L3**

17. November 2018 Question 12

An organisation decided to distribute gift bags of sweets to a Grade R class at a certain school. There is a mystery gift in exactly $\frac{1}{4}$ of the total number of bags.

Each learner in the class may randomly select two gift bags of sweets, one after the other. The probability that a learner selects two bags of sweets with a mystery gift is.

Calculate the number of gift bags of sweets with a mystery gift inside. (6) **L4**

18. June 2019 question 11

Two learners from each grade at a high school (Grades 8, 9, 10, 11 and 12) are elected to form a sports committee.

18.1 In how many different ways can the chairperson and the deputy chairperson of the sports committee be elected if there is no restriction on who may be elected? (2) **L1**

18.2 A photographer wants to take a photograph of the sports committee. In how many different ways can the members be arranged in a straight line if:

18.2.1 Any member may stand in any position? (1) **L1**

18.2.2 Members from the same grade must stand next to each other and the Grade 12 members must be in the centre? (3) **L3**

19. November 2019 question 11.1

Events A and B are independent. P(A) = 0,4 and P(B) = 0,25.

19.1 Represent the given information on a Venn diagram. Indicate on the Venn diagram the probabilities associated with each region. (3) **L3**

19.2 Determine P[A or (not B)]. (2) **L2**

20. November 2020 question 11

Harry shoots arrows at a target board. He has a 50% chance of hitting the bull's eye on each shot.

20.1 Calculate the probability that Harry will hit the bull's eye in his first shot and his second shot. (2) **L2**

20.2 Calculate the probability that Harry will hit the bull's eye at least twice in his first three shots. (3) **L3**

20.3 Glenda also has a 50% chance of hitting the bull's eye on each shot. Harry and Glenda will take turns to shoot an arrow and the first person to hit the bull's eye will be the winner. Calculate the probability that the person who shoots first will be the winner of the challenge. (3) **L4**

21. June 2021 question 11.3

A three-digit number is made up by using three randomly selected digits from 0 to 9. No digit may be repeated.

21.1 Determine the total number of possible three-digit numbers, greater than 100, that can be formed. (2) **L2**

21.2 Determine the total number of possible three-digit numbers, both even and greater than 600, that can be formed. (4) **L2**

22. June 2022 question 10.3

There are 120 passengers on board an aeroplane. Passengers have a choice between a meat sandwich or a cheese sandwich, but more passengers will choose a meat sandwich. There are only 120 sandwiches available to choose from. The probability that the first passenger chooses a meat sandwich and the second passenger chooses a cheese sandwich is $\frac{18}{85}$. Calculate the probability that the first passenger will choose a cheese sandwich. (5) **L4**

23. November 2010 question 5

In Gauteng number plates are designed with 3 alphabetical letters, excluding the 5 vowels, next to one another and then any 3 digits, from 0 to 9, next to one another.

The GP is constant in all Gauteng number plates, for example TTT 012 GP.

Letters and digits may be repeated on a number plate.

23.1 How many unique number plates are available? (3) **L2**

23.2 What is the probability that a car's number plate will start with a Y? (3) **L2**

23.3 What is the probability that a car's number plate will contain only one 7? (3) **L3**

23.4 How many unique number plates will be available if the letters and numbers are not repeated? (3) **L2**

24.

There are 15 girls in a mixed class. If two learners from the class are selected at random to represent the class on the RCL, the probability that both will be girls is 0,35. How many boys are there in the class? (5) **L4**

25. **Gr. 11 November 2017 question 9**

A survey was done among 80 learners on their favourite sport. The results are shown below:

- 52 learners like rugby (R)
- 42 learners like volleyball (V)
- 5 learners like chess only (C)
- 14 learners like rugby and volleyball, but not chess
- 12 learners like rugby and chess, but not volleyball
- 15 learners like volleyball and chess, but not rugby
- x like all three types of sport
- 3 learners do not like any sport

25.1 Draw a Venn diagram to represent the information above. (5) **L2**

25.2 Show that $x = 8$. (2) **L2**

25.3 How many learners like only rugby? (1) **L1**

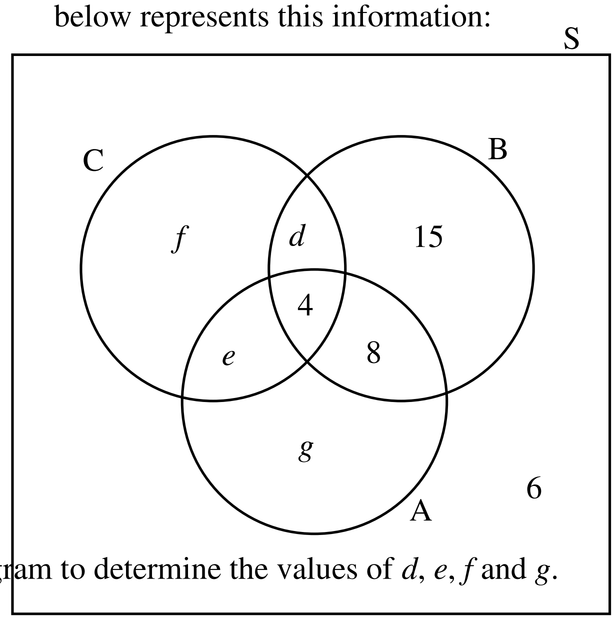
25.4 Calculate the probability that a learner, chosen randomly, likes at least two different types of sport. (3) **L3**

26. **March 2015 question 10**

Research was conducted about driving under the influence of alcohol. Information obtained from traffic authorities in 54 countries about the methods used to measure alcohol levels in a person are summarised below:

- 4 countries use all three methods (A, B and C.)
- 12 countries use the alcohol content of breath (A) and blood-alcohol concentration (B).
- 9 countries use blood-alcohol concentration (B) and certificates issued by doctors (C)
- 8 countries use A and C
- 21 countries use A
- 32 countries use B
- 20 countries use C
- 6 countries use none of these methods.

The partially completed Venn diagram below represents this information:



26.1 Use the given information and the Venn diagram to determine the values of d, e, f and g. (4) **L2**

26.2 For a randomly selected country, calculate:

26.2.1 P(A and B and C) (1) **L1**

26.2.2 P(A or B or C) (1) **L1**

26.2.3 P(only C) (1) **L1**

26.2.4 P(that a country uses exactly two methods) (1) **L2**

27.

Determine the probablity of getting at least one six when rolling a six-sided dice three times. (4) **L4**

STATISTICS/DATA HANDLING
GUIDELINES, SUMMARY NOTES, & STRATEGIES

Definition:
Data Handling is a process during which data (information) is collected, recorded, and presented.

Terminology:

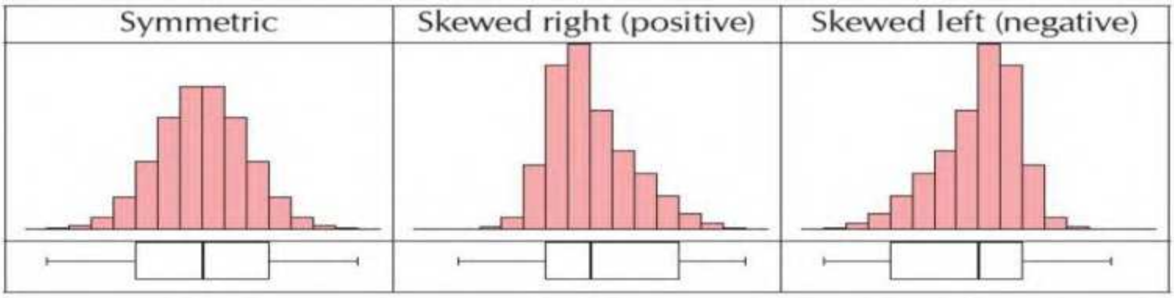
- Data – information that is being analysed.
  - **Population** – data is collected on the entire group of elements.
  - **Sample** – data is collected on a specified set from a larger group of elements.
  - **Ungrouped data** – a set of random data elements gathered for analysis.
  - **Grouped data** – data elements aggregated into different classes, groups, or intervals.
  - **Univariate data** – single set of data that distinguished by specific characteristics.
  - **Bivariate data** – data set that compares two related variables.
- Measures of central tendency – single numbers around which all data items seem to be spread.
  - The **Mean**, also known as the average, is the sum of all the data values in a set, divided by number of all elements in the set i.e $\bar{x} = \dfrac{\sum f x}{n}$ or $\bar{x}_{est} = \dfrac{\sum f m}{\sum f}$; where f is the frequency and m is the midpoint of a class interval.
  - The **Median,** (Q_2) is the most middle data item in an ordered data set.

    Position of median $= \dfrac{1}{2}(n+1)$
  - The **Mode** is the most frequent data item in a set. In grouped data, the modal group will have the highest frequency. Data sets may have no mode, two modes (bimodal), three modes (trimodal), etc.
- Measures of dispersion – numbers that describe the spread of the data.
  - The **Range** is the difference between the maximum and the minimum data values in a given data set.
  - The **Inter-Quartile-Range** (IQR) is the difference between the third and first quartiles, i.e. $IQR = Q_3 - Q_1$
  - **Standard Deviation** (σ) is a measure of how dispersed data is around the mean. The square of the standard deviation is the **variance** (σ^2).
- Quartiles – numbers that divide data into quarters in an ordered data set.
  - **Lower quartile**, (Q_1), is a data item below which a quarter of the data lies in an ordered data set.

    Position of lower quartile $\dfrac{1}{4}(n+1)$
  - **Upper quartile**, (Q_3) is a data item above which a quarter of the data lies in an ordered data set.

    Position of upper quartile $= \dfrac{3}{4}(n+1)$
- Percentiles – numbers below which a certain percentage of data item lies in an ordered data set.
  - Position of percentile $= \dfrac{percentile}{100} \times number\ \ of\ \ data\ \ items\ \ in\ \ a\ \ set$
- Five Number Summary – five numbers that separate a data set into quarters.
  - Minimum value
  - Lower quartile (Q_1)
  - Median (Q_2)
  - Upper quartile (Q_3)
  - Maximum value

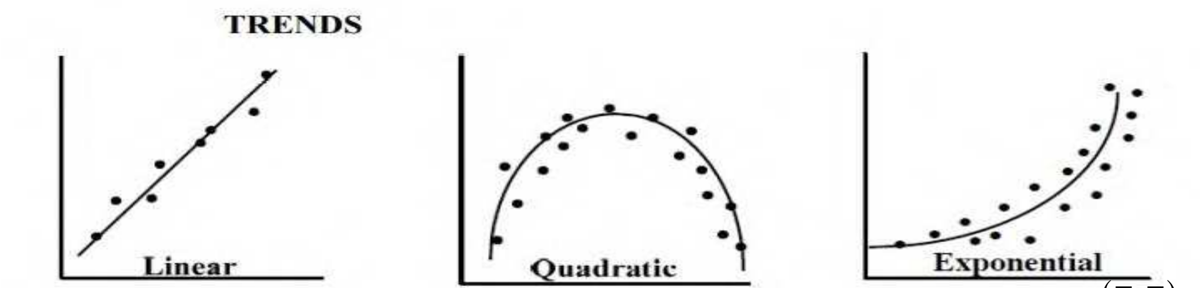
- Box – and – Whisker Diagram (drawn using the five number summary)
  - ❖ It is important in analysing the distribution of data in a given set.
  - ❖ If mean – median = 0, then the distribution is symmetric.
  - ❖ If mean – median > 0, then the distribution is positively skewed.
  - ❖ If mean – median < 0, then the distribution is negatively skewed.



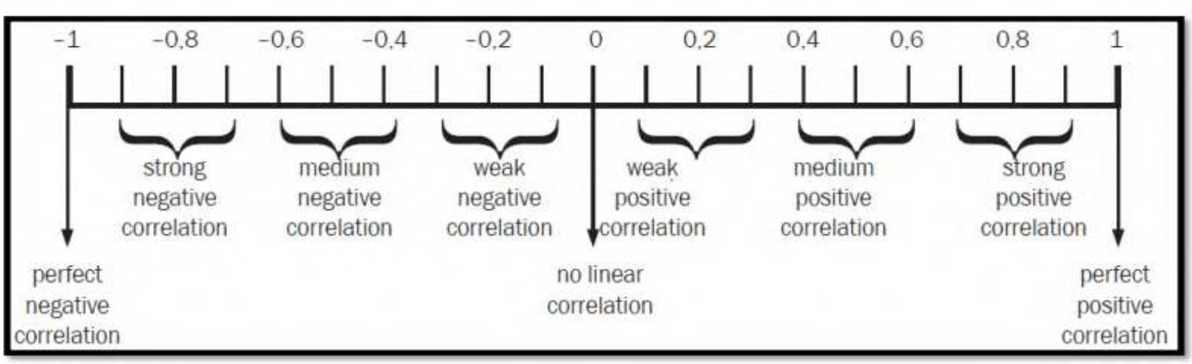
  - ❖ In a symmetrical data set approximately 68% of the data will fall within one standard deviation of the mean $[\overline{x}-\sigma;\overline{x}+\sigma]$ and approximately 95% of the data will lie within two standard
  - ❖ deviations of the mean $[\overline{x}-2\sigma;\overline{x}+2\sigma]$
- Outliers – data items that are a lot bigger or smaller than the rest of the elements in the data set.

They are determined as follows:

  - ❖ Lower outliers are numbers $< Q_1 - 1.5 \times IQR$
  - ❖ Upper outliers are numbers $> Q_1 + 1.5 \times IQR$
- Graphical representations
  - ❖ **Histogram** – represents grouped data as condensed bars whose widths and lengths represent class intervals and frequency respectively.
  - ❖ **Ogive (Cumulative Frequency Curve)** – an *s*-shaped smooth curve drawn by plotting upper limits of class intervals of a grouped data against cumulative frequency of a set.
  - ❖ **Scatter plot** – representation of bivariate data as discrete data points.
- Bivariate data summaries
  - ❖ **Regression line (line of best fit) -** a line drawn on the scatter plot that shows a general trend that bivariate data seem to follow.



  - ❖ **Least squares regression line** – is a straight line that passes through the mean point $(\overline{x};\overline{y})$ relating bivariate data.
  - ❖ **Corelation Coefficient (r)** – indicates the strength of the relationship between the variables in bivariate data. It lies between −1 and 1.



REVISION QUESTIONS

1 **February/March 2014 Question 1**

The tuck shop at Great Future High School sells cans of soft drinks. The Environmental Club at the school decided to have a can-collection project for three weeks to make learners aware of the effects of litter on the environment.

The data below shows the number of cans collected on each school day of the three-week project.

58 83 85 89 94 97 98 100 105 109 112 113 114 120 145

1.1 Calculate the mean number of cans collected over the three-week period. (2) **L1**

1.2 Calculate the standard deviation. (2) **L2**

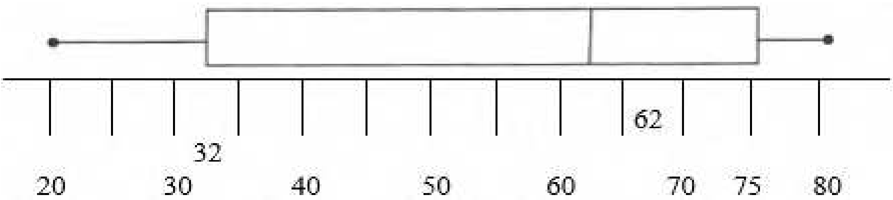
1.3 Determine the lower and upper quartiles of the data. (2) **L2**

1.4 Draw a box and whisker diagram to represent the data. (3) **L2**

1.5 On how many days did the number of cans collected lie outside ONE standard deviation of the mean? (3) **L2**

2 **February/March 2016 Question 1**

The box and whisker diagram below shows the marks (out of 80) obtained in a History test by a class of nine learners.



2.1 Comment on the skewness of the data. (1) **L1**

2.2 Write down the range of the marks obtained. (2) **L1**

2.3 If the learners had to obtain 32 marks to pass the test, estimate the percentage of the class that failed the test. (2) **L2**

2.4 In ascending order, the second mark is 28, the third mark is 36 and the sixth mark is 69. The seventh and the eighth marks are the same. The average mark for this test is 54.

	28	36			69			

Fill in the marks of the remaining learners in ascending order. (6) **L3**

3 **May/June 2022 Question 1**

The table below shows the mass (in kg) of the school bag of 80 learners.

MASS (kg)	**FREQUENCY**	**CUMULATIVE FREQUENCY**
$5 < m \leq 7$	6	
$7 < m \leq 9$	18	
$9 < m \leq 11$	21	
$11 < m \leq 13$	19	
$13 < m \leq 15$	11	
$15 < m \leq 17$	4	
$17 < m \leq 19$	1	

3.1 Write down the modal class of the data. (1) **L1**

3.2 Complete the cumulative frequency column in the table (2) **L1**

3.3 Draw a cumulative frequency graph (ogive) for the given data. (3) **L2**

3.4 Use the graph to determine the median mass for this data. (2) **L2**

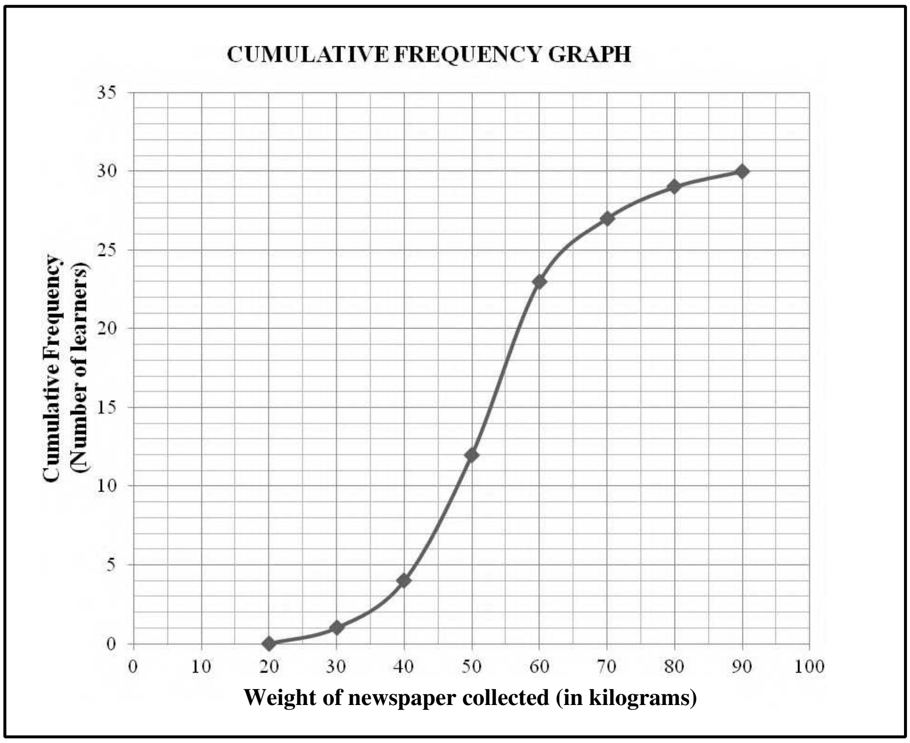
3.5 The international guideline for the mass of a school bag is that it should not exceed 10% of a learner's body mass.

3.5.1 Calculate the estimated mean mass of the school bag (2) **L2**

3.5.2 The mean mass of this group of learners was found to be 80kg. On average, are these school bags satisfying the international guideline with regards to mass? Motivate your answer. (2) **L3**

4 **November 2012 Question 4**

As part of an environmental awareness initiative, learners of Greenside High School were requested to collect newspapers for recycling. The cumulative frequency graph (ogive) below shows the total weight of the newspapers (in kilograms) collected over a period of 6 months by 30 learners.



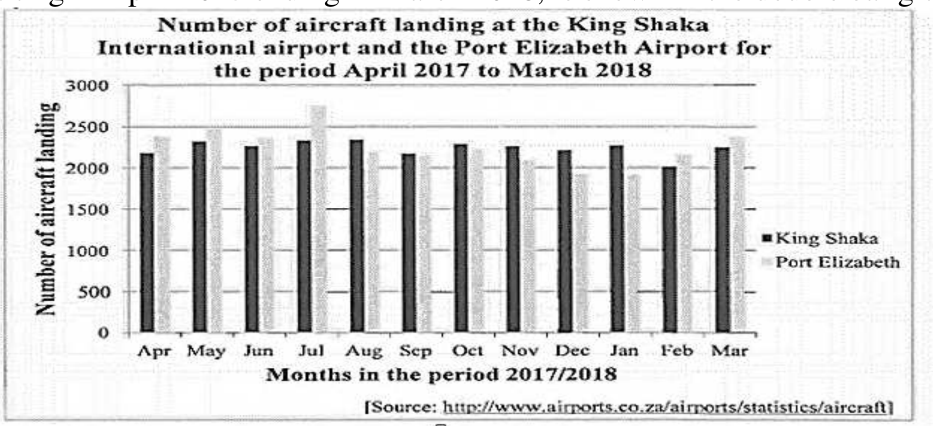
4.1 Determine the modal class of the weight of the newspapers collected (1) **L1**

4.2 Determine the median weight of the newspapers collected by this group of learners. (1) **L1**

4.3 How many learners collected more than 60 kilograms of newspaper? (2) **L2**

5 **November 2020 Question 2**

The number of aircraft landing at the King Shaka International and the Port Elizabeth Airport for the period starting in April 2017 ending in March 2018, is shown in the double bar graph below.



5.1 The number of aircraft landing at the Port Elizabeth Airport exceeds the number of aircraft landing at the King Shaka International Airport during some months of the given period. During Which month is this difference the greatest? (1) **L1**

5.2 The number of aircraft landing at the King Shaka International Airport during these months are given below. Calculate the mean for the data.

2 182	2 323	2 267	2 334	2 346	2 175
2 293	2 263	2 215	2 271	2018	2 254

(2) **L2**

5.3 Calculate the standard deviation for the number of aircraft landing at the King Shaka International Airport for the given period. (2) **L2**

5.4 Determine the number of months in which the number of aircraft landing at the King Shaka International Airport were within one standard deviation of the mean. (3) **L2**

5.5 Which one of the following statements is CORRECT?

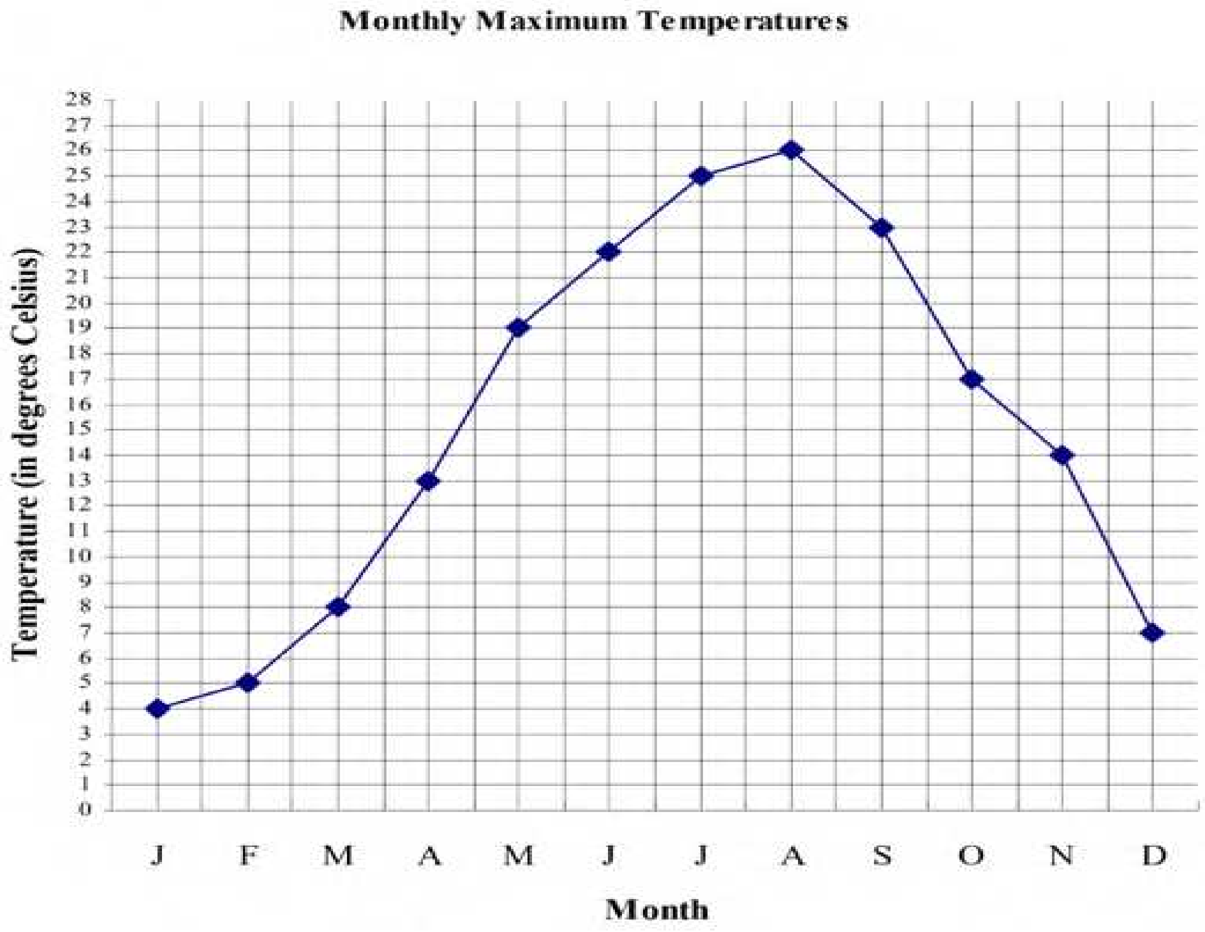
a) During December and January, there were more landings at the Port Elizabeth Airport than at the King Shaka International Airport.

b) There was a greater variation in the number of aircraft landings at the King Shaka International than at the Port Elizabeth for the given period.

c) The standard deviation of the number of landings at the Port Elizabeth Airport will be higher than the standard deviation of the number of landings at the King Shaka International Airport (1) **L3**

6 **February/March 2010 Question 1**

The graph below shows the monthly maximum temperatures in a certain city.



6.1 What is the range of the monthly maximum temperatures? (2) **L1**

6.2 Calculate the mean monthly maximum temperature. (3) **L1**

6.3 Calculate the standard deviation of the monthly maximum temperature. (2) **L2**

6.4 It is predicted that one hundred years from now, global warming is likely to increase the city's monthly maximum temperature by 5° C in December, January and February. It will also result in an increase of 1° C in the other months of the year.

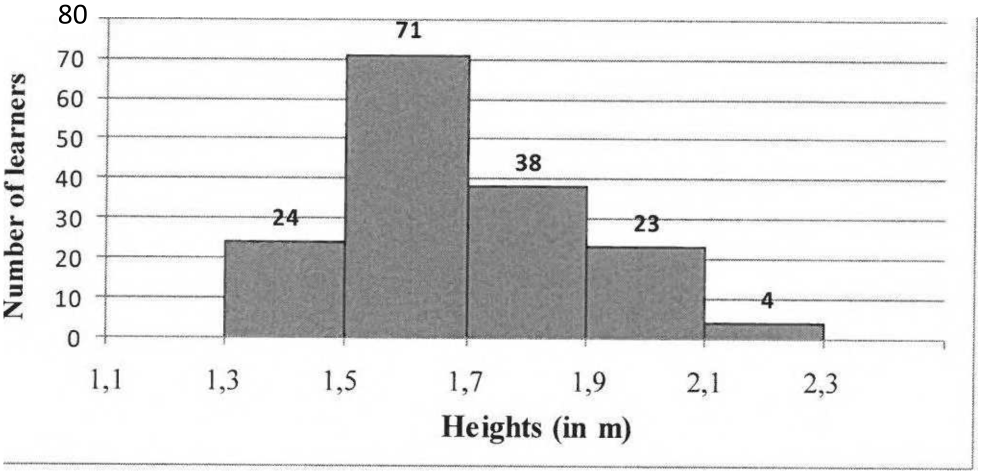
6.4.1 By how much does the mean increase? (2) **L3**

6.4.2 Describe the effect that the predicted increases in temperature will have on the standard deviation. Justify your answer. (2) **L4**

7 **November 2016 Question 2**

The heights of 160 learners in a school are measured. The height of the shortest learner is 1,39 m and the height of the tallest learner is 2,21 m. The heights are represented in the histogram below.

Histogram



7.1 Describe the skewness of the data. (1) **L1**

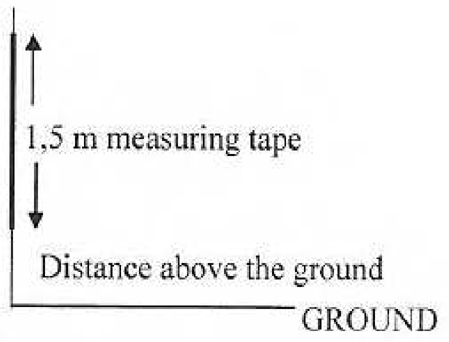
7.2 Calculate the range of the heights. (2) **L1**

7.3 Draw and complete a cumulative frequency table. (2) **L2**

7.4 Draw an ogive (cumulative frequency curve) to represent the data. (4) **L2**

7.5 Eighty learners are less than x metres in height. Estimate x. (2) **L2**

7.6 The person taking the measurements only had 1,5 m measuring tape available. In order to compensate for the short measuring tape, he decided to mount the tape on a wall at a height of 1m above the ground. After recording the measurements, he discovered that the tape was mounted at 1,1 m above the ground instead of 1 m. How does this error influence the following?



7.6.1 Mean of the data set. (1) **L3**

7.6.2 Standard deviation of the data set. (1) **L4**

8 **November 2015 Question 2**

A group of 30 learners each randomly rolled two dice once and the sum of the values on the uppermost faces of the dice was recorded. The data is shown in the frequency table below.

Sum of the values on uppermost faces	Frequency
2	0
3	3
4	2
5	4
6	4
7	8
8	3
9	2
10	2
11	1
12	1

8.1 Calculate the mean of the data. (2) **L2**

8.2 Determine the median of the data. (2) **L2**

8.3 Determine the standard deviation of the data. (2) **L2**

8.4 Determine the number of times that the sum of the recorded values of the dice is within ONE standard deviation from the mean. Show your calculations. (3) **L2**

9 **FS/September 2020 Question 1**

The table below gives the average exchange rate and the average monthly oil price for the year 2010.

	Jan	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sept**	**Oct**	**Nov**	**Dec**
Exchange rate in R/S	**7.5**	**7.7**	**7.2**	**7.4**	**7.7**	**7.7**	**7.6**	**7.3**	**7.1**	**7.0**	**6.9**	**6.8**
Oil price in $	**69.9**	**68.0**	**72.9**	**70.3**	**66.3**	**67.1**	**67.9**	**68.3**	**71.3**	**73.6**	**76.0**	**81.0**

9.1 Draw a scatterplot to represent the exchange rate (in R/S) versus the oil price (in $). (3) **L2**

9.2 Determine the equation of the least square regression line. (3) **L2**

9.3 Calculate the value of the correlation coefficient. (1) **L1**

9.4 Comment on the strength of the relationship between the exchange rate (in R/S) and the oil price (in $). (2) **L1**

9.5 Determine the mean oil price. (1) **L2**

9.6 Determine the standard deviation of the oil price. (1) **L2**

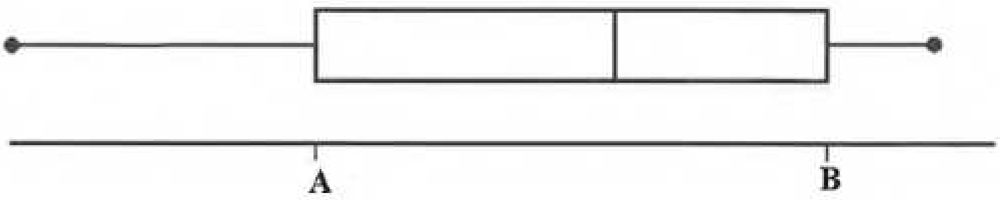
9.7 Generally, there is a concern from the public when the oil price is higher than two standard deviations from the mean. In which months would the public have been concerned? (2) **L3**

10 **February/March 2018 Question 1.2 – 1.3**

An organisation decided that it would set up blood donor clinics at various colleges. Students would donate blood over a period of 10 days. The number of units of blood donated per day by students of college X is shown in the table below.

DAYS	1	2	3	4	5	6	7	8	9	10
UNITS OF BLOOD	45	59	65	73	79	82	91	99	101	106

The number of units of blood donated by students of college X is represented in the box and whisker diagram below.



10.1 Describe the skewness of the data. (1) **L1**

10.2 Write down the values of **A** and **B**, the lower quartile and the upper quartile of the data, respectively. (2) **L2**

10.3 It was discovered that there was an error in counting the number of units of blood donated by college X each day. The correct mean of the data is 95 units of blood. How many units of blood were NOT counted over the ten days? (1) **L4**

11 **February/March 2012 P3 Question 2**

A large company employs several people. The table below shows the number of people employed in each position and the monthly salary paid to each person in that position.

POSITION	**NUMBER EMPLOYED IN POSITION**	**MONTHLY SALARY PER PERSON (IN RAND)**
Managing director	1	150 000
Director	2	100 000
Manager	2	75 000
Foreman	5	15 000
Skilled workers	30	10 000
Semi-skilled workers	40	7 500
Unskilled workers	65	6 000
Administration	5	5 000

11.1 Calculate the total number of people employed at this company. (1) **L1**

11.2 Calculate the total amount needed to pay salaries for ONE month. (2) **L1**

11.3 Determine the mean monthly salary for an employee in this company. (2) **L1**

11.4 Is the mean monthly salary calculated in QUESTION 2.3 a good indicator of an employee's monthly salary? Motivate your answer. (2) **L2**

12 **February/March 2013 P3 Question 3**

12.1 The height of each learner in a class was measured and it was found that the mean height of the class was 1,6 m. At the time, three learners were absent. However, when the heights of the learners who were absent were included in the data for the class, the mean height did not change. If the heights of two of the learners who were absent are 1,45 m and 1,63 m, calculate the height of the third learner who was absent (3) **L4**

12.2 There are 184 students taking Mathematics in a first-year university class. The marks, out of 100, in the half-yearly examination are normally distributed with a mean of 72 and a standard deviation of 9.

12.2.1 What percentage of students scored between 72 and 90 marks? (2) **L3**

12.2.2 Approximately how many students scored between 45 and 63 marks? (3) **L3**

13 Consider the following set of four ordered positive whole numbers and their frequency.

Scores	$x+3$	$2x$	$x-1$	6
Frequency	4	3	2	2

13.1 Determine the median score. (1) **L1**

13.2 Determine the mean in terms of x. (3) **L1**

13.3 If only the scores are taken into consideration (without the frequency), determine the standard deviation if it is given that $x = 5$. (2) **L2**

14 **February/March 2014 Question 3**

The scatter plot below shows the age and the time taken for each of the first ten swimmers of a swimming club to complete an open water swimming event. The time taken is rounded to the nearest half-minute.

14.1 Write down the coordinates of an outlier in the scatter plot. (1) **L1**

14.2 Which of the following functions will best fit the data: linear, quadratic or exponential? (1) **L1**

14.3 Give an explanation for the trend observed in this set of data. (2) **L2**

14.4 If the two worst (longest) times are disregarded from the set of data, how will this affect the following:

14.4.1 The standard deviation of the original set of data. (1) **L2**

14.4.2 The mean of the original set of data. (1) **L2**

15 A group of learners from Mr Smith's class wrote a Mathematics test which was scored out of 75 marks. The results were represented in the table below.

MARKS	**FREQUENCY**	**CUMULATIVE FREQUENCY**
$5 < x \leq 15$	3	
$15 < x \leq 25$	6	
$25 < x \leq 35$	m	21
$35 < x \leq 45$	4	
$45 < x \leq 55$	7	
$55 < x \leq 65$	9	
$65 < x \leq 75$	n	
Total	**51**	

15.1 How many learners wrote the test? (1) **L1**

15.2 Determine the value of m and n. (2) **L2**

15.3 Complete the given table on the diagram sheet. (2) **L1**

15.4 Draw a cumulative frequency curve (Ogive) to represent above data (3) **L2**

15.5 Hence, or otherwise estimate the value of the median for the above data (2) **L2**

16 **Feb./March 2012 Question 4**

In the grid below a, b, c, d, e, f and g represent values in a data set written in an increasing order. No value in the data set is repeated.

a	***b***	***c***	***d***	***e***	***f***	***g***

Determine the value of a, b, c, d, e, f and g if:

- The maximum value is 42
- The range is 35
- The median is 23
- The difference between the median and the upper quartile is 14
- The interquartile range is 22
- $e = 2c$
- The mean is 25

(7) **L3**

17 **Feb./March 2009 P2 Question 12**

A motor company did research on how the speed of a car affects the fuel consumption of the vehicle. The following data was obtained:

Speed in km/h	60	75	115	85	110	95	120	100	70
Fuel consumption in ℓ/100 km	11,5	10	8,4	9,2	7,8	8,9	8,8	8,6	10,2

17.1 Represent the data as a scatter plot. (3) **L2**

17.2 Suggest whether a linear, quadratic or exponential function would best fit the data. (1) **L1**

17.3 What advice can the company give about the driving speed in order to keep the cost of fuel to a minimum? (2) **L2**

TOPIC ANALYTICAL GEOMETRY
GUIDELINES, SUMMARY NOTES, & STRATEGIES

DISTANCE BETWEEN TWO POINTS

The distance formula can be used to determine the length of a line segment between two points or the coordinates of a point when the length is known.

The formula to calculate the length of a line segment between two points $A(x_A; y_A)$ and $B(x_B; y_B)$ is given by the formula:

$$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2 \text{ or}$$

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

MIDPOINT OF A LINE SEGMENT

The formula for point M, the midpoint of a line segment AB joining the points $A(x_A; y_A)$ and $B(x_B; y_B)$ is given by the formula:

$$M(x_M; y_M) = M\left(\frac{x_B + x_A}{2}; \frac{y_B + y_A}{2}\right)$$

GRADIENT OF A LINE

The gradient of a line between any two points on the line is the ratio:

$$m = \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$$

A formula to calculate the gradient of a line joining two points $A(x_A; y_A)$ and $B(x_B; y_B)$ is given by the formula:

The gradient of line AB: $m_{AB} = \dfrac{y_B - y_A}{x_B - x_A}$

VERTICAL LINES

The vertical line always cuts through the *x*-axis. It is parallel to the *y*- axis and perpendicular to the *x*-axis. The equation of a line cutting the *x* axis at *a:*

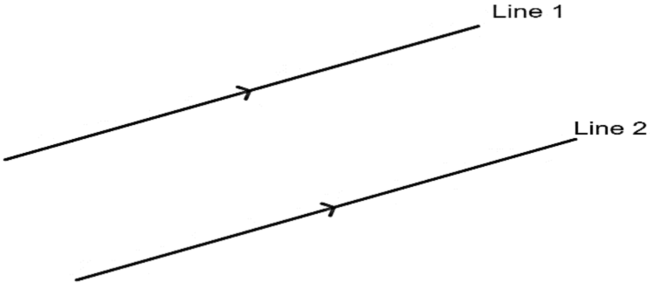
$x = a$

HORIZONTAL LINES

The horizontal line cuts through the *y*- axis. The line is parallel to the *x*-axis. It is parallel to the *x*-axis and perpendicular to the *y*- axis. The equation of a line cutting through the y axis at b:

$x = b$

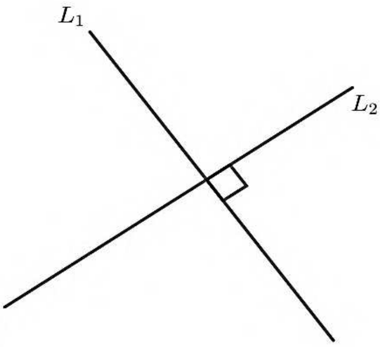
PARALLEL LINES



If $L_1 \parallel L_2$ then $m_1 = m_2$

PERPENDICULAR LINES

If $L_1 \perp L_2$ then $m_1 \times m_2 = -1$

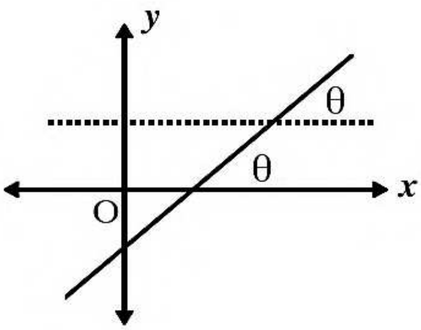


COLLINEAR POINTS

Points that are **collinear** lie on the same line. The gradient between each pair of points is the same. For example, if the points A, B and C are collinear, then:
$\text{Gradient}_{AB} = \text{Gradient}_{BC} = \text{Gradient}_{AC}$

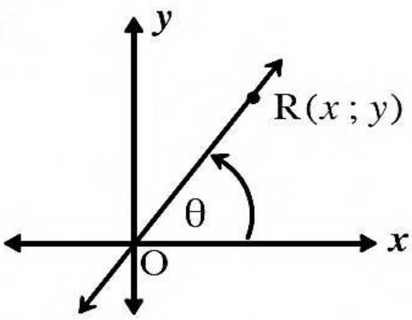
INCLINATION OF A LINE

The inclination of a line is the angle formed with the horizontal in an anti-clockwise direction. On the Cartesian plane, the inclination of a line is calculated by finding the **angle formed at the x-axis** measured in anti-clockwise direction. θ is the angle of inclination of line AB.



Formula for finding the angle of inclination of a line

If $R(x; y)$ is a point on the terminal arm of θ, then by definition, $\tan\theta = \frac{y}{x}$. But with $O(0;0)$, the gradient of line OR $= \frac{y-0}{x-0} = \frac{y}{x}$.



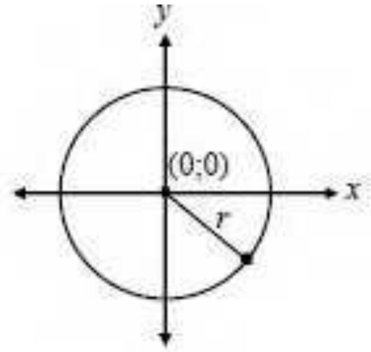
$$\boxed{\tan\theta = \text{Gradient}_{OR}\text{, where } \theta \text{ is the angle of inclination of line OR.}}$$

EQUATION OF A CIRCLE

✓ **Key notes:**

- The diameter is twice the radius
- The radius is the same throughout the circle.
- The tangent is perpendicular to the radius
- A normal is a line perpendicular to the tangent at the point of contact – the normal is not the radius but can go through the circle or be outside the circle.
- A secant cuts the circle twice.
- A chord touches the circle twice internally and divides the circle into segments
- A sector is the middle piece between two radii.
- A chord divides a circle's circumference into different arcs.
- A circumference is the distance around the circle.

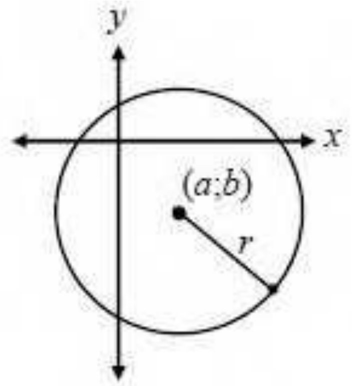
CIRCLE WITH CENTRE AT THE ORIGIN



$$r^2 = x^2 + y^2$$

- This formula should remind you of Pythagoras.
- r is the radius and x and y is the coordinate at a point through the circle.

CIRCLE WITH ANY CENTRE



$$r^2 = (x-a)^2 + (y-b)^2$$

- Essentially a circle with any centre is simply a circle with a centre at the origin that has been shifted left or right and up or down.
- r is the radius
- a is the x-coordinate of the centre
- b is the y-coordinate of the centre.

THE EQUATION OF A CIRCLE IS NOT GIVEN IN THE FORM

$$r^2 = (x-a)^2 + (y-b)^2$$

We need to be able to complete the square in order to find the co-ordinates of the centre of the circle as well as the length of the radius.

❖ Step 1: Rewrite the equation: The x and y terms are written separately and the constant term is moved to the right hand side of the equation.

❖ Step 2: Halve the co-efficient of x and add the square of the result on both sides of the equation. Repeat the same process for y.

❖ Step 3: Factorise

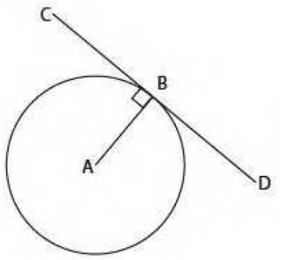
EQUATION OF THE TANGENT TO THE CIRCLE

✓ A tangent is a straight line that is drawn perpendicular to the circle's radius and touching the circle at only one point.

✓ To work out the equation of the tangent use the straight line formula:
$y = mx + c$

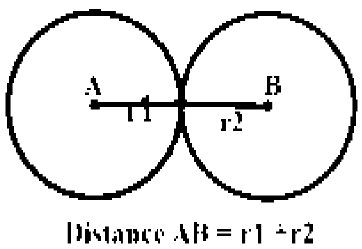
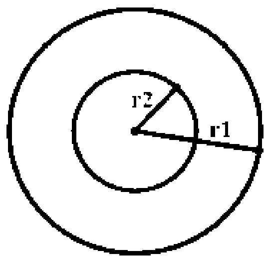
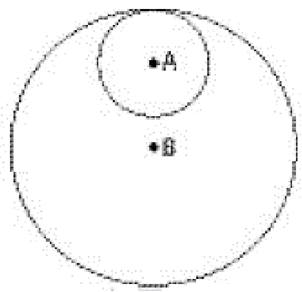
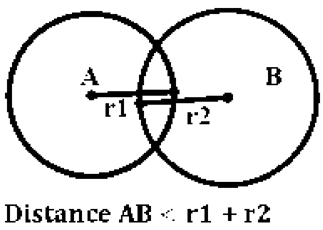
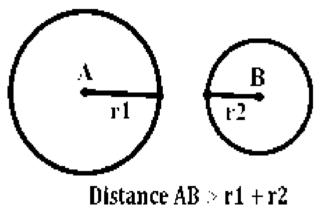
- ✓ x and y: are the coordinates that the tangent touches the circle at
- ✓ m: is the gradient – you find the gradient by working out the gradient of the gradient of the radius from the centre of the circle to the point where the tangent touches the circle. Then find its negative inverse – this is the gradient of the tangent.
- ✓ c: is the y-intercept – substitute x, y and m into the straight line equation and solve for c.

In the diagram below CBD is a tangent to the circle with centre A.



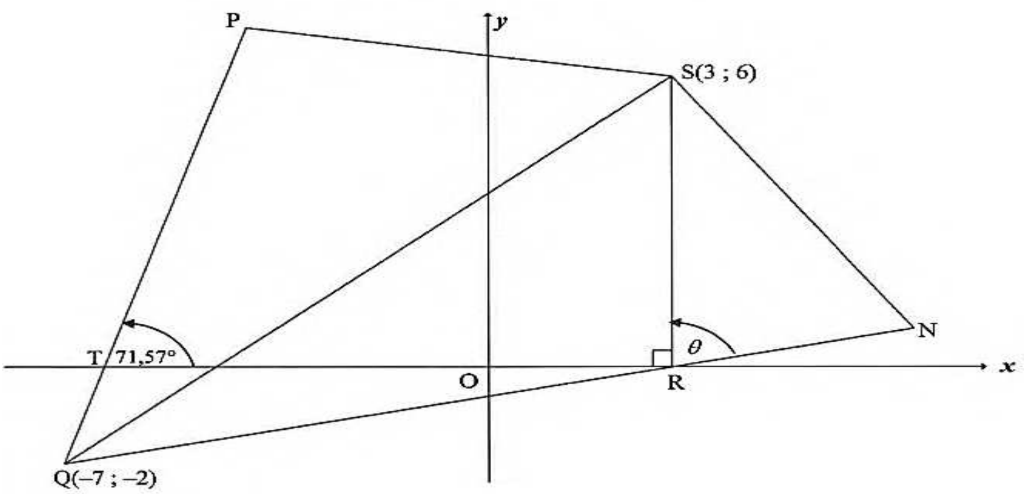
- ✓ A tangent is a straight line in the form :$y = mx + c$.
- ✓ In order to find the equation of a tangent it is important to know that: mradius × mtangent = –1 **this means the radius and the tangent form of a 90° angle at the point of** contact of the tangent.
- ➢ If we have 2 solutions it proves that the line intersects the circle of 2 places and is therefore not a tangent.

Let the centre of the one circle be A and the other B. Calculate the distance AB using the distance formula. Then add R (the radius of the one circle) to *r* the radius of the other

		
How to prove that 2 circles touch each other	**R1 > r2 only if both circles have same centre**	**In this case AB = R − *r*.**
		
If AB < R+ *r* the circles generally intersect at two points.	**If AB > R + *r* the two circles never touch.**	

Feb-March 2018 Questions

1 In the diagram, P, Q (-7; -2), R and S (3; 6) are vertices of a quadrilateral. R is a point on the x-axis. QR is produced to N such that QR = 2RN. SN is drawn. $P\hat{T}O = 71,57°$ and $S\hat{R}N = \theta$

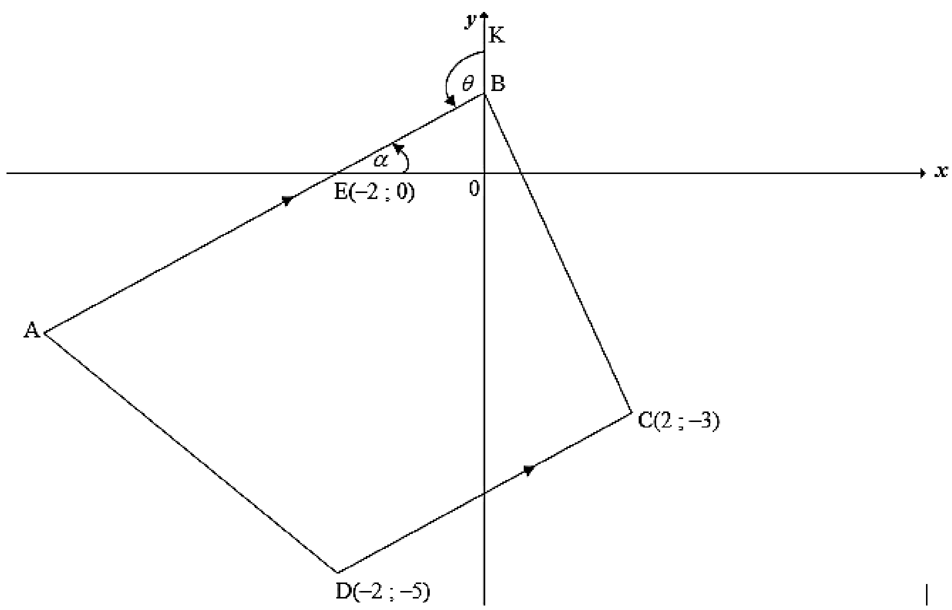


Determine:

1.1	The equation of SR.	(1)	**L1**
1.2	The gradient of QP to the nearest integer.	(2)	**L2**
1.3	The equation of QP in the form $y = mx + c$.	(2)	**L2**
1.4	The length of QR. Leave your answer in surd form.	(2)	**L2**
1.5	$\tan(90° - \theta)$.	(3)	**L2**
1.6	The area of ΔRSN without using a calculator.	(6)	**L3**
		[16]	

MAY-JUNE 2019 QUESTION 3

2. In the diagram, A, B, C $(2 ; a - 3)$ and $D(-2 ; -5)$ are vertices of a trapezium with AB $\parallel$ DC. E $(-2 ; 0)$ is the x-intercept of AB. The inclination of AB is α. K lies on the y-axis and $K\hat{B}E = \theta$



2.1 Determine:

2.1.1 The midpoint of EC. (2) **L1**

2.1.2 The gradient of DC. (2) **L1**

2.1.3 The equation of AB in the form $y = mx + c$. (3) **L2**

2.1.4 The size of θ. (3) **L3**

2.1.5 Prove that AB $\perp$ BC. (3) **L2**

2.2 The points E, B and C lie on the circumference of a circle. Determine:

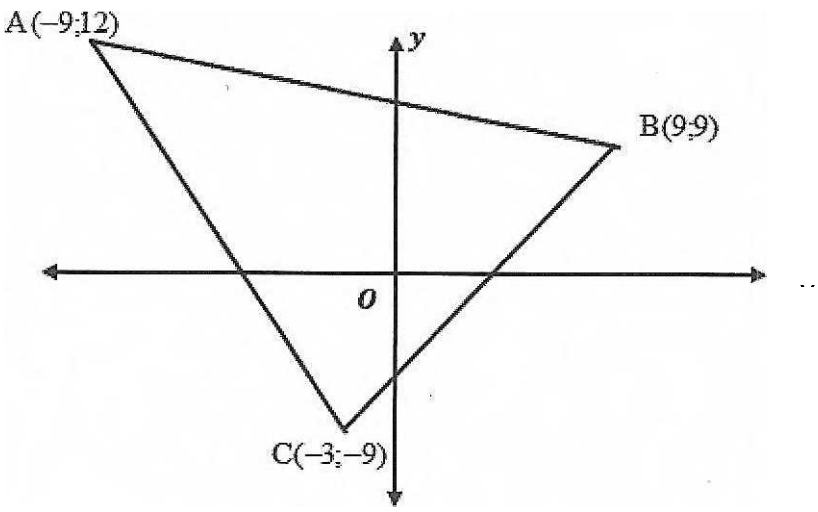
2.2.1 The centre of the circle. (1) **L1**

2.2.2 The equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$. (4) **L2**

[18]

LIMPOPO TRIAL 2019 QUESTION3

3 In the diagram A(-9 ;12) , B(9;9) and C(3;-9)are the vertices of ΔABC. K (m; n) is a point in the second quadrant.



3.1 Calculate the gradient of AB. (2) **L1**

3.2 Calculate the size of $\hat{B}$, rounded off to two decimal digits. (5) **L3**

3.3 Determine the coordinates of M, the midpoint of BC. (2) **L1**

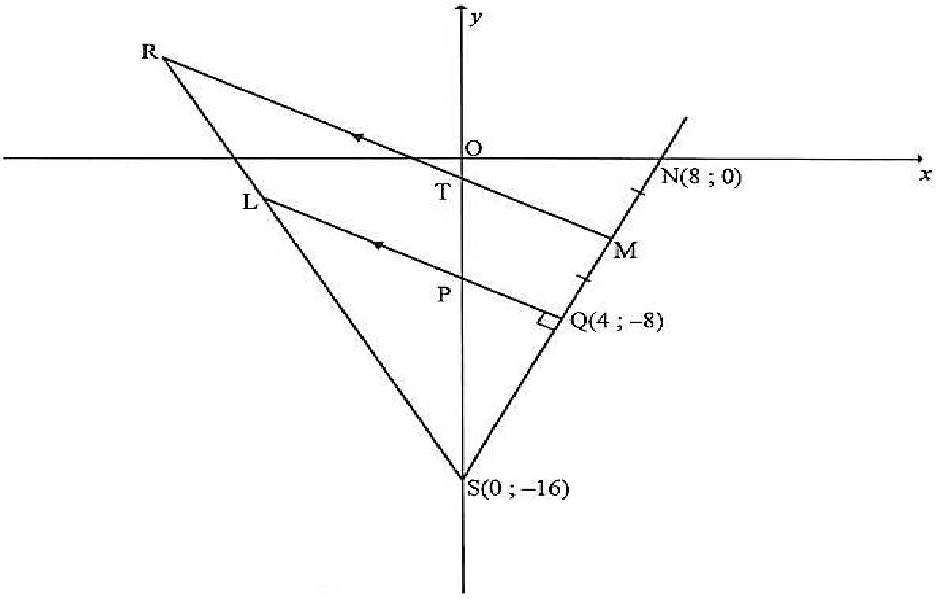
3.4 Determine the equation of AM. (3) **L2**

3.5 Determine the coordinates of K, if A, K and M are collinear and $BK = 5\sqrt{5}$ units (8) **L3**

[20]

May-June 2021 Question 3

4 In the diagram, S(0;-16), L and Q(4;-8) are the vertices of ΔSLQ having LQ perpendicular to SQ. SL and SQ are produced to points R and M respectively such that $RM \parallel LQ$. SM produced cuts the x-axis at $N(8;0)$. $QM = MN$ T and P are the y-intercepts of RM and LQ respectively.



4.1 Calculate the coordinates of M. (2) **L1**

4.2 Calculate the gradient of NS. (2) **L1**

4.3 Show that the equation of line LQ is $y = -\frac{1}{2}x - 6$ (3) **L2**

4.4 Determine the equation of a circle having centre at O, the origin, and also passing through S. (2) **L2**

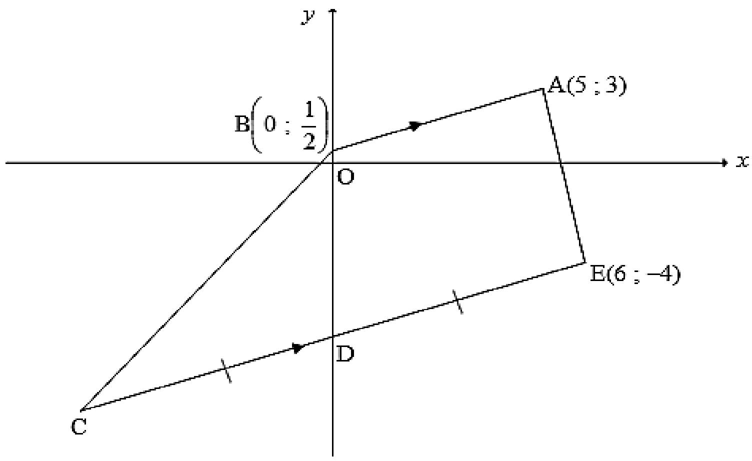
4.5 Calculate the coordinates of T. (3) **L3**

4.6 Determine $\frac{LS}{RS}$. (3) **L3**

4.7 Calculate the area of PTMQ. (4) **L3**

MAY-JUNE 2022 QUESTION3

5. In the diagram, A (5; 3). $B\left(0\ ;\ \frac{1}{2}\right)$. C and $E(6;-4)$ are the vertices of a trapezium having $BA \parallel CE$. D is the y-intercept of CE and $CD = DE$.



5.1 Calculate the gradient of AB. (2) **L1**

5.2 Determine the equation of CE in the form $y = mx + c$. (3) **L2**

5.3 Calculate the Coordinates of C. (3) **L2**

5.2 Calculate the area of quadrilateral ABCD. (4) **L3**

5.3 If point K is the reflection of E in the y-axis:

5.3.1 Write down the coordinates of K (2) **L2**

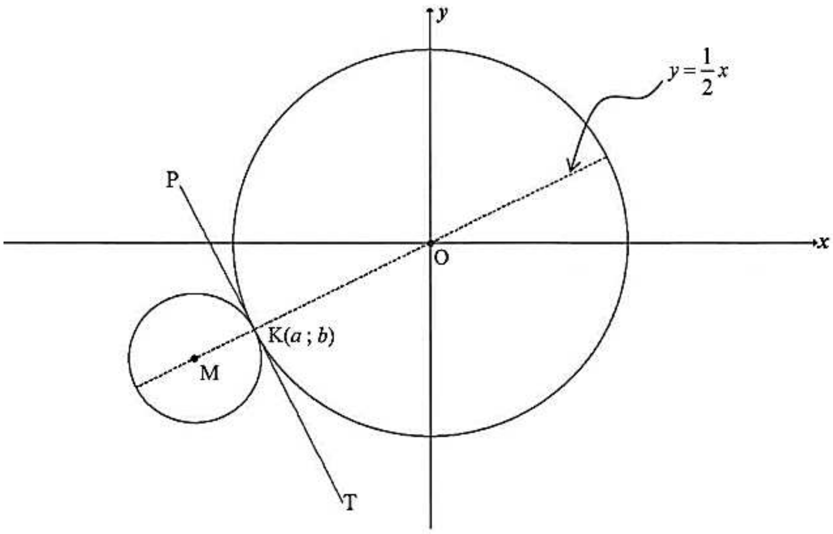
5.3.2 Calculate the perimeter of ΔKEC (4) **L2**

5.2.3 Calculate the size of $K\hat{C}E$ (3) **L2**

[21]

FEB/ MARCH 2018 QUESTION 4

6 In the diagram, PKT is a common tangent to both circles at K (a; b). The centres of both circles lie on the line $y = \frac{1}{2}x$. The equation of the circle centred at O is $x^2 + y^2 = 180$. The radius of the circle is three times that of the circle centred at M.



6.1 Write down the length of OK **in surd form.** (1) **L1**

6.2 Show that K is the point $(-12;-6)$ (4) **L2**

6.3 Determine:

6.3.1 The equation of the common tangent, PKT, in the form $y = mx + c$ (3) **L2**

6.3.2 The coordinates of M (6) **L2**

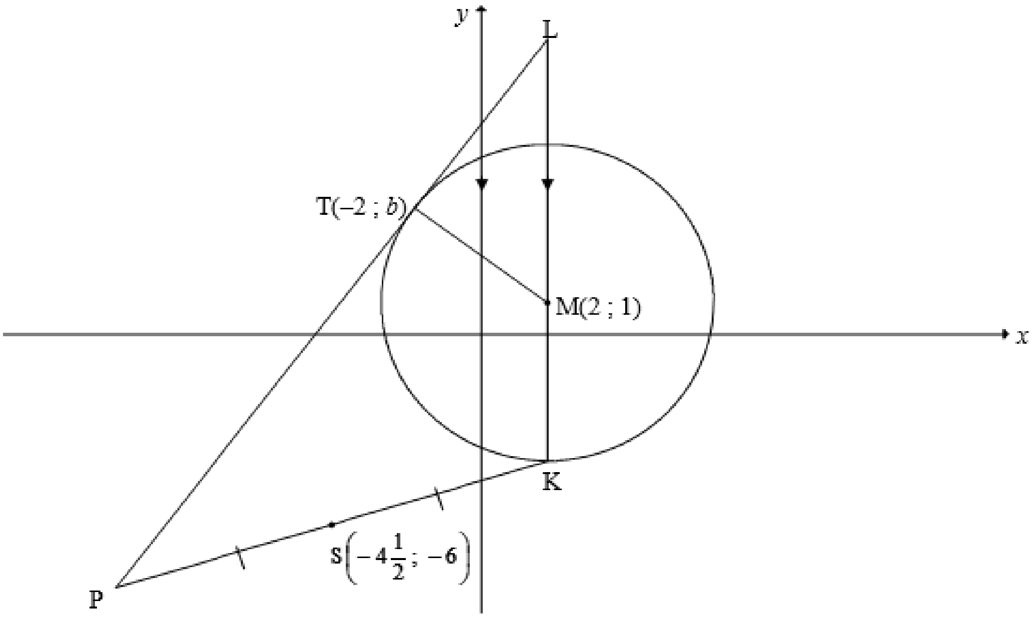
6.3.3 The equation of the smaller circle in the form $(x-a)^2 + (y-b)^2 = r^2$ (2) **L2**

6.4 For which value(s) of r will another circle, with equation $x^2 + y^2 = r^2$, intersect the circle centred at M at two distinct points? (3) **L3**

6.5 Another circle, $x^2 + y^2 + 32x + 16y + 240 = 0$ is drawn. Prove by calculation that this circle does NOT cut the circle with centre $M(-16;-8)$ (5) **L4**

MAY/JUNE 2019 QUESTION4

7 In the diagram, the circle is centred at M (2; 1). Radius KM is produced to L, a point outside the circle, such that KML $\parallel$ y-axis. LTP is a tangent to the circle at T (–2; b). $S\left(-4\frac{1}{2}\,;\,-6\right)$ is the midpoint of PK.



7.1 Given that the radius of the circle is 5 units, show that $b = 4$. (4) **L2**

7.2 Determine:

7.2.1 The coordinates of K (2) **L2**

7.2.2 The equation of the tangent LTP in the form $y = mx + c$ (4) **L2**

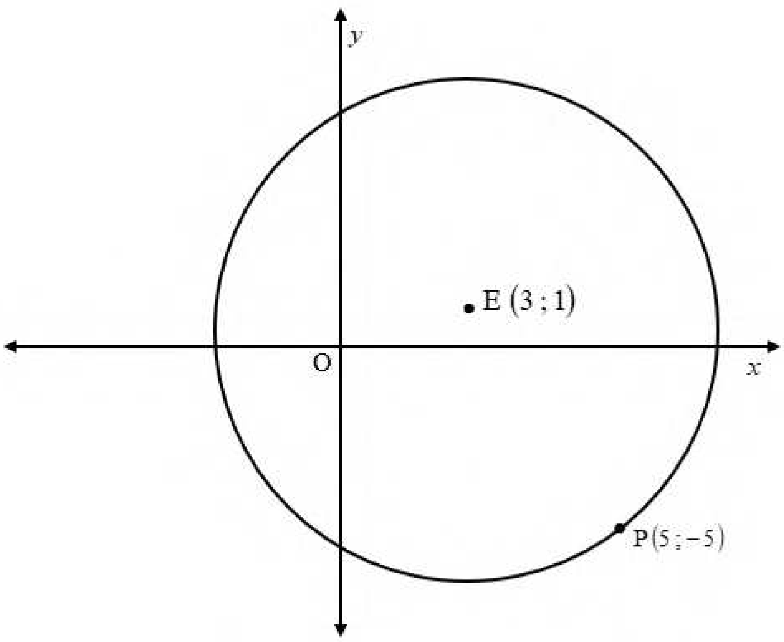
7.2.3 The area of ΔLPK (7) **L3**

7.3 Another circle with equation $(x-2)^2 + (y-n)^2 = 25$ is drawn. Determine, with an explanation, the value(s) of n for which the two circles will touch each other externally. (4) **L3**

[21]

8 **FS PREP EXAM 2019 QUESTION4**

In the diagram below, the circle centred at $E(3\,;1)$ passes through point P(5; -5).



8.1 Determine the equation of:

8.1.1 The circle in the form $x^2 + y^2 + Ax + By + C = 0$. (4) **L2**

8.1.2 The tangent to the circle at $P(5\,;-5)$ in the form $y = mx + c$ (5) **L2**

8.2 A smaller circle is drawn inside the circle. Line EP is a diameter of the small circle. Determine the:

8.2.1 Coordinates of the centre of the smaller circle. (3) **L2**

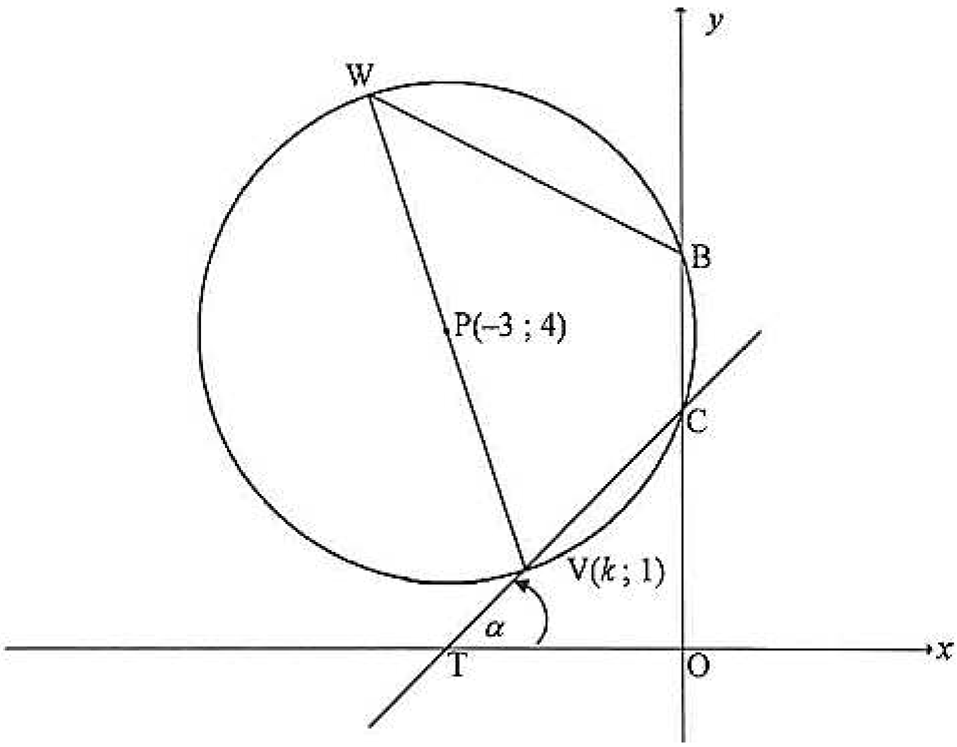
8.2.2 Length of the radius. (3) **L2**

8.3 Hence, or otherwise, determine whether point $C(9\,;3)$ lies inside or outside the circle centre at E. (3) **L3**

[18]

MAY/ JUNE 2021 QUESTION4

9 In the diagram, P (-3; 4) is the centre of the circle. $V(k\,;1)$ and W are the endpoints of a diameter. The circle intersects the y-axis at B and C. BCVW is a cyclic quadrilateral. CV is produced to intersect the x- axis at T. $O\hat{T}C = \alpha$



9.1 The radius of the circle is $\sqrt{10}$. Calculate the value of k if point V is to the right of point P. Clearly show ALL calculations. (5) **L3**

9.2 The equation of the circle is given as $x^2 + 6x + y^2 - 8y + 15 = 0$ Calculate the length of BC. (4) **L2**

9.3 If $k = -2$, calculate the size of:

9.3.1 α (3) **L2**

9.3.2 $V\hat{W}B$ (2) **L3**

9.4 A new circle is obtained when the given circle is reflected about the line $y = 1$.Determine the:

9.4.1 Coordinates of Q, the centre of the new circle (2) **L2**

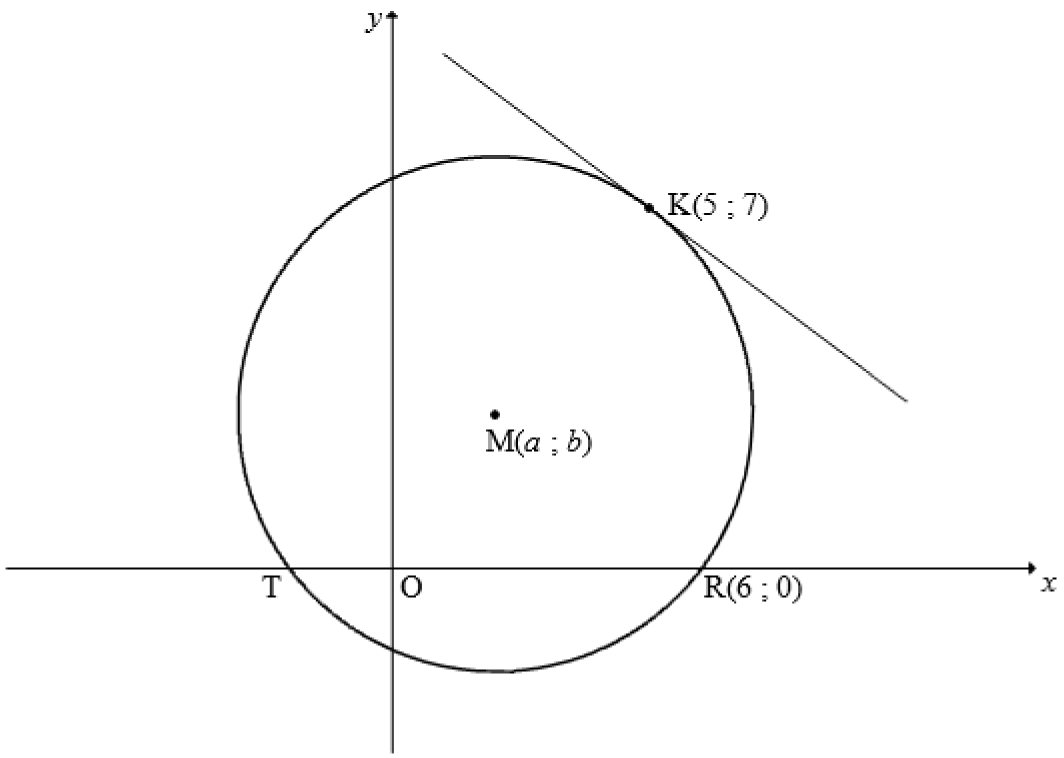
9.4.2 Equation of the new circle in the form $(x-a)^2 + (y-b)^2 = r^2$ (2) **L2**

9.4.3 Equations of the lines drawn parallel to the y-axis and passing through the points of intersection of the two circles. (2) **L3**

[20]

MAY / JUNE 2022 QUESTION 4

10 In the diagram, the circle centred at M $(a\,;\,b)$ is drawn. T and R $(6\,;\,0)$ are the x-intercepts of the circle. A tangent is drawn to the circle at K$(5\,;\,7)$.



10.1. M is a point on the line $y = x + 1$.

10.1.1 Write b in terms of a. (1) **L1**

10.1.2 Calculate the coordinates of M. (5) **L3**

10.2 If the coordinates of M are $(2\,;\,3)$, calculate the length of:

10.2.1 The radius of the circle (2) **L2**

10.2.2 TR (2) **L2**

10.3 Determine the equation of the tangent to the circle at K. Write your answer in the form $y = mx + c$. (5) **L2**

10.4. A horizontal line is drawn as a tangent to the circle M at the point N$(c\,;\,d)$, where $d < 0$.

10.4.1 Write down the coordinates of N. (2) **L2**

10.4.2 Determine the equation of the circle centred at N and passing through T. Write your answer in the form $(x-a)^2 + (y-b)^2 = r^2$. (3) **L2**

[20]

TOPIC TRIGONOMETRY

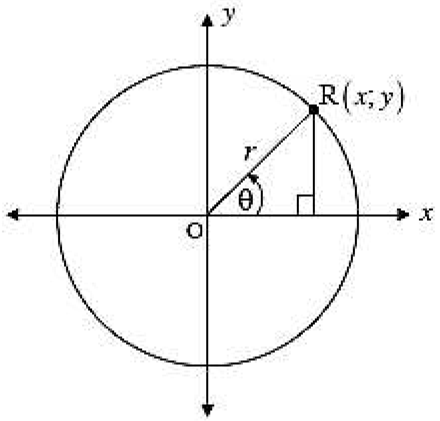
GUIDELINES, SUMMARY NOTES, & STRATEGIES

1. Definitions of trig ratios:

In a right angled triangle: $\sin\theta = \frac{opposite}{hypotenuse}$; $\cos\theta = \frac{adjacent}{hypotenuse}$ and $\tan\theta = \frac{opposite}{adjacent}$

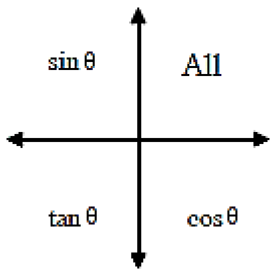
SOH CAH TOA helps you to remember these definitions.

In a Cartesian plane: $\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$; $\tan\theta = \frac{y}{x}$ and $r^2 = x^2 + y^2$



2. CAST Rule:

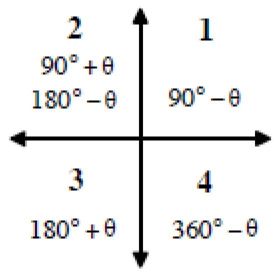
All trig ratios are positive in the 1st quadrant. **All**
Only $\sin\theta$ is positive in the 2nd quadrant. **Students**
Only $\tan\theta$ is positive in the 3rd quadrant. **Take**
Only $\cos\theta$ is positive in the 4th quadrant. **Care**



3. Reduction Formulae:

If θ is an acute angle, $180° - \theta$ and $90° + \theta$ will lie in the 2nd quadrant, $180° + \theta$ will lie in the 3rd quadrant, $360° - \theta$ will lie in the 4th quadrant, $360° + \theta$ and $90° - \theta$ will lie in the 1st quadrant.

For $90° - \theta$ and $90° + \theta$ the ratio changes to its co-function. Co-function of cos is sin and co-function of sin is cos.



Trigonometric identities:

Square identity: $\sin^2\theta + \cos^2\theta = 1$

Quotient identity: $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Compound Angles:

$\sin(\theta \pm \beta) = \sin\theta\cos\beta \pm \cos\theta\sin\beta$

$\cos(\theta \pm \beta) = \cos\theta\cos\beta \mp \sin\theta\sin\beta$

Double Angles:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

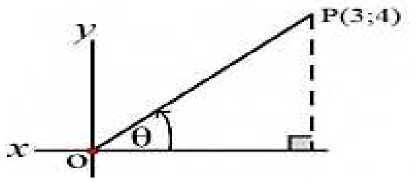
$\sin 2\theta = 2\sin\theta\cos\theta$

REVISION QUESTIONS

1. In the diagram alongside P(3; 4) is a point in the Cartesian plane.

OP makes an angle θ with positive x- axis.

Without using a calculator, determine:



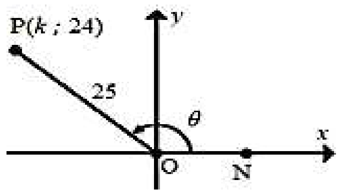
1.1 OP (1) **L1**

1.2 $\sin\theta + \cos\theta$ (2) **L1**

2. GR11 NOV 2013:

In the diagram alongside, P(k; 24) is a point in the second quadrant such that: $OP = 25$ units. N is a point on the positive x-axis and $P\hat{O}N = \theta$.

Without calculating the size of θ, determine the value of The following:



2.1 k (1) **L1**

2.2 $\sin\alpha$ if $\theta + \alpha = 360°$ (3) **L2**

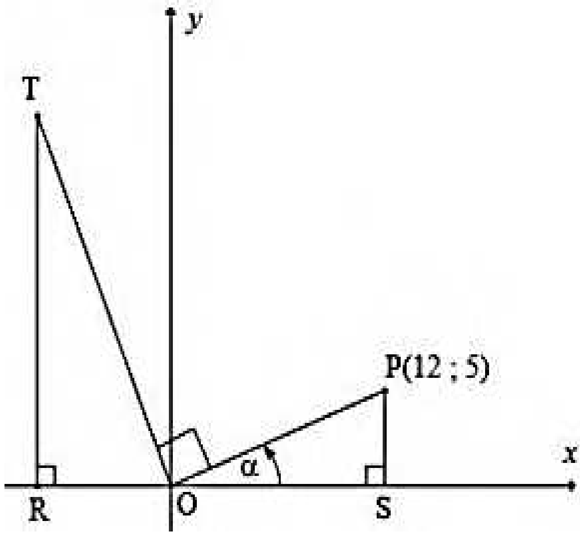
2.3 $\tan\theta$ (1) **L1**

2.4 $\cos^2\theta - \sin^2\alpha$ (3) **L2**

3. FEB/MARCH 2012:

In the diagram alongside, P is the point (12; 5). $OT \perp OP$. PS and TR are perpendicular to the x-axis. $P\hat{O}S = \alpha$ and $OR = 7{,}5$ units.

Determine the following without use of a calculator.



3.1 $\cos\alpha$ (2) **L2**

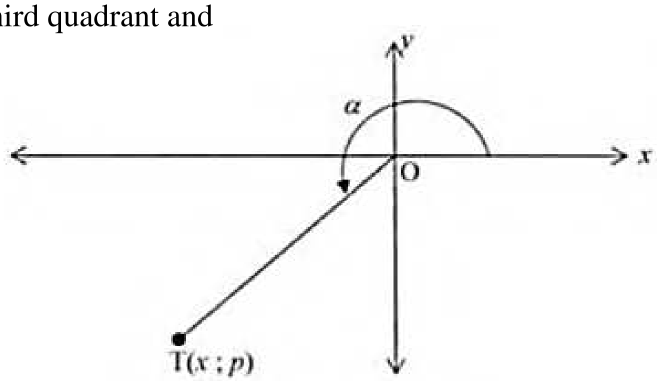
3.2 $T\hat{O}R$, in terms of α (2) **L2**

The length of

3.3 OT (4) **L3**

4. **FEB/ MARCH 2015**

In the diagram alongside, T(x; p) is a point in the third quadrant and it is given that $\sin\alpha = \dfrac{p}{\sqrt{1+p^2}}$.



4.1 Show that $x = -1$ (2) **L2**

4.2 Write $\cos(180° + \alpha)$ in term of p in its simplest form (2) **L2**

4.3 Show that $\cos 2\alpha$ can be written as $\dfrac{1-p^2}{1+p^2}$ (3) **L3**

5. If $\sin\theta = -\dfrac{12}{13}$ and $\theta \in (90°; 270°)$, calculate without using a calculator and with the aid of a diagram.

5.1 $\cos^2\theta - \sin^2\theta$ (3) **L2**

5.2 $\dfrac{\tan(360° - \theta)}{\cos(90° + \theta)}$ (3) **L2**

6. If $4\tan\theta = 3$ and $180° < \theta < 360°$, determine without using a calculator and with the aid of a diagram.

6.1 $\sin\theta + \cos\theta$ (4) **L2**

6.2 $\tan 2\theta$ (5) **L3**

7. If $\tan\alpha = \dfrac{12}{5}$, $\sin\alpha < 0$ and $\cos\beta = -\dfrac{8}{17}$ with $\beta \in (0°; 180°)$. Without using a calculator and with an aid of a diagram, determine the value of:

7.1 $\sin(\alpha + \beta)$ (5) **L3**

7.2 $\cos 2\alpha - \cos 2\beta$ (4) **L2**

8. If $\sin 2\theta = \dfrac{\sqrt{5}}{3}$ with $90° < 2\theta < 270°$, determine without the use of a calculator the value of:

8.1 $\sin\theta\cos\theta$ (2) **L2**

8.2 $\sin\theta$ (3) **L3**

9. If $\sin 36° = k$, determine the following in terms of k.

9.1 $\cos(-36°)$ (2) **L2**

9.2 $\cos 72°$ (3) **L2**

9.3 $\sin 72°$ (3) **L2**

9.4 $\cos 126.\tan 1116°$ (3) **L2**

10. If $\sin 12° = p$, determine the following in terms of p.

10.1 $\tan 12°$ (2) **L1**

10.2 $\sin 24°$ (2) **L2**

10.3 $\sin 57°$ (4) **L3**

10.4 $\sin 6°$ (3) **L3**

11. If $\cos 55° = p$ determine the value of $\cos 5°$ in terms of p. (4) **L3**

12. Given $\cos 20° = p$ and $\sin 14° = q$. Without using a calculator, calculate the value of the following in terms of p or q.

12.1 $\sin 20°$ (2) **L1**

12.2 $\cos 6°$ (4) **L3**

13. If $\sin 38°\cos 10° = p$ and $\cos 38°\sin 10° = q$, determine in terms of p and q the value of:

13.1 $\sin 48°$ (3) **L2**

13.2 $\sin 28°$ (3) **L2**

14. Evaluate the following trigonometric expressions without using a calculator

14.1 $\dfrac{\sin 210°.\tan 330°}{\sin^2 225°}$ (5) **L2**

14.2 $\dfrac{\sin 315°.\tan 210°.\sin 190°}{\cos 100°.\sin 120°}$ (6) **L2**

14.3 $\sin 15°$ (4) **L2**

14.4 $\cos 615°$ (5) **L3**

14.5 $\sqrt[3]{\dfrac{\sin 225°.\cos 315°.\sin(-210°)}{\sin 120°.\tan 30°}}$ (6) **L2**

14.6 $\dfrac{2\sin 165°.\cos 345°}{\cos 45°.\cos 15° + \sin 45°.\sin 15°}$ (6) **L3**

15. Simplify the following trigonometric expressions into a single trigonometric ratio.

15.1 $\dfrac{\sin(-x).\sin(x-180°).\sin 35°}{\cos(360° + x).\cos(90° - x).\cos 55°}$ (6) **L2**

15.2 $\sin x.\tan x + \cos x$ (3) **L2**

15.3 $\dfrac{\cos x}{1-\sin x} - \dfrac{\cos x}{1+\sin x}$ (4) **L2**

15.4 $\dfrac{\sin 2x + \sin x}{\cos 2x + \cos x + 1}$ (5) **L3**

15.5 $\dfrac{1-\cos 2x - \sin x}{\sin 2x - \cos x}$ (5) **L3**

16. Prove the following trigonometric identities.

16.1 $\dfrac{\sin^3 x + \sin x \cos^2 x}{\cos x} = \tan x$ (2) **L2**

16.2 $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$ (3) **L2**

16.3 $\dfrac{\cos 2\theta + \sin^2\theta}{1+\sin\theta} = 1 - \sin\theta$ (2) **L2**

16.4 $\tan\alpha + \dfrac{1}{\tan\alpha} = \dfrac{2}{\sin 2\alpha}$ (5) **L3**

16.5 $\cos 4x = 1 - 8\sin^2 x + 8\sin^4 x$ (4) **L4**

16.6 $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (4) **L3**

17. Given the trigonometric identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = \dfrac{2}{\cos x}$

17.1 Prove the given identity. (5) **L3**

17.2 For which value(s) of x in the interval $0° \le x \le 360°$ will the identity be undefined? (2) **L2**

18. Given: $\dfrac{2\tan x - \sin 2x}{2\sin^2 x} = \tan x$

18.1 Prove the above identity. (6) **L3**

18.2 For which value(s) of x will the above identity be undefined in the interval $-180° \le x \le 180°$? **L3**

19. Given that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

19.1 Prove the above identity. (3) **L3**

19.2 Hence, determine the value of $\tan 22,5°$ without the use of a calculator. (3) **L2**

20. Determine the general solution of the following trigonometric equations.

20.1 $\cos\theta = 0,4$ (2) **L2**

20.2 $\sin(2\theta + 16°) = -0,67$ (3) **L2**

20.3 $\sqrt{3}\cos\theta - 3\sin\theta = 0$ (3) **L2**

20.4 $2\cos^2\theta = \cos\theta$ (4) **L2**

20.5 $2\sin^2\theta - \sin\theta = 1$ (4) **L2**

20.6 $\cos(2\theta + 45°) = \cos(20° - \theta)$ (3) **L2**

20.7 $\cos(\theta + 30°) = \sin 2\theta$ (5) **L3**

20.8 $\cos 2\theta - 5\cos\theta = 1 - \cos^2\theta$ (6) **L4**

TOPIC TRIGONOMETRIC GRAPHS

GUIDELINES, SUMMARY NOTES, & STRATEGIES

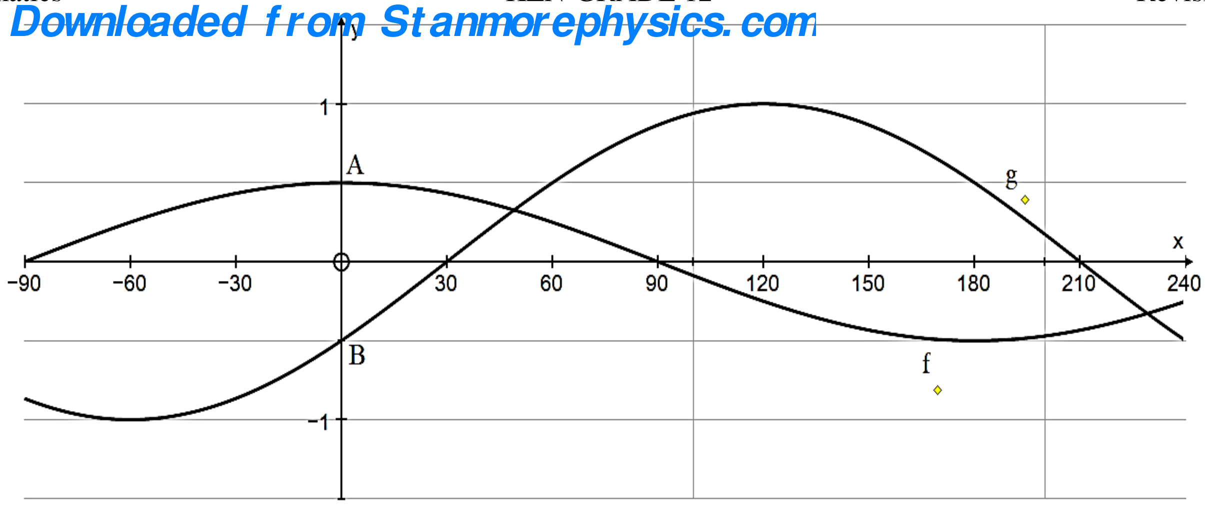
- The focus of trigonometric graphs is on the relationships, simplification and determining points of intersection by solving equations, although characteristics of the graphs should not be excluded.
- Candidates must be able to use and interpret functional notation. In the teaching process learners must be able to understand how f (x) has been transformed to generate $f(-x)$, $-f(x)$, $f(x+a)$ $f(x)+a$, $a.f(x)$ and $x = f(y)$ where a $\in$ R.

REVISION QUESTIONS

1 **Durban girls –September 2019 Question 7**

In the diagram below, the graphs of :

$f(x) = 2\sin x$ and $g(x) = \cos(x - 30°)$, are drawn for $x \in [-180°;180°]$



1.1 Write down the range of g. (1) **L1**

1.2 Write down the period of h if $h(x) = f\left(\frac{1}{2}x\right)$ (1) **L2**

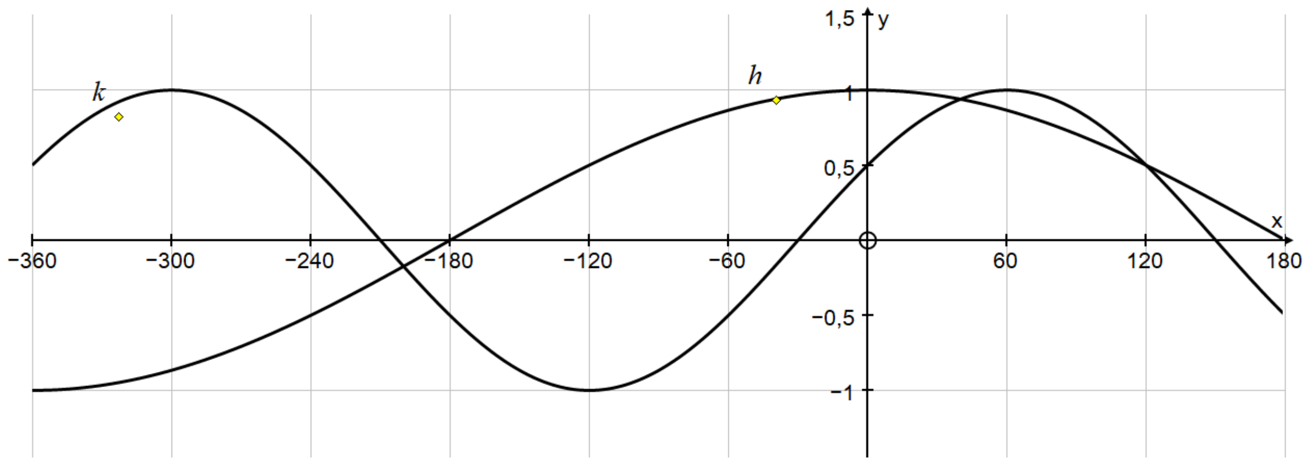
1.3 Use your graph to write down the values of x in the interval $x \in [-180°;180°]$ for which :

1.3.1 $\frac{f(x)}{g(x)} = 1$ (2) **L3**

1.3.2 $f'(x).g(x) < 0$ (2) **L3**

2 **Westville boys high –September 2019 Question 5**

The diagram below shows the graphs of $h(x) = \cos px$ and $k(x) = \sin(x+q)$ for $x \in [-360°;180°]$.



2.1 Write down the period of h. (1) **L1**

2.2 Determine the value of p and q (2) **L1**

2.3 For which values of x in the interval $x \in [-360^0;-60^0]$ is $\frac{h(x)}{k(x)} \leq 0$? (3) **L3**

3. **DBE Feb 2018 Question 5.5**

Consider $g(x) = -4\cos(x + 30°)$

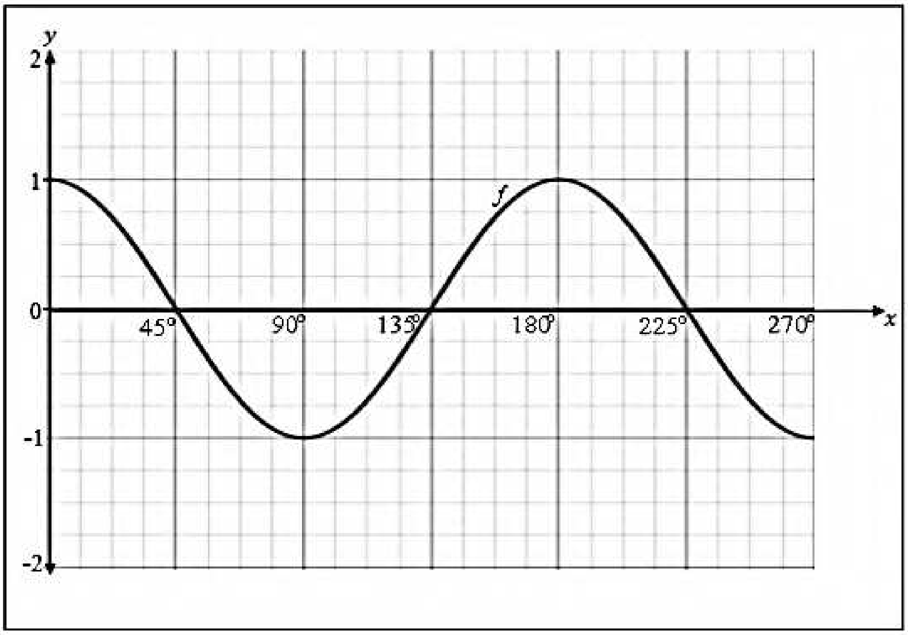
3.1 Write down the maximum value of $g(x)$. (1) **L1**

3.2 Determine the range of $g(x) + 1$ (2) **L1**

3.3 The graph of g is shifted $60°$ to the left and reflected about x-axis to form a new graph h. Determine the equation of h in its simplest form. (3) **L2**

4

In the diagram below, the graph of $f(x) = \cos 2x$ is drawn for the interval $x \in [0°; 270°]$.



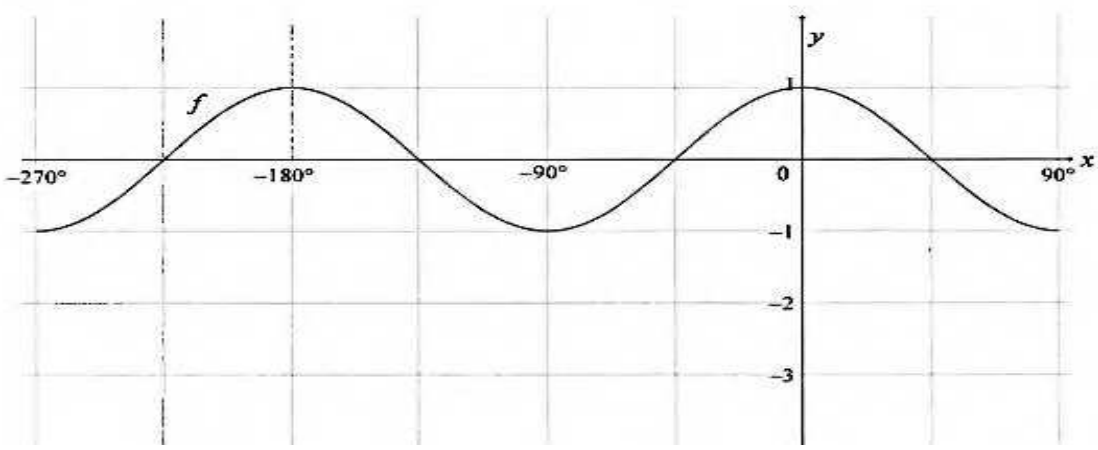
4.1 Draw the graph of $g(x) = -\frac{1}{2}\tan x$ for the interval $x \in [0°; 270°]$. Show all intercept with the axes and asymptotes. (4) **L2**

4.2 Write down the range of $h(x) = 3 - f(x)$. (1) **L2**

4.3 Use the graph to determine the value(s) of x in the interval $x \in [135°; 270°]$ for which $\frac{f(x)}{g(x)} \geq 0$. (2) **L2**

5

In the diagram, the graph of $f(x) = \cos 2x$ is drawn for the interval $x \in [-270°; 90°]$.



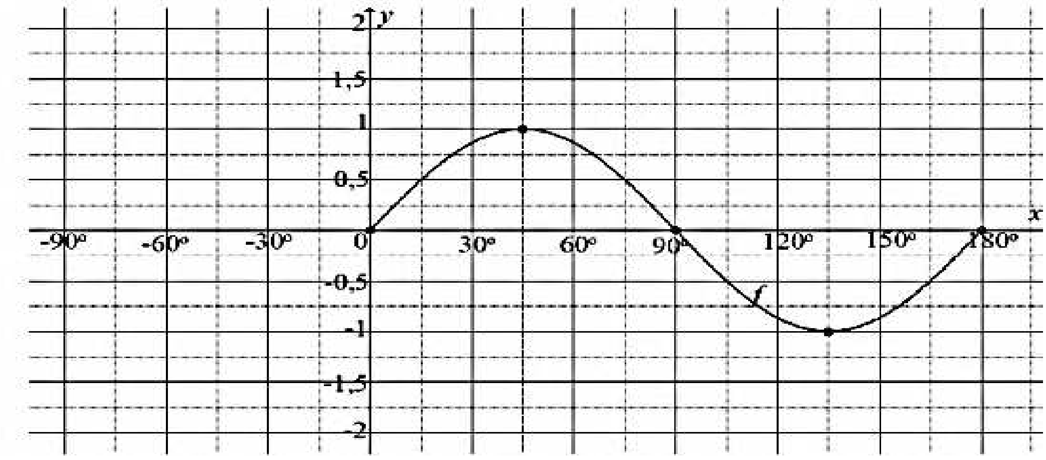
5.1 Draw the graph of $g(x) = 2\sin x - 1$ for the interval $x \in [-270°; 90°]$ on the grid given. Show all the intercepts with the axis as well as turning points. (4) **L2**

5.2 Let A be a point of intersection of the graphs of f and g. Show that the x-coordinate of A satisfies the equation $\sin x = \frac{-1+\sqrt{5}}{2}$ (4) **L3**

5.3 Hence, calculate the coordinates of the points of intersection of graphs of f and g on the interval $x \in [-270°; 90°]$. (4) **L3**

6 **EC SEPT 2019**

The graph below shows the part of the function $f(x) = \sin 2x$ for $0° \leq x \leq 180°$.



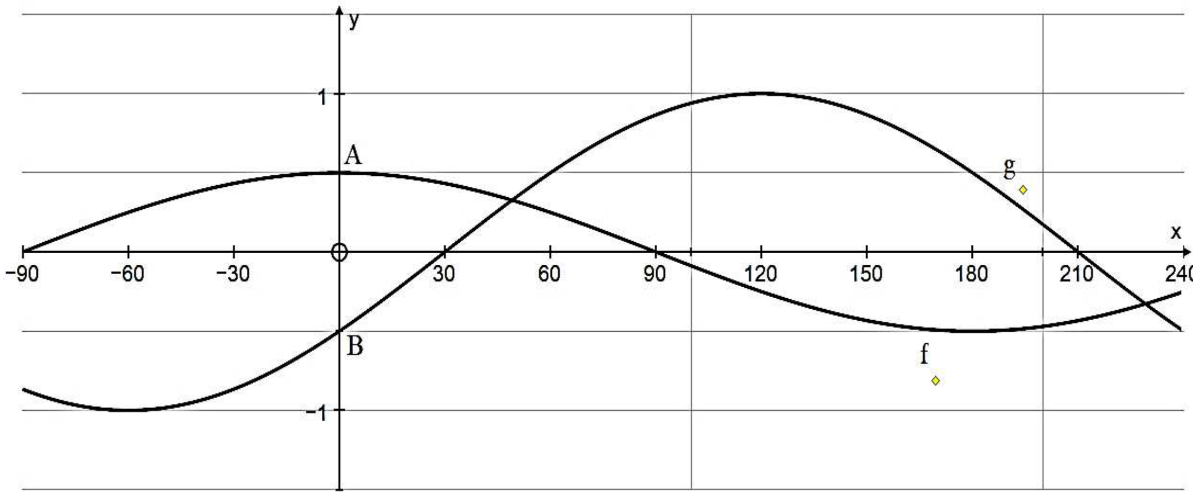
6.1 Complete the graphs of f for the interval $-90° \leq x \leq 180°$. (1) **L2**

6.2 Draw the graphs of $g(x) = \cos(x - 30^0)$ for the interval $-90° \leq x \leq 180°$. Clearly show the intercepts with the axis, the coordinates of the turning points and the end points of the graphs. (4) **L2**

6.3 Calculate the solutions to the equation: $\sin 2x = \cos(x - 30°)$ for the interval $-90° \leq x \leq 180°$. (6) **L3**

7. **QUESTION 7 May/June 20222**

In the diagram below, the graphs of $f(x) = \frac{1}{2}\cos x$ and $g(x) = \sin(x - 30°)$ are drawn for the interval $x \in [-90° ; 240°]$. A and B are the y-intercepts of f and g respectively.



7.1 Determine the length of AB. (2) **L1**

7.2 Write down the range of $3f(x) + 2$. (2) **L1**

7.3 Read off from the graphs a value of x for which $g(x)-f(x)=\frac{\sqrt{3}}{2}$. (2) **L2**

7.4 For which values of x, in the interval $x\in[-90°\ ;\ 240°]$, will:

7.4.1 $f(x).g(x)>0$ (2) **L2**

7.4.2 $g^{/}(x-5°)>0$ (2) **L4**

TOPIC **TRIGONOMETRY: TWO AND THREE DIMENSIONAL**

GUIDELINES, SUMMARY NOTES, & STRATEGIES

THE SINE-RULE

In any ΔABC it is true that:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Important: Use the Sine Rule when given **two angles and a side** in a triangle, also when **two sides and a non-included angle** are given.

It is advisable that when calculating **sides** have the **sides as numerators**: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ and when calculating **angles, have the angles as numerators:** $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.

THE COSINE-RULE

In any ΔABC it is true that: $a^2=b^2+c^2-2bc.\cos A$, $b^2=a^2+c^2-2ac.\cos B$ and $c^2=a^2+b^2-2ab.\cos C$

Important: Use the Cosine Rule when given **two sides and an included angle,** also when you are given **all the three sides.**

It is advisable that when calculating **sides** use: $a^2=b^2+c^2-2bc.\cos A$, $b^2=a^2+c^2-2ac.\cos B$ $c^2=a^2+b^2-2ab.\cos C$, also when calculating **the angles** use: $\cos A=\frac{b^2+c^2-a^2}{2bc}$, $\cos B=\frac{a^2+c^2-b^2}{2ac}$ and $\cos C=\frac{a^2+b^2-c^2}{2ab}$.

THE AREA-RULE

In any ΔABC it is true that:

Area of ΔAB $=\frac{1}{2}bc.\sin A=\frac{1}{2}ac.\sin B=\frac{1}{2}ab.\sin C$

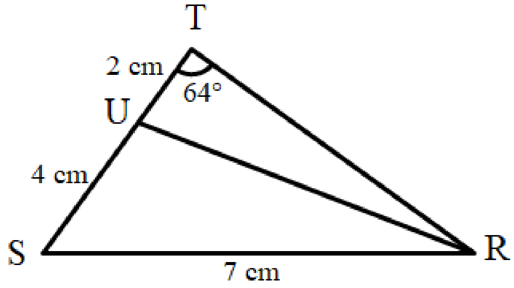
Important: Use the Area Rule when given **two sides and an included angle.**

STRATEGIES

Note: When solving 3D problems separate all the triangle so that they will be 2D and easy to solve. It is also advisable that write all your findings back to the diagrams to help you with the next question.

REVISION QUESTIONS

1 In the diagram below, ΔTSR is drawn with U on TS. US = 4 cm, UT = 2 cm, SR = 7 cm and $\hat{T} = 64°$.



1.1 Calculate the value of $\hat{S}$ *(correct to ONE decimal place).* (4) **L3**

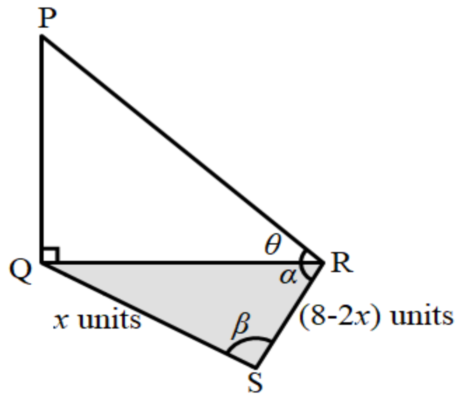
1.2 If $\hat{S} = 65,6°$, calculate the following:

1.2.1 The area of ΔUSR. (3) **L2**

1.2.2 The length of UR. (3) **L2**

2 **KZN STEPAHEAD DOCUMENT 2022**

In the diagram alongside, PQ is a vertical mast. R and S are two points on the same horizontal plane as Q, such that: $Q\hat{R}S = \alpha,\ Q\hat{S}R = \beta,\ SR = 8 - 2x,\ QS = x.$



2.1 Show that: $PQ = \dfrac{x \sin\beta \tan\theta}{\sin\alpha}$ (5) **L3**

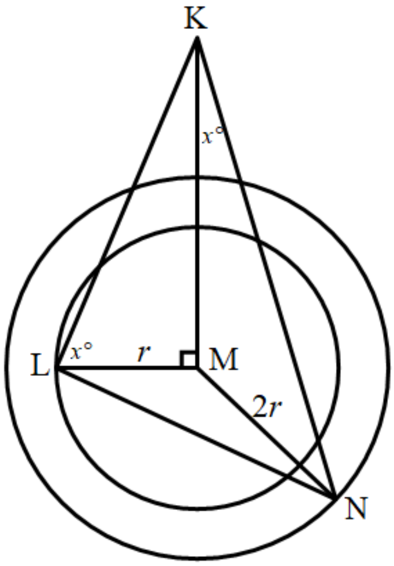
2.2 If $\beta = 60°$, show that the area of $\Delta QSR = 2\sqrt{3}.x - \dfrac{\sqrt{3}}{2}x^2$. (3) **L2**

2.3 Determine the value of x for which the area of ΔQSR will be maximum. (3) **L3**

2.4 Calculate the length of QR if the area of ΔQSR is maximum. (3) **L2**

3 **Gauteng September 2020**

In the figure alongside, KM is a vertical flag post set in the centre of two circles which lie on the same horizontal plane. $M\hat{K}N = M\hat{L}K = x°$. The radius of the inner circle ML = r units and the radius of the outer circle MN = $2r$ units.

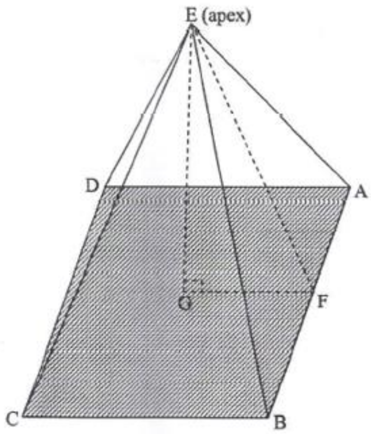


3.1 Calculate the value of x. (6) **L4**

3.2 If $r = 5$units and $L\hat{M}N = 110°$, calculate the length of LN. (2) **L1**

4 **November 20013 question 13**

The Great Pyramid at Giza in Egypt was built around 2 500 BC. The Pyramid has a square base (ABCD) with sides 232,6m. The distance from each corner of the base to the apex (E) was originally 221,2 m.



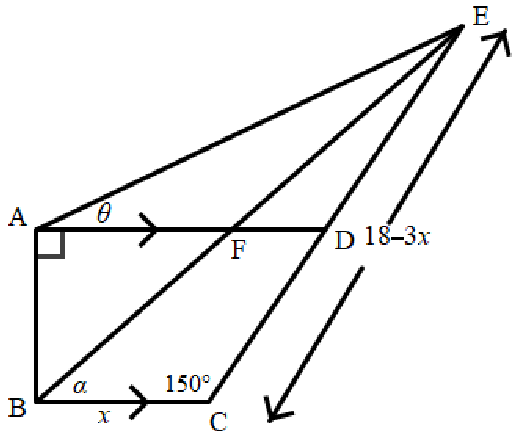
4.1 Calculate the size of the angle at the apex of a face of the pyramid (e.g. $B\hat{E}A$ or $C\hat{E}B$). (3) **L2**

4.2 Calculate the angle each face makes with the base (e.g. $E\hat{F}G$, where $EF \perp AB$ in Δ AEB). (6) **L4**

5 **KZN September 2016 question 7**

ABCD is a trapezium with AD // BC, $B\hat{A}D = 90°$ and $B\hat{C}D = 150°$ .CD is produced to E. F is a point on AD such that BFE is straight line, and

$C\hat{B}E = \alpha$. The angle of elevation if E from A is θ, BC = x and CE = $18 - 3x$



5.1 Show that: $BE = \dfrac{AB\cos\theta}{\sin(\alpha - \theta)}$ (5) **L3**

5.2 Show that the area of Δ BCE $= \dfrac{9}{2}x - \dfrac{3}{4}x^2$. (3) **L3**

5.3 If $x = 3$, calculate the area of Δ BCE. (2) **L1**

TOPIC EUCLIDEAN GEOMETRY

GUIDELINES, SUMMARY NOTES, & STRATEGIES

WAYS IN WHICH EUCLIDEAN GEOMETRY IS TESTED

1. Completing a statement of a theorem in words.
2. Determining the value of an angle in **two ways:** numerical and / or in terms of the variable(s)
3. **Proofs in riders:** Direct and indirect proofs
4. **Similarity** and **Proportionality** Theorems
    - Proportionality theorem: Question involving parallel lines in proportions, Areas (common angle vs. common vertex/same height)
    - Similarity theorem: AAA, ratios after similar triangles.
5. **Examinable proofs** to be known

1. COMPLETING A STATEMENT OF A THEOREM IN WORDS.

- Know by heart all the theorems and be able to complete the statement.

2. DETERMINING THE VALUE OF AN ANGLE

- Know all the theorems about **lines**, **triangles** and **circles (Centre** group, **non-centre** group, **tangent** group and **cyclic quad** group).
- Every statement must come with a reason and reasons must be stated according to the list of acceptable reasons from the exam guidelines
  E.g. base $\angle' s$ of an iso. Δ (unacceptable) the acceptable reason is: $\angle' s$ opp = sides

3. PROOFS IN RIDERS

Know how theorems and their converses are being formed in diagrams.

- When given 3 points on the circumference look out for a possibility of a triangle. If one side is produced then you may expect exterior angle of a triangle. If there is a tangent on the circle then there is a possibility of having a Tan Chord Theorem
- When given 4 or 5 points on the circumference then there is a possibility that 4 points may be joined and then there is a cyclic quad. In a case that one side is produced then you may expect exterior angles of a cyclic quad.
- Start with a given angle linking with what is required to prove
- Visualization: Mind picture of diagrams of theorems

DIRECT AND INDIRECT PROOFS IN RIDERS.

- In Geometry we mostly use angles to prove in questions.

1. Direct proof question: Prove $A = B$

2. Indirect proof question: Prove that a line // to another line.

Remember in Euclidean geometry- we mostly use angles to prove. This question is not asking about the angles directly. Here we need to prove sides but using angles i**ndirectly. Why indirectly?** Because we mostly use angles to prove.

∴ First, we need to change this question to be direct, and then prove. If we say it must be direct we mean that it must ask to prove angles 1^{st}, then conclude by stating the sides that are parallel

4. SIMILARITY AND PROPORTIONALITY THEOREMS

PROPORTIONALITY THEOREM

- Identify parallel lines, and use ratios for proportion.
- **Useful strategies in solving problems involving ratio in <u>areas</u> of triangles:**

CASE 1: If triangles share a **common angle** use area rule. Area = $\frac{1}{2}$a.bsinC

CASE 2: If triangles share a common vertex or height use Area = $\frac{1}{2}bh$

CASE 3: If none of the cases above apply then identify a common triangle and relate the two triangles in question to it, then use any of the two methods mentioned above. **OR**

Required Area = Area of big Δ − other known Area

SIMILARITY THEOREM

CASE 1: Prove that triangles are similar e.g. $\Delta ABC \,|||\, \Delta DEF$

- Angles and / or sides in proportion can be used to prove that two triangles are similar.
- Always name the triangles you are referring to when proving similar triangles

CASE 2: Prove that $\frac{AB}{PQ} = \frac{AC}{PR}$. First prove: ΔABC|||ΔPQR and then deduce the proportion of the sides.

CASE 3: Prove that: KN. PX = NR. YP. Find two triangles in which KN, PX, NR and Y, (or sides equal to these), and thus prove that: ΔKNR|||ΔYPX, then deduce what you were asked to prove. Identify triangles. This method is used when proved similarity don't give asked ratios.

CASE 4: Prove: Proportion with square, with division, with + in between, there is a possibility that two similarities were used or Pythagoras theorem was used.

e.g. $\frac{CF^2}{EF^2} = \frac{BD}{DE}$

5. EXAMINABLE PROOFS

Five grade 11 proofs to be known for exam purposes:

5.1 Line from the centre ⊥ chord
5.2. **NEW**: line from centre to midpt of chord
5.3. Angle at the centre is 2× angle at the circumference.
5.4. Opposite angles of a cyclic quad are supplementary.
5.5. Tan chord theorem.

Two grade 12 proofs:

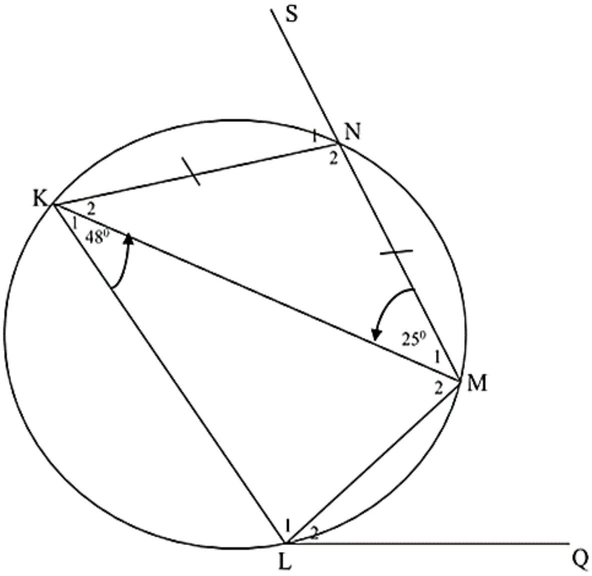
5.6. Line drawn parallel to one side of a triangle, divides the other two sides proportionally: **Proportionality theorem**
5.7 If two triangles are equiangular, then their corresponding sides are in proportion: **Similarity theorem**

NB!!!!!

- Do not make any assumption e.g. do not assume that a line is a tangent or a diameter, unless you are told that it is.
- Look for key words in the statement such as centre, // lines, tangents, cyclic quads, bisects, etc.
- Continuously update the diagram as you read the statement and as you find the angles.
- When proving theorems, no construction no marks.
- You will not always be told that you have a cyclic quadrilateral. Therefore check lines joining four points on the circumference.
- For every statement there **must** be a reason.

REVISION QUESTIONS

1 In the figure, LQ is a tangent to the circle KLMN at L. MN is produced to S. MN = KN , $\hat{K}_1 = 48°$ and $\hat{M}_1 = 25°$ determine with reasons the size of the following angles :

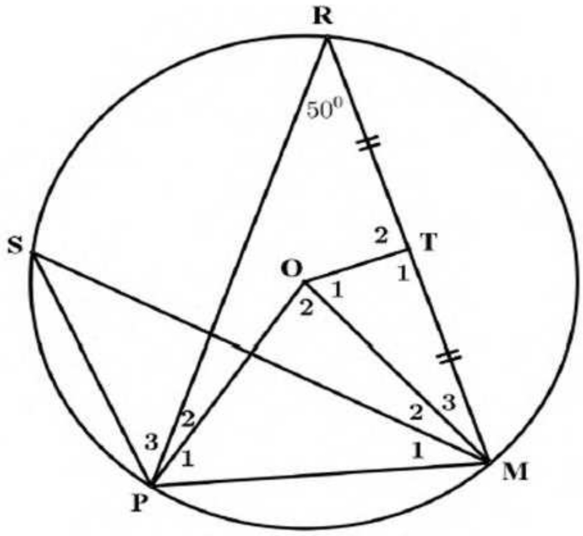


1.1 $\hat{L}_2$ (1) **L1**

1.2 $\hat{N}_1$ (4) **L2**

1.3 $\hat{M}_2$ (2) **L2**

2 In the figure, O is the centre of the circle RMPS. T is the midpoint of RM. and $\hat{R} = 50°$. Calculate with reasons the size of the angles that follows.



.

2.1 $\hat{T}_1$ (1) **L1**

2.2 $\hat{O}_2$ (4) **L1**

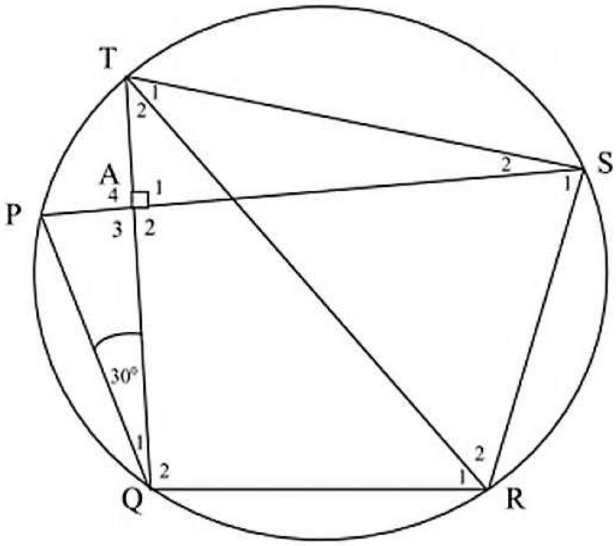
2.3 $\hat{S}$ (3) **L1**

2.4 $\hat{P}_1$ (3) **L2**

2.5 Is TOPM a cyclic quadrilateral? Give a reason for your solution. (2) **L2**

MPUMALANGA 2019 TRIAL

3. In the figure, TR is a chord of the circle PQRST. QAT ⊥ PAS. $\hat{Q}_1 = 30°$ and $\hat{P} = \hat{S}_1$



3.1 Name 3 angles each equals to 60° (4) **L2**

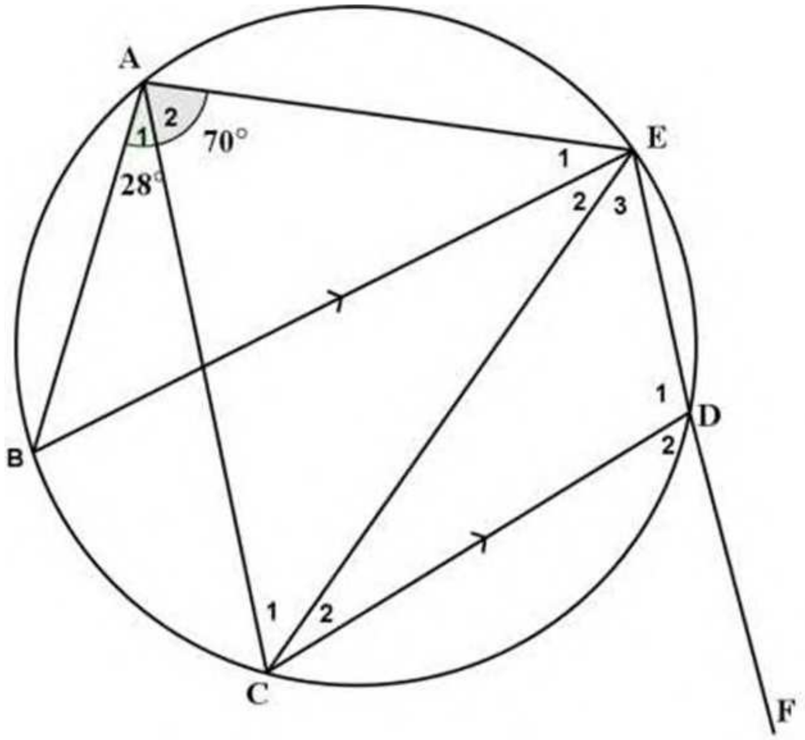
3.2 Calculate the size of $Q\hat{R}S$ (2) **L2**

3.3 Prove that PS$\parallel$ QR (2) **L3**

3.4 Prove that TR is a diameter of the circle (3) **L3**

SAICA NOVEMBER 2017

4 In the diagram below: BE $\parallel$ CD. EDF is a straight line $\hat{A}_1 = 28^\circ$ and $BAE = 70^\circ$



Calculate giving reasons the values of:

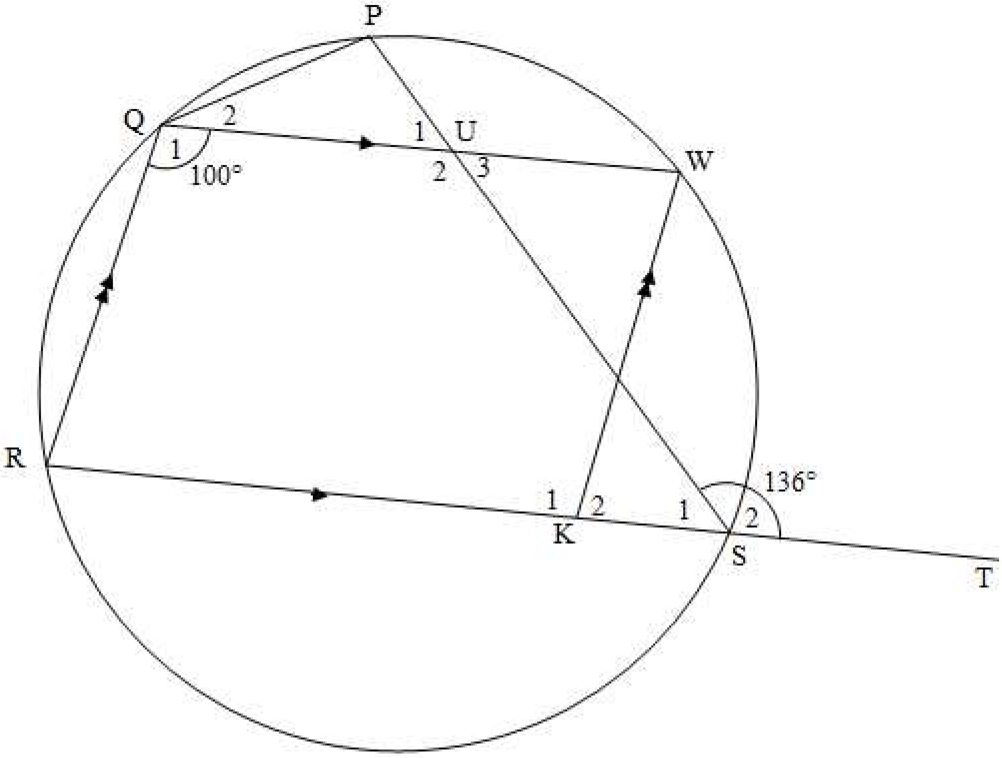
4.1 C_2 (3) **L2**

4.2 D_2 (2) **L1**

4.3 E_3 (2) **L2**

DBE NOVEMBER 2019

5 In the diagram, PQRS is a cyclic quadrilateral. Chord RS is produced to T. K is a point on RS and W is a point on the circle such that QRKW is a parallelogram. PS and QW intersect at U. $P\hat{S}T = 136°$ and $\hat{Q}_1 = 100°$.

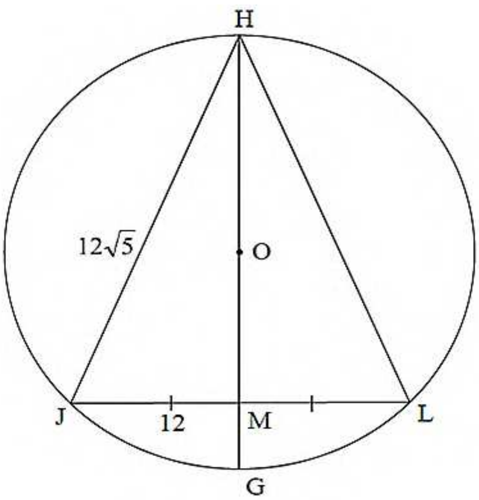


Determine, with reasons, the size of:

5.1	$\hat{R}$	(2)	**L1**
5.2	$\hat{P}$	(2)	**L1**
5.3	$P\hat{Q}W$	(3)	**L2**
5.4	$\hat{U}_2$	(2)	**L2**

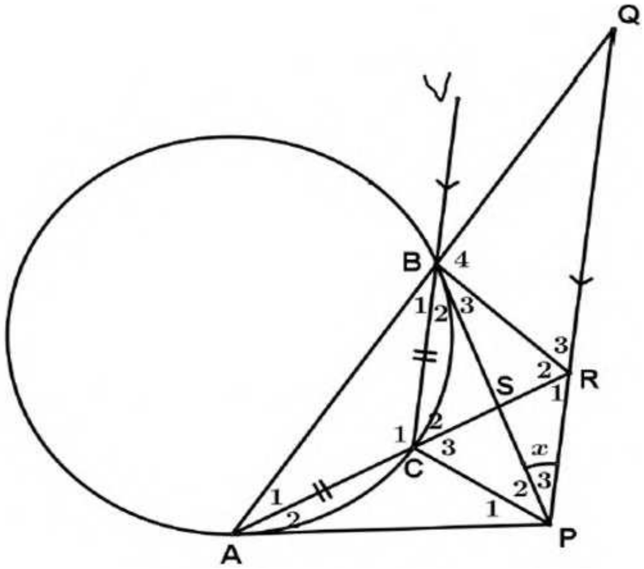
NOVEMBER 2020(2)

6. In the diagram along, a circle centred at O is drawn. H, J, G and L are points on the circle. ΔHJL is drawn. HOG bisects JL at M. HJ $= 12\sqrt{5}$ units and JM$=12$ units.



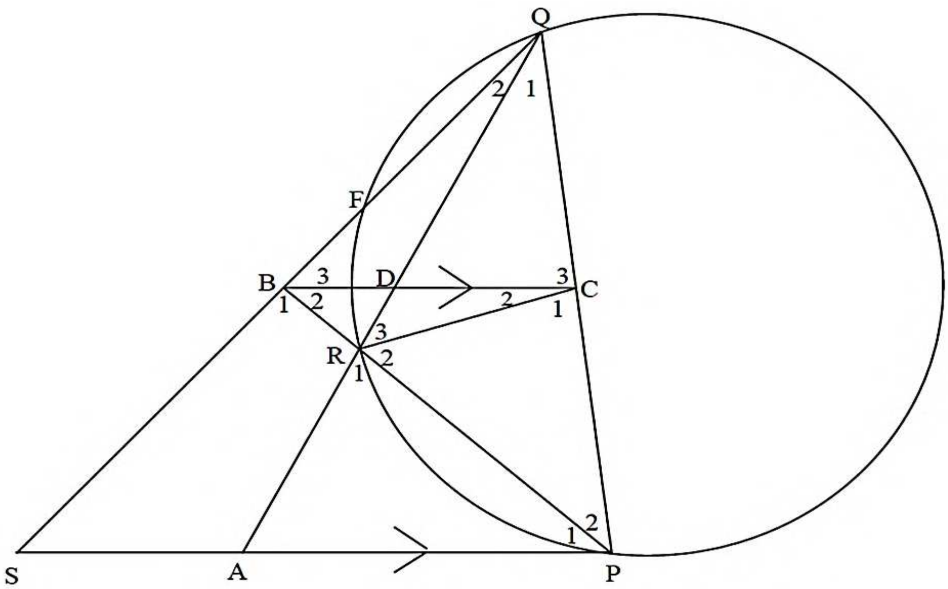
6.1	If MG = 6 units and OM = x, write HM in terms of x.	(2)	**L2**
6.2	Calculate, giving reasons, the length of the radius of the circle	(5)	**L3**

7. In the figure, AP and PB are tangents to the circle at A and B. C is a point on the circle such that AC = CB. AB is extended to Q such that PQ|| CB. AC is extended to meet PQ at R. BR is joined. Let $B\hat{P}R = x$



7.1	Write down with reasons 5 other angles each equal to x.	(6)	**L2**
7.2	Prove that ABRP is a cyclic quadrilateral	(4)	**L3**

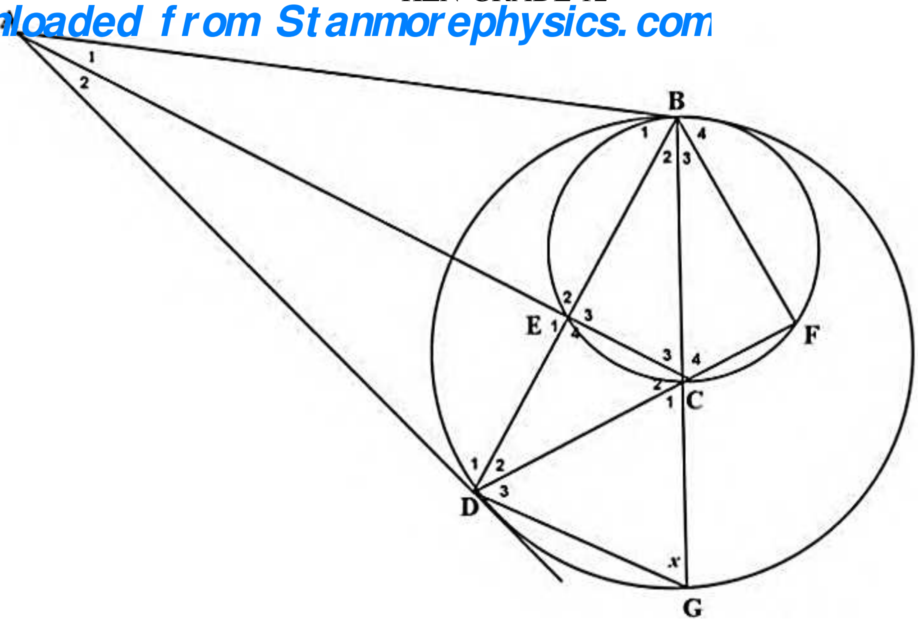
8. In the diagram below, SP is a tangent to the circle at P and PQ is a chord. Chord QF produced meets SP at S and chord RP **bisects** $Q\hat{P}S$. PR produced meets QS at B. BC|| SP and cuts the chord QR at D. QR produced meets SP at A. Let $\hat{B}_2 = x$



8.1	Name, with reasons, 3 angles equal to x	6	**L2**
8.2	Prove that PC = BC	2	**L2**
8.3	Prove that RCQB is a cyclic quadrilateral.	2	**L3**
8.4	Prove that ΔPBS ||| ΔQCR	5	**L3**
8.5	Show that PB.CR = QB.CP	4	**L4**

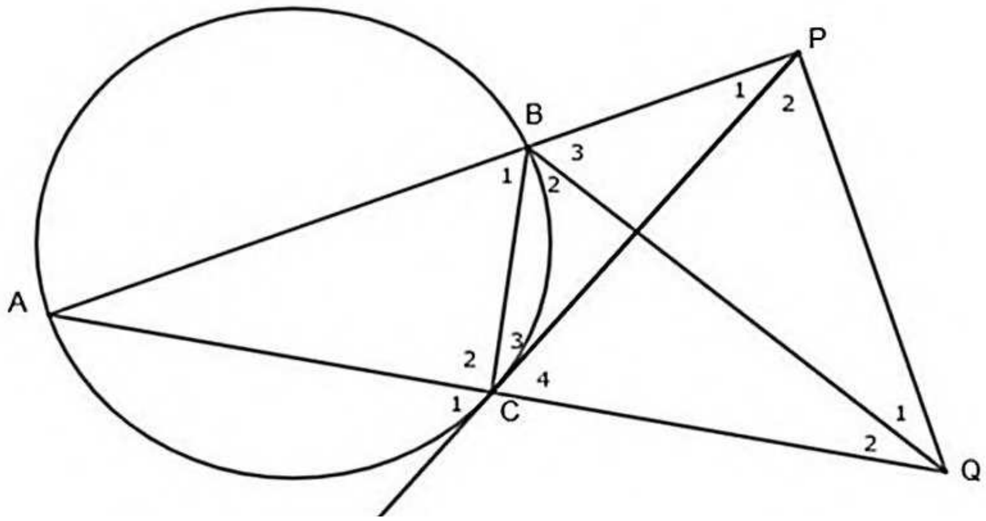
9. In the diagram below: AB is a common tangent to the two circles at B. AD is a tangent to the bigger circle at D. CD = CG. $\hat{G} = x$.

prove that:



9.1	AEC ∥ DG	(4)	**L3**
9.2	ABCD is a cyclic quadrilateral.	(5)	**L3**
9.3	$\frac{BE}{ED} = \frac{BC}{CD}$	(3)	**L4**

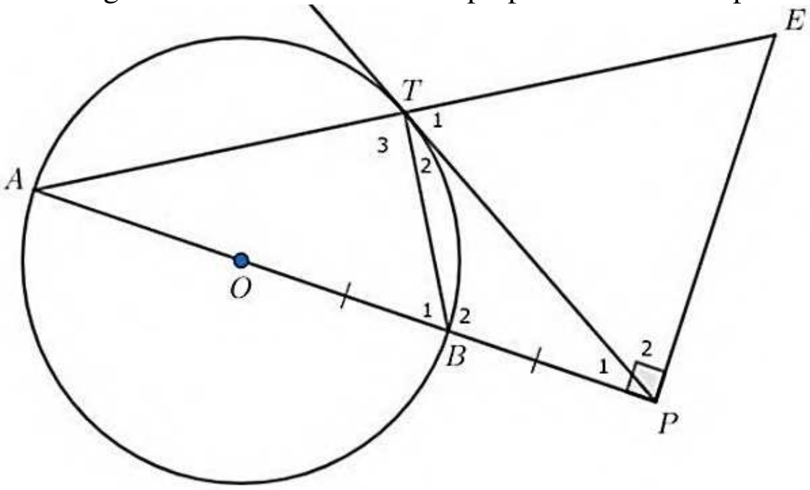
10. A, B and C are concyclic. AB produced meets the tangent through C at P. AC is produced to Q so that PQ = PC.



10.1	If $\hat{C}_1 = x$, determine, with reasons THREE other angles each equal to x.	(6)	**L2**
10.2	Prove that: (a) BCQP is a cyclic quadrilateral.	(2)	**L3**
10.3	(b) PQ is a tangent to circle ABQ.	(4)	**L3**
10.4	(c) $PQ^2 = PA.PB$.	(5)	**L3**

SACAI 2020

11 In the diagram along is AB the diameter of the circle with centre O. BP = OB and PT is a tangent to the circle at T. EP is perpendicular to AP.p



Prove that:

11.1 TBPE is a cyclic quadrilateral. (2) **L2**

11.2 PT = PE (4) **L3**

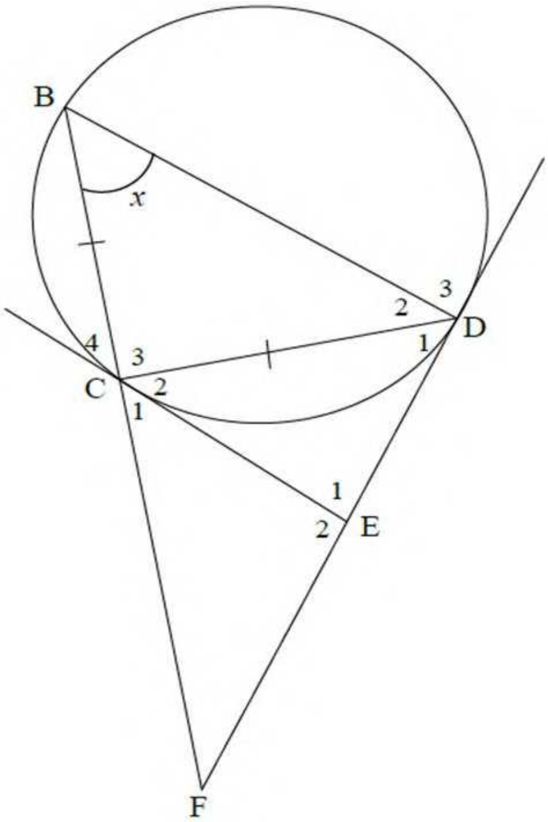
11.3 AT.AE = AB. AP (4) **L3**

11.4 $PT^2 = PB.PA.$ (4) **L3**

11.5 $2PE^2 = AT.AE.$ (4) **L4**

DBE NOVEMBER 2020(2)

12 In the diagram below, B, C and D are points on a circle such that BC = CD. EC and ED are tangents to the circle at C and D respectively. BC produced meets tangent DE produced at F. $\hat{B} = x$.



Prove, giving the reasons.

12.1 $\hat{E}_1 = 180° - 2x$ (5) **L2**

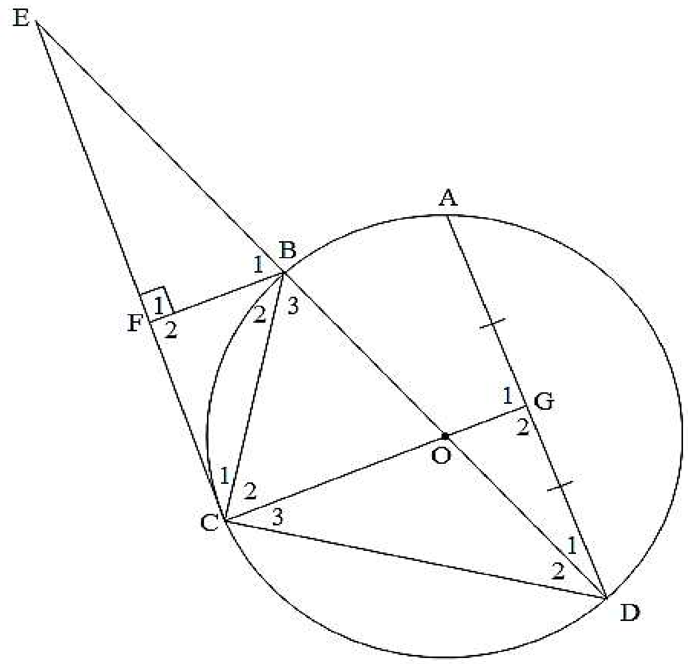
12.2 ΔECD ||| ΔCBD (3) **L2**

12.3 $CD^2 = CE . BD$. (3) **L3**

12.4 $\frac{CF^2}{EF^2} = \frac{BD}{DE}$ (6) **L4**

MAY/JUNE 2022

13 In the diagram, O is the centre of a circle passing through A, B, C and D. EC is a tangent to the circle at C. Diameter DB produced meets tangent EC at E. F is a point on EC such that BF ⊥ EC. Radius CO produced bisects AD at G. BC and CD are drawn.



Prove, with reasons, that:

13.1 FB || CG (3) **L2**

13.2 ΔFCB ||| ΔCDB (5) **L2**

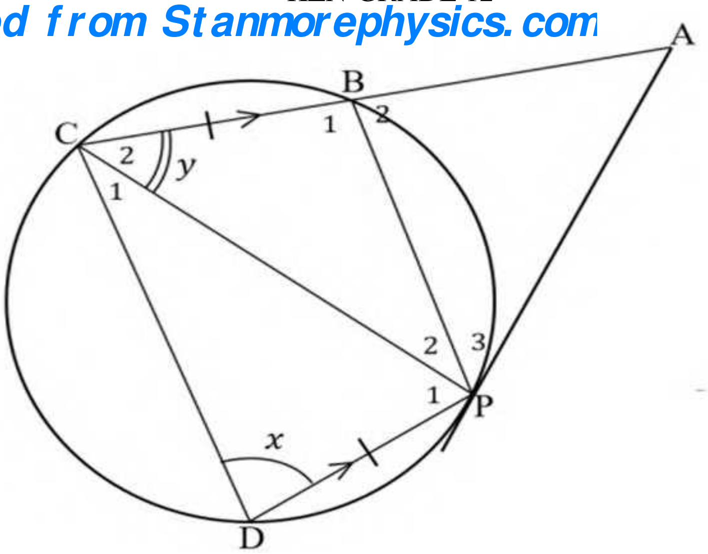
13.3 Give a reason why $\hat{G}_1 = 90°$. (1) **L1**

13.4 Prove, with reasons, that $CD^2 = CG.DB$. (5) **L3**

13.5 Hence, prove that DB = CG + FB. (5) **L4**

WC 2019 TRIAL

14 AP is a tangent to the circle at P. CB ∥ DP and CB = DP. CBA is a straight line. Let $\widehat{D} = x$ and $\hat{C}_2 = y$



Prove with reasons that,

14.1 ΔAPC ||| ΔABP. (4) **L2**

14.2 $AP^2 = AB.AC$ (2) **L3**

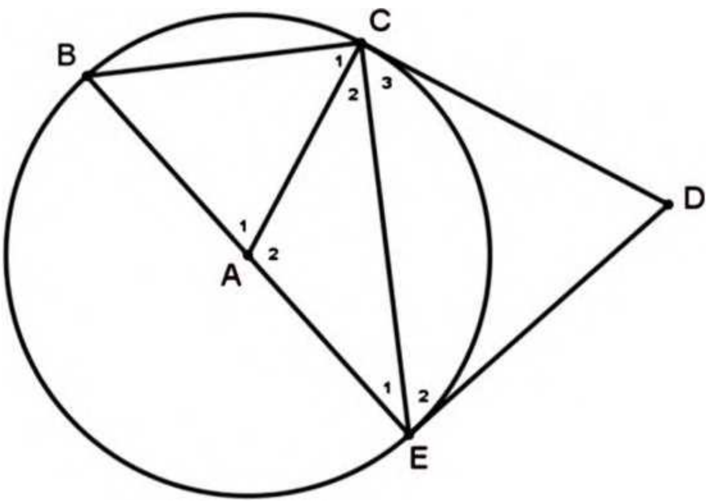
14.3 ΔAPC ||| ΔCDP. (4) **L3**

14.4 $AP^2 + PC^2 = AC^2$ (4) **L4**

IEB NOVEMBER 2019

15 In the diagram below:

- *DC* and *DE* are tangents to the circle at *C* and *E* respectively.
- *A* is the centre of the circle.
- *B* lies on the circle and *BAE* is a straight line.

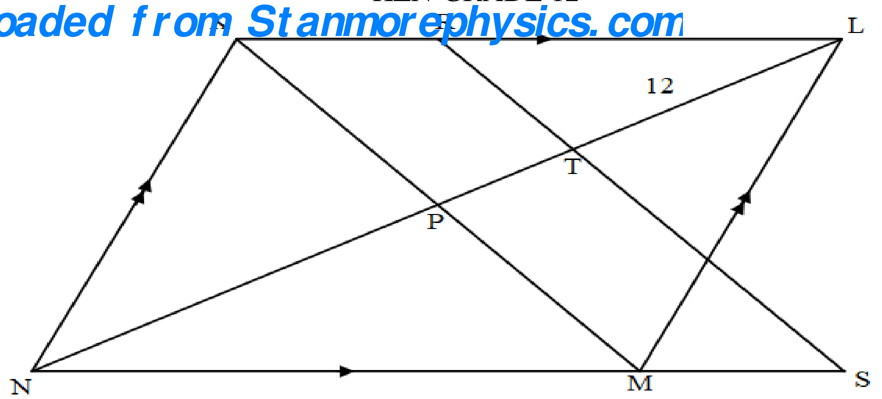


15.1 Prove that $\Delta ABC \parallel\mid \Delta DEC$. (5) **L3**

15.2 Hence, show that $AE.EC = BC.DE$. (3) **L2**

MAY/JUNE 2022

16 In the diagram, the diagonals of parallelogram KLMN intersect at P. NM is produced to S. R is a point on KL and RS cuts PL at T. NM **:** MS = 4 **:** 1,NL = 32 units and TL = 12 units.

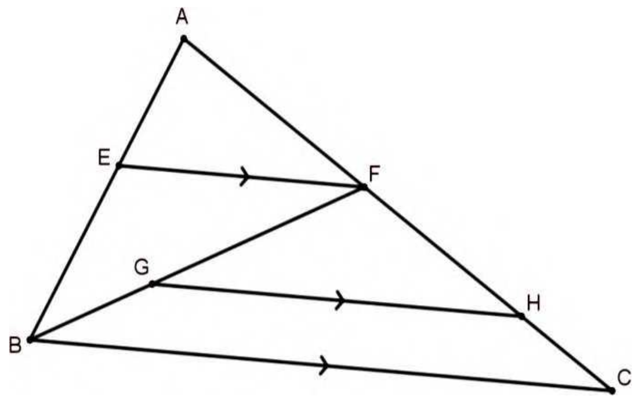


16.1 Determine, with reasons, the value of the ratio NP **:** PT in simplest form. (4) **L2**

16.2 Prove, with reasons, that KM ‖ RS. (2) **L2**

16.3 If NM = 21 units, determine, with reasons, the length of RL. (4) **L3**

17 In the diagram below: E lies on AB and F on AC in Δ APC with $EF \parallel BC$. G lies on FB and H on FC in Δ FBC with $GH \parallel BC$. 5AE=4EB and $\frac{FG}{FB}=\frac{5}{8}$.

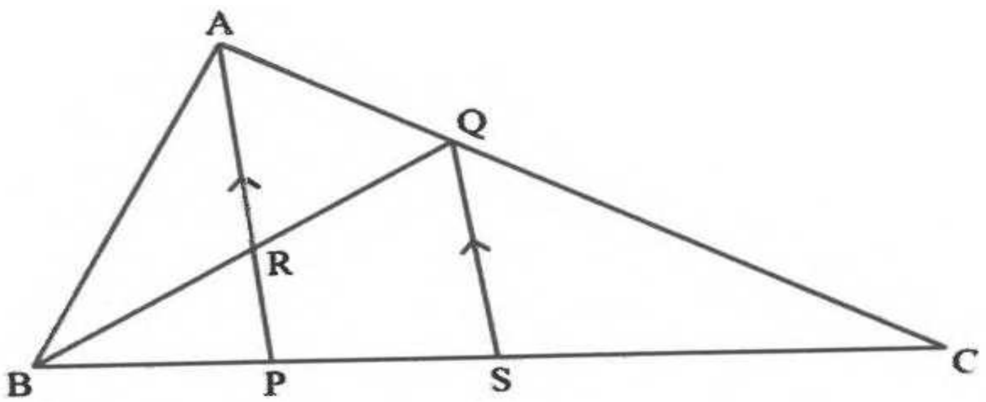


17.1 Calculate the value of $\frac{AF}{AC}$ (2) **L2**

17.2 Calculate the value of $\frac{HF}{AF}$ (4) **L3**

KZN JUNE 2022

18 In ΔABC, AQ : QC = 1 : 3. AP//QS with P and S on BC and Q on AC. BQ intersects AP in R. BP = $\frac{1}{3}BC$.



Determine with reasons, the following:

18.1 $\frac{BP}{PS}$ (5) **L3**

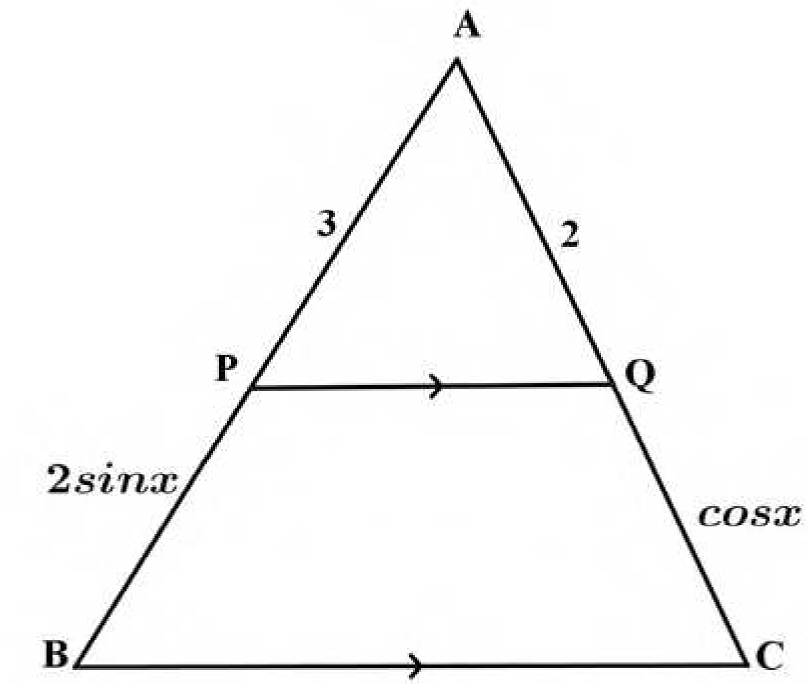
18.2 $\frac{BR}{QR}$ (2) **L2**

18.3 $\dfrac{\text{Area } \Delta \text{ APC}}{\text{Area } \Delta \text{ BQC}}$ (3) **L3**

18.4 $\dfrac{\text{Area } \Delta \text{ ABC}}{\text{Area } \Delta \text{ APC}}$ (3) **L3**

18.5 $\dfrac{\text{Area } \Delta \text{ QSC}}{\text{Area } \Delta \text{ ABP}}$ (4) **L4**

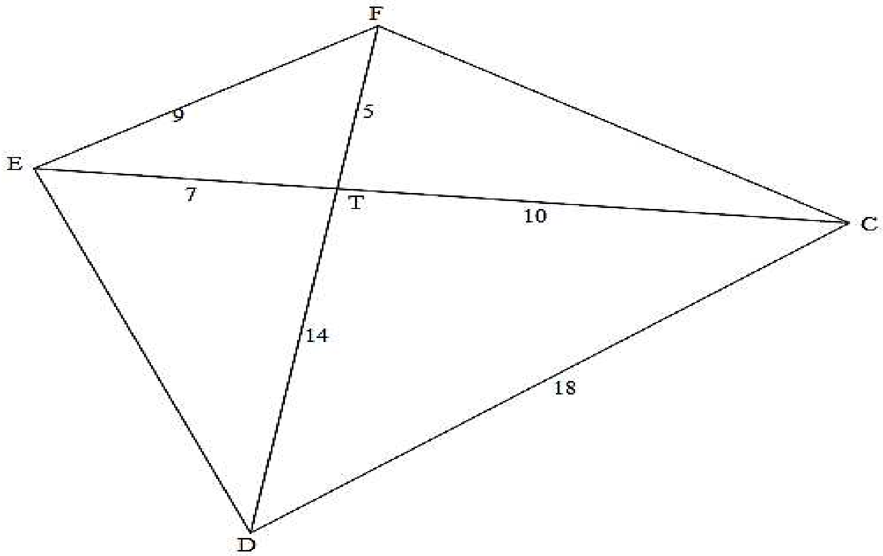
19 In ΔABC $PQ \parallel BC$. $AP=3$, $PB=2\sin x$, $AQ=2$, $QC=\cos x$



19.1 Calculate the value of x, $x \in [-180°\ ;\ 180°]$ (4) **L3**

DBE NOVEMBER 2019

20 In the diagram, the diagonals of quadrilateral CDEF intersect at T. EF = 9 units, DC = 18 units, ET = 7 units, TC = 10 units, FT = 5 units and TD = 14 units.



Prove with reasons:

20.1 $E\hat{F}D = E\hat{C}D$ (4) **L3**

20.2 $D\hat{F}C = D\hat{E}C$ (3) **L4**

ANSWERS: ALGEBRA

1.1.1	$x=0$ or $x=2$
1.1.2	$x=2,77$ or $x=-1.27$
1.1.3	$x=-1$ or $x=-2$
1.1.4	$x<-1$ or $x>3$
1.2	$x=2$ or $x=-\frac{1}{5}$ $y=7$ or $y=\frac{13}{5}$
1.3	$x=\frac{1}{5}$
2.1.1	$x=0$ or $x=-6$
2.1.2	$x=-2,39$ or $x=-0,28$
2.1.3	$-8\leq x\leq 8$
2.1.4	$x=4$ or $x\neq -1$
2.2	$x=\frac{2}{3}$ or $x=1$ $y=\frac{4}{3}$ or $y=-1$
3.1.1	$x=6$ or $x=-3$
3.1.2	$x=-4,37$ or $x=3,41$
3.1.3	$x\leq 5$
3.1.4	$x=7$ or $x\neq -2$
3.1.5	$x=9$
3.2	$x=5$; $y=15$
4.1.1	$x=\pm 1$
4.1.2	$x=-2.41$ or $x=-2.41$
4.1.3	$x=16$ or $x=1$
4.1.4	$x=\frac{10}{9}$
4.2	$x=-2$ or $x=3$ $y=-10$ or $y=5$
5.1.1	$x=2$ or $x=-5$
5.1.2	$x=-3$ or $x=1$
5.1.3	$-1\leq x\leq 4$
5.1.4	$x=3$
5.2	$y=-\frac{3}{2}$ or $y=-1$ $x=0$ or $x=1$
5.3	$p\leq 4$
6.1.1	$x=-0,37$ or $x=5,37$
6.1.2	$x\leq -3$ or $x\geq 12$
6.1.3	$x=4$
6.1.4	$x=10$ or $x\neq 5$

6.2.1	$n=-5$
6.2.2	$m=\frac{3}{2}$
6.2.3	$m\in R$
7.1.1	$x=0$ or $x=-3$
7.1.2	$x=9$ or $x\neq 4$
7.1.3	$x=1$
7.2	$p=2$
8.1.1	$x=2$ or $x=-1$
8.1.2	$x=-1,37$ or $x=2,89$
8.1.3	$x\leq -1$ or $x\geq \frac{2}{3}$
8.2.1	$\frac{x}{y}=-4$ or $\frac{x}{y}=2$
8.2.2	$y=-2$ or $y=2$ $x=8$ or $x=4$
8.3	27
9.1.1	$x=5$ or $x=-4$
9.1.2	$x=0,73$ or $x=4,77$
9.1.3	$x<-\frac{1}{5}$ or $x>4$
9.1.4	$x=3$
9.2	$x=2$ or $x=-1$ $y=3$ or $y=-3$
9.3.1	$k=-2$; $k=2$
9.3.2	$k=-3$
9.4	$a=2$; $b=1006$
10.1.1	$x=3$ or $x=-2$
10.1.2	$x=-0,17$ or $x=-5,83$
10.1.3	$x\leq 0$ or $x\geq 3$
10.1.4	$x=2$
10.2	$y=1$ or $y=2$ $x=4$ or $x=5$
10.3.1	No. $TP(1;5)$ is above 3.
10.3.2	$k>2$
11.1.1	$x=-6$ or $x=1$
11.1.2	$x=0,80$ or $x=-1,55$
11.1.3	$-\frac{1}{2}<x<\frac{1}{2}$
11.1.4	$x=2$
11.2	$x=14$ or $x=1$ $y=-2$ or $y=11$
11.3	$k=26$

12.1.1	$x=0$ or $x=5$
12.1.2	$x=1{,}15$ or $x=-0{,}35$
12.1.3	$x=-1$ or $x\neq\frac{7}{4}$
12.1.4	$x=\pm2$
12.2	$y=-6$ or $y=2$ $x=-2$ or $x=6$
12.3	$a=4$; $b=4$

13.1.1	$x=\frac{3}{5}$ or $x=-\frac{3}{2}$
13.1.2	$x=6{,}742$ or $x=6{,}742$
13.1.3	$x<-3$ or $x>2$
13.1.4	$x=7$ or $x\neq-6$
13.2.1	14
13.2.2	52
13.3	$y=7$ or $y=-2$ $x=11$ or $x=-1$

ANSWERS: SEQUENCES AND SERIES

1.	1.1	59
	1.2	$T_n=-n^2+17n-1$
	1.3	-271
2.	2.1	50
	2.2	$T_n=-2n^2+24n-20$
	2.3	52
	2.4	Invalid due to a negative answer
3.		$T_n=-\frac{1}{2}n-\frac{9}{2}n+12$
4.	4.1	T_{484} and T_{485}
	4.2	$c=30$
5.	5.1	$x=2$
	5.2	$T_5=17$
6.	6.1	$r=\frac{2}{3}$
	6.2	$k=\frac{4}{3}$
	6.3	$n=8$
7.		$5;3;1$ or $5;7;9$
8.	8.1	$n=20$
	8.2	$S_{20}=630$
	8.3	1200
9.		324000
10.	10.1	$x=100$
	10.2	Proof
	10.3	$S_\infty=1000$
11.	11.1	7
	11.2	41
12.		$T_5=-7$
13.	13.1	$T_{51}=105$
	13.2	$S_{51}=2805$
	13.3	$5+7+9+...+10003$
	13.4	2001

14.	14.1	$1+\frac{1}{2}+\frac{1}{4}+...$
	14.2	$\frac{1023}{512}$
15.	15.1	$S_\infty=\frac{1}{2}$
	15.2	Reasoning
16.		$4<t<6$
17	17.1	$x=\frac{4}{3}$
	17.2	$\sum_{n=1}^{\infty}3(\frac{2}{3})^{n-1}$
18.		$r=\frac{1}{3}$
19.		$k=-398099$
20	20.1	$r=2p-1$
	20.2	$0<p<1$
21	21.1	$S_{800}=80800$
	21.2	$T_{100}=1402$
	21.3	$n=45$
22.	22.1	8
	22.2	4
	22.3	Proof
	22.4	$\frac{1-p^2}{p}$
23.		$p=10$
24.		$x=30^\circ$
25.		$p=3$
26.		$n=14$
27.	27.1	32%; $33{,}60\%$; $35{,}28\%$
	27.2	54,73
	27.3	509,35
	27.4	42,45

28.		$r=\frac{3}{2}$
29.	29.1	$T_n=\frac{1}{3}(2)^{n-1}$
	29.2	85
30.		-189
31.		Proof
32.	32.1	Proof
	32.2	$r=\frac{3}{4}$

33.		$y=-20{,}175$
		$T_n=\frac{2n+3}{3n^2-2n+1}$
34.		27 or 2 years 3 months
35.		Proof
36.		$0^\circ < \theta < 30^\circ$
37.		$d=3$
38.		$S_\infty=27$

ANSWERS TOPIC: FUNCTIONS

1		
	1.1	**Graph**
	1.2	$y=-20\frac{1}{4}$
	1.3	$-20\frac{1}{4}< k <-14$
	1.4	$h(x)=-2x$
2.		
	2.1	$S(2;0)$
	2.2	$y=-2(x+1)^2+18$
	2.3	$T(-3;10)$
	2.4	$x<-3$ or $x>2$ $(-\infty;-3)\cup(2;\infty)$
	2.5.1	$x<-1$ or $(-\infty;-1)$
	2.5.1	graph
3		
	3.1	$C(0;-3)$
	3.2	$AB=4units$
	3.3	$D(1;-4)$
	3.4	$m=-1$
	3.5	$OCB=45°$
	3.6	$-4<k<-3$ or $(-4;-3)$
	3.7	$x>1$
4		graph
5		
	5.1	$A(-2;0)$ and $B(4;0)$
	5.2	$C(1;18)$
	5.3	$y\le 18$ or $y\in(-\infty;18]$
	5.4	$p=-1$ $q=-3$
	5.5	$y=\frac{1}{2}x-2$
	5.6	$x=4$ or $x=-2$
	5.7	$k<-12{,}5$
6		
	6.1	$E(-3;16)$
	6.2	$k=12$
	6.3	$y=-x+7$
	6.4	$P\left(-\frac{5}{2};\frac{63}{4}\right)$
	6.5	$-5<x<-1$ or $(-5;-1)$
7		
	7.1	$(0;3)$
	7.2	$C(-1;4)$
	7.3	$A(-3;0)$
	7.4	$CE=\sqrt{5}/2{,}24units$
	7.5	$k=7$
	7.6	$y=\frac{x-6}{2}$ or $y=\frac{x}{2}-3$
	7.7	$x\ge -6$
8		
	8.1	$x=-2$ $y=3$
	8.2	$\left(0;\frac{3}{2}\right)$
	8.3	$\left(-\frac{7}{3};0\right)$
	8.4	graph
9		$x=1$ $y=1$
10	10.1	$x=2$ $y=1$
	10.2	graph
11		
	11.1	$A(4;3)$

	11.2	$B\left(0;\frac{5}{2}\right)$
	11.3	$C(2;0)$
	11.4	$m=-\frac{3}{4}$
	11.5	$y=-x+7$
12		
	12.1	$q=1$
	12.2	$a=2 \quad p=-2$
	12.3	graph
13		
	13.1	proof
	13.2	$y\in\Re; \quad y\neq 1$
	13.3	$y=x+3$
	13.4	$K'(-5;2)$
14		
	14.1	$x\in\Re; \quad x\neq -1$
	14.2	$x=-\frac{3}{2}$
	14.3	$k=2$
	14.4	$C(2;4)$
	14.5	$y=-(x-2)^2+4$
	14.6	$x\leq -\frac{3}{2} \quad or \quad -1<x<0 \quad or \quad x>4$
	14.7	One real root
15		
	15.1	$a=4 \quad p=-1 \quad q=-8$
	15.2	$k=3$
	15.3	$0\leq x\leq 1$
	15.4	$-8<k<-4$
	15.5	$y=-x-\frac{1}{2}$
	15.6	$Q\left(\frac{15}{2};-\frac{3}{2}\right)$
16		
	16.1	$(0;2)$
	16.2	*Graph*
	16.3	$m=-\frac{7}{6}$
	16.4	$y=3$
17		
	17.1	$y>0 \quad or \; y\in(0;\infty)$
	17.2	$y=\log_{\frac{1}{2}} x$
	17.3	*yes* *any valid reason*
	17.4.	

		17.4.1	$a=\frac{1}{4}$
		17.4.2	$M'\left(2;\frac{1}{4}\right)$
	17.5	$M''\left(-1;\frac{9}{4}\right)$	
18			
	18.1	$y=\log_3 x$	
	18.2	$P'(2;9)$	
	18.3	$q=-16 \quad p=2$	
19			
	19.1	$T(0;1)$	
	19.2	$a=3$	
	19.3	$h(x)=\left(\frac{1}{3}\right)^x$	
	19.4	$1<x<3$	
20			
	20.1	$y=\log_b x$	
	20.2	$y=x$	
	20.3	$P(1;0)$	
	20.4	$y=-x+1$	
	20.5	$b=\frac{1}{4}$	
21			
	21.1	$0<x\leq 1 \quad or \quad (0;1]$	
	21.2	$p=2$	
	21.3	$y=\left(\frac{4}{3}\right)^x$	
	21.4	$y>0 \quad or \quad y\in(0;\infty)$	
	21.5	$B''\left(-2;\frac{16}{9}\right)$	
22			
	22.1	$a=\frac{1}{3}$	
	22.2	$\left(0;\frac{22}{3}\right)$	
	22.3		
		22.3.1	7 *unit down*
		22.3.2	$y=\log_{\frac{1}{3}} x$
23			
	23.1	$U(1;0)$	
	23.2	$x=1$ $y=1$	
	23.3	$T(-1;0)$	
	23.4	$y=5^x$	

	23.5	$y=-6$
	23.6	$V(2,41;2,41)$
	23.7	$T'(3;2)$
24		
	24.1	$r=2$
	24.2	$p=-3$

	24.3	$A\left(-3\,;\frac{17}{8}\right)$
	24.4	$-3<x\leq 0$ or (-3 ;0]
	24.5	$h(x)=\frac{3}{x+1}+2$

ANSWERS
TOPIC 1: FINANCIAL MATHEMATICS

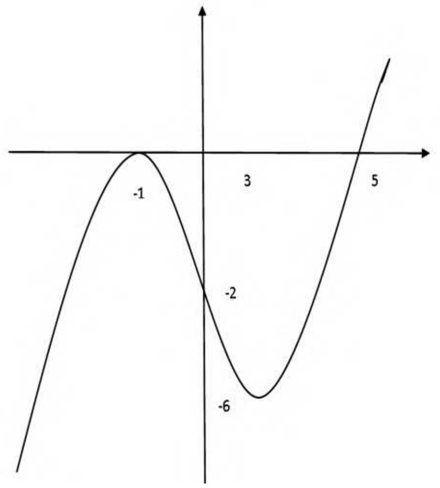
1	R3 037,50
2.1	155 payments
2.2	R3 230,50
2.3	R3 278,96
2.4	R773 278,96
3.1	R19 694,79
3.2	R1 588 473,03
3.3	R1 181 687,40
3.4	R770 160,43
4	12%
5.1	234 payments
5.2	R10 632,39
6.1	R1 034 939,44
6.2	R2 944 096.27
6.3	R1 909 156,83
6.4	R26 666,85
6.5	R27 070,32

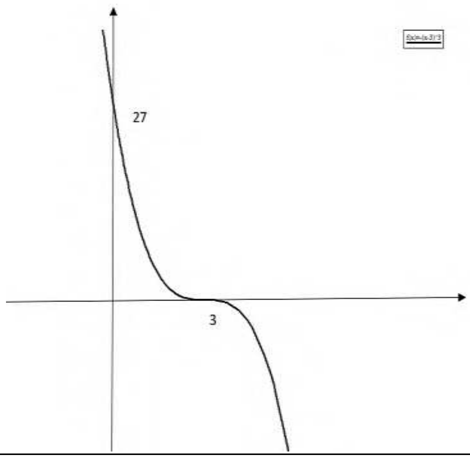
7.1	R74 883,86
7.2	R168 306,21
7.3	R1 184,62
8	R57 934,44
9	8,24 years
10.1	18,48%
10.2	R678 635,11
10.3	R6 510,36
11	4 years
12	66,04 months
13	1,8:1
14	R791 000
15.1	R718 305,71
15.2	R273421,38
16.1	R7 982,73
16.2	R216 021,16
16.3	27 months

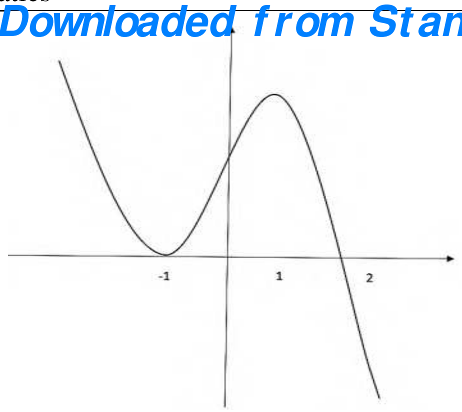
ANSWERS: CALCULUS

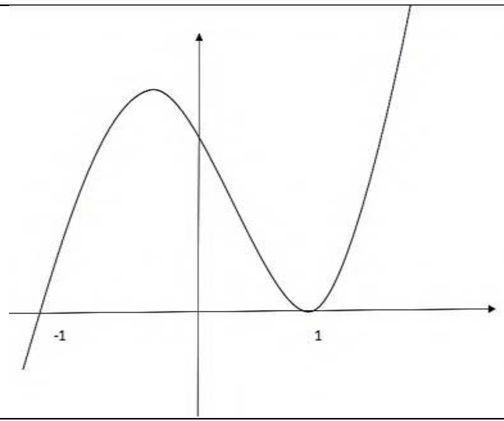
1.1	$-10x$
1.2	$2x+2$
1.3	$\frac{2}{x^2}$
1.4	$-6x$
1.5	$-2x$
1.6	$2ax$
1.7	-7
2.1	$98x-42$
2.2	$\frac{5}{2}x^{\frac{3}{2}}-6x^{\frac{1}{2}}+\frac{5}{2x^{\frac{3}{2}}}$
2.3	$-2x-5$
2.4	$2x+2$
2.5	$-\frac{1}{4x^{\frac{1}{2}}}+\frac{2}{x^3}$
2.6	3
2.7	$3x^2-4-\frac{4}{x^2}$
2.8	$9x^2a^4-a^5$

2.9	$\frac{22}{x^{\frac{1}{2}}}-\frac{8}{x^5}$
2.10.1	$2ax$
2.10.2	x^2+1
2.11.1	$9x^2+12x+1$
2.11.2	2
3.1	$y=3x-4$
3.2	-6
3.3	$\left(-\frac{1}{2};-\frac{1}{2}\right)$
3.4	It is a tangent at $(5;-5)$
3.5	$a=-2$ $b=7$
3.6	$y=2x+3$
3.7	$b=-6$ $c=12$
3.8	$y=\frac{1}{2}x+7$
4.1	$D(0;12)$

4.2	$B\left(-\frac{4}{3};\frac{500}{27}\right)$ $C(2;0)$
4.3	Proof
4.4	$C'(-2;1)$
4.5.1	$k=0$ or $k=\frac{500}{27}$
4.5.2	$k=12$
5.1	Showing
5.2	$L\left(-\frac{2}{3};\frac{400}{27}\right)$ $M(4;-36)$
5.3	$g(x)=4x-24$
5.4	$x=-3$ $AM=7units$
5.5.1	$x<-\frac{2}{3}$ or $x>4$
5.5.2	$x<\frac{5}{3}$
6.1	
6.2	$x=1$
6.3	$-1<x<3$
7.1	Proof
7.2	$x=2.12$
7.3	f is decreasing between A and B
7.4	$x>\frac{2}{3}$
7.5	Max = 10
8.1	Proof
8.2	$B\left(\frac{5}{3};-\frac{256}{27}\right)$
8.3.1	$-1\le x\le 3$
8.3.2	$x<-1$ or $x>\frac{5}{3}$

8.3.3	$x>\frac{1}{3}$
8.4	Length=2.52 *units*
9.1	Proof
9.2	$x=0$ $-\frac{1}{2}$ $y=24x-44$ $x=3$ $f(x)$ or $x=-\frac{7}{3}$ $f'(x)$ $\rightarrow$ $x>3$ $x=\frac{1}{2}$
9.3	$x<-\frac{7}{3}$ or $x>0$
9.4	$x=0$ or $x=-3$
10.1	A (-2;20)
10.2	$x>-\frac{1}{2}$
10.3	$y=24x-44$
11	
12.1	-4
12.2	$x=-3$ or $x=1$
12.3	$x=-1$
12.4.1	$x<-3$ or $x>1$
12.4.2	$-3<x<1$
13.1	$x=0$ / $x=3$
13.2.1	$f(x)$ & $f'(x)$
13.2.2	$f(x)\rightarrow$ point of inflection $f'(x)\rightarrow$ min TP
13.3	Length = 9 *units*
13.4	$x>3$

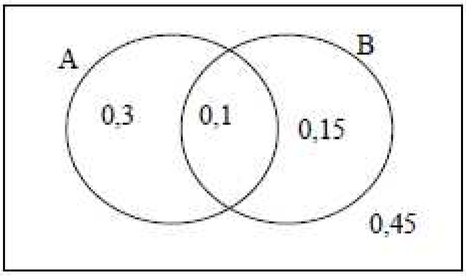
14	
15.1	Proof
15.2	$x=\frac{1}{2}$
16.1	(0;1)
16.2	(1;0) / (-1;0)
16.3	$\left(-\frac{1}{3};\frac{32}{27}\right)$ / (1;0)
18.1	Proof
18.2	$x=7$
18.3	73,5 *square units*
19.1	Proof
19.2	$x=10$
19.3	360*square units*
20.1	$\frac{2048\pi}{3}$
20.2	Proof
20.3	8 :27

16.4	
16.5	$\frac{-1}{3}<x<1$
17.1	$A=-x^2+3x+10$
17.2	$x=\frac{1}{3}$
21.1	16 *m*
21.2	3 seconds
21.3	10 *m/s*
22.1	36 *cm*
22.2	Once
22.3	52 *m*
23.1	3:2
23.2	$t=5$

ANSWERS
TOPIC: PROBABILITY

1.1	0,99
1.2	0,01
2.1	24
2.2	8
3.1	479 001 600 different ways
3.2	362 880 different ways
3.3	144 different ways.
4.1	5040
4.2	120
4.3	720
5.1.1	0,46
5.1.2	$\frac{59}{90}$
5.2	the events are not independent
6.1	120
6.2	$\frac{1}{10}$
7.1	823 543
7.2	5040
7.3	1440

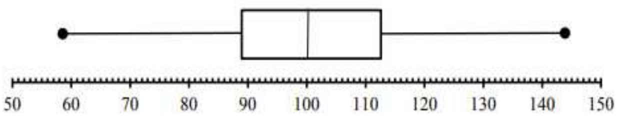
8	There are 3 orange balls in the bag
9.	160
10.1	160
10.2	$\frac{3}{8}=0{,}375$
10.3	$b=30$
11.1	20 160
11.2	4.76%
12.1	$a=20$ $b=128$
12.2	No. n(M and not watching) $\neq 0$
12.3.1	80%
12.3.2	7,5%
13.1	3 628 800
13.2	120 960
13.3	0.000198 or $\frac{1}{5040}$
14	$y=0{,}29$
15.1	80%
15.2	10.5%
15.3	71.5%

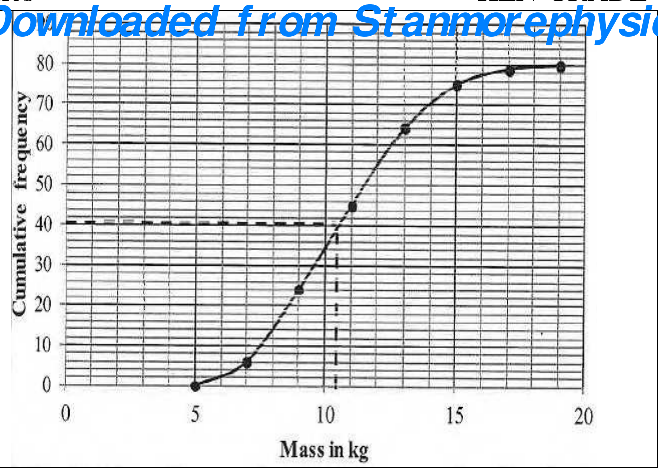
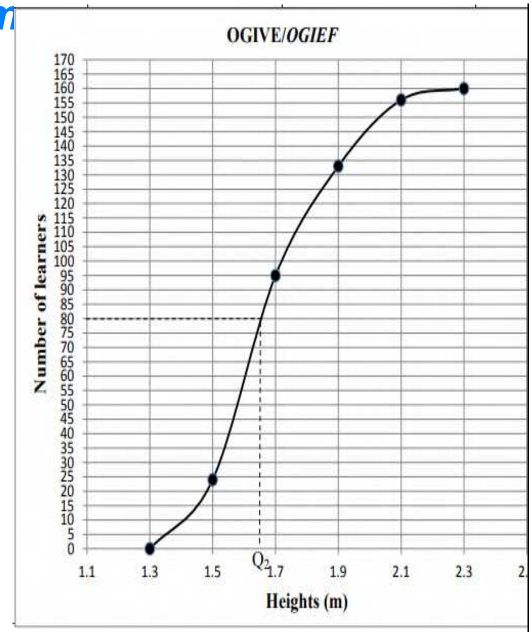
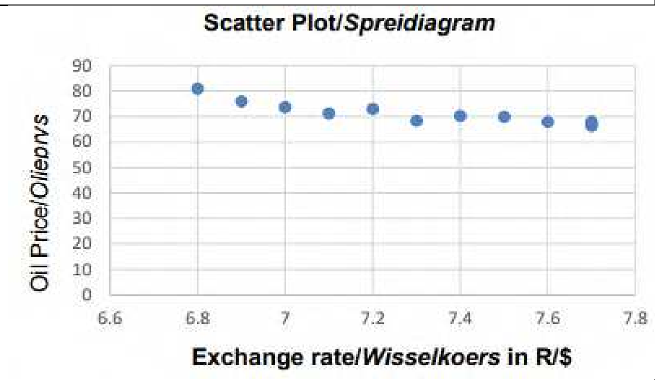
16.1	0.3
16.2	0.2
16.3	Not independent.
17.	15 bags
18.1	90
18.2.1	3 628 800
18.2.2	768
19.1	
19.2	0.85

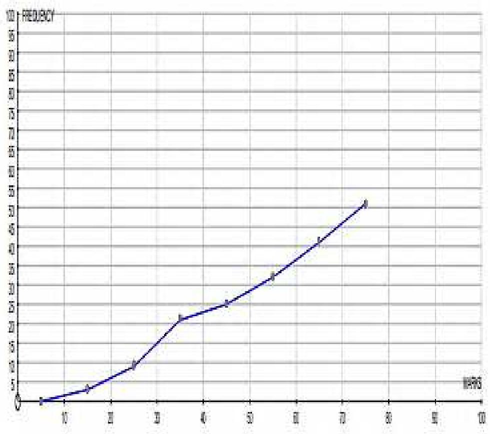
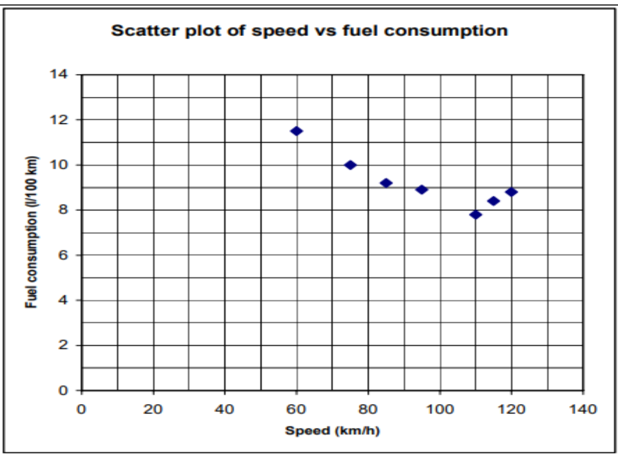
20.1	$\frac{1}{4}$
20.2	$\frac{1}{2}$
20.3	$\frac{2}{3}$
21.1	648
21.2	144
22.	$\frac{3}{10}$
23.1	9 261 000
23.2	$\frac{1}{21}$
23.3	$\frac{243}{1000}$

ANSWERS

TOPIC: STATISTICS

<table>
<tr><td></td><td></td><td>5.1</td><td>July</td></tr>
<tr><td>1.1</td><td>101,47</td><td>5.2</td><td>2 245, 08 aircraft landings</td></tr>
<tr><td>1.2</td><td>19,07</td><td>5.3</td><td>86, 30</td></tr>
<tr><td>1.3</td><td>Q1 = 89 ; Q3 = 113</td><td>5.4</td><td>There were 9 months</td></tr>
<tr><td>1.4</td><td></td><td>5.5</td><td>The standard deviation of the number of landings at the Port Elizabeth Airport will be higher than the standard deviation of the number of arrivals at the King Shaka International Airport or C</td></tr>
<tr><td>1.5</td><td>2 days</td><td>6.1</td><td>Range = 26 – 4 =22</td></tr>
<tr><td>2.1</td><td>The data is skewed to the left/Negatively skewed</td><td>6.2</td><td>Mean = 15,25</td></tr>
<tr><td>2.2</td><td>60</td><td>6.3</td><td>Standard deviation = 7,6</td></tr>
<tr><td>2.3</td><td>25% of the learners failed.</td><td>6.4.1</td><td>Increase in mean is 2°C per month.</td></tr>
<tr><td rowspan="2">2.4</td><td rowspan="2"><table><tr><td>20</td><td>28</td><td>36</td><td>41</td><td>62</td><td>69</td><td>75</td><td>75</td><td>80</td></tr></table></td><td>6.4.2</td><td>Range of the data will decrease, This will result in the SD getting smaller.</td></tr>
<tr><td>7.1</td><td>Positively skewed/Skewed to the Right.</td></tr>
<tr><td>3.1</td><td>$9 < m \leq 11$</td><td>7.2</td><td>0, 82 m</td></tr>
<tr><td>3.2</td><td><table><tr><th>Mass (in kg)</th><th>Frequency</th><th>Cumulative frequency</th></tr><tr><td>$5 < m \leq 7$</td><td>6</td><td>6</td></tr><tr><td>$7 < m \leq 9$</td><td>18</td><td>24</td></tr><tr><td>$9 < m \leq 11$</td><td>21</td><td>45</td></tr><tr><td>$11 < m \leq 13$</td><td>19</td><td>64</td></tr><tr><td>$13 < m \leq 15$</td><td>11</td><td>75</td></tr><tr><td>$15 < m \leq 17$</td><td>4</td><td>79</td></tr><tr><td>$17 < m \leq 19$</td><td>1</td><td>80</td></tr></table></td><td>7.3</td><td><table><tr><th>Intervals
Klasse</th><th>Cumulative frequency
Kumulatiewe frekwensie</th></tr><tr><td>$1,3 \leq x < 1,5$</td><td>24</td></tr><tr><td>$1,5 \leq x < 1,7$</td><td>95</td></tr><tr><td>$1,7 \leq x < 1,9$</td><td>133</td></tr><tr><td>$1,9 \leq x < 2,1$</td><td>156</td></tr><tr><td>$2,1 \leq x < 2,3$</td><td>160</td></tr></table></td></tr>
</table>

3.3		7.4	
3.4	10,5 kg		
3.5.1	10,68		
3.5.2	Learners' bags are heavier than the stipulated international guidelines. 10, 68 kg is greater than 8 kg.		
4.1	Modal class is $50 \leq x < 60$ **OR** 50 to 60	7.5	1,65 (accept any value between 1,6 and 1,69)
4.2	Median position is 15 learners (grouped data). Approximate weight is about 53 kg. (Accept from 52 kg to 54 kg)	7.6.1	The mean would change by 0,1 m
		7.6.2	No influence/change
4.3	7 learners collected more than 60 kg		
8.1	6,73	13.1	$2x$
8.2	Median is 7	13.2	$\frac{12x+22}{11}$
8.3	Standard deviations is 2,26		
8.4	19 times		
9.1		13.3	8 ; 10 ; 4 and 6 $sd(\sigma) = \sqrt{5}$
		14.1	(41 ; 26)
		14.2	quadratic
		14.3	The younger or older the participants are, the longer they will take to complete the item. They do not have the required strength, fitness and stamina. **OR** t would appear that swimmers close to 19 years completed the item in the shortest time. Swimmers of that age are normally in good physical condition and have lots of stamina.
9.2	$y = 158,67 - 11,96x$		
9.3	$r = -0,91$		
9.4	Exchange rate increase, oil price decrease OR Strong Negative correlation		
9.5	$y = 71,05$	14.4.1	The standard deviation will become smaller/decrease.
9.6	Standard deviation: $\sigma = 4,09$		
9.7	December	14.4.2	The mean will become smaller/decrease.
10.1	Skewed to the left or negatively skewed		
10.2	A = 65; B = 99	15.1	51
10.3	Units not counted:150	15.2	$m = 12$ and $n = 10$
11.1	150		
11.2	R1 590 000		

11.3	R10 600		
11.4	No, 110 of the 150 employees earn R7 500 or less **OR** Only 10 employees in this company earn more than R10 600. The majority (140) of the employees earn below this amount. It is therefore not a good indicator of the average monthly amount earned by an employee.	15.3	CUMUATIVE FREQUENCY 3 9 **21** 25 32 41 51
12.1	1, 72		
12.2.1	90 lies at 2 standard deviations to the right of the mean. 48% of the students scored between 72 and 90 marks.	15.4	
12.2.2	approximately 29 students scored between 45 and 63 marks.		
16	$a=7$ | $b=15$ | $c=17$ | $d=23$ | $e=34$ | $f=37$ | $g=42$		
17.1			
17.2	Quadratic	15.5	46 (Accept 45 – 47)
17.3	The quadratic pattern shows that the best fuel consumption occurs when the car is driven at 110 km/h. In order for the company to keep its fuel bill to a minimum, drivers should be urged to travel at this speed where possible.		

ANSWERS
TOPIC: ANALYITICAL GEOMETRY

		Feb/ March - 2018	
1.1		$x=1$	(1)
1.2		$m_{QP}=3$	(2)
1.3		$y=3x+19$	(2)
1.4		R (3; 0) $QR=\sqrt{104}$ or $2\sqrt{26}$	(2)
1.5		$\tan(90^0-\theta)=m_{QR}$	(3)

		$=\frac{1}{5}$	
1.6		$RN=\sqrt{26}$ $SR=6$ Area $\Delta RSN = 15$ units^2	(6)
2.1		**May/June – 2019**	

	2.1.1	Midpoint of EC = $(0\ ;\ \frac{-3}{2})$	(2)
	2.1.2	$M_{DC} = \frac{1}{2}$	(2)
	2.1.3	$y = \frac{1}{2}x + 1$	(3)
	2.1.4	$\theta = 90^0 + 26.57^0$ $\theta = 116.57^0$	(3)
2.2		Proof	(3)
	2.2.1	Centre of circle = $(0\ ;\ \frac{-3}{2})$	(1)
	2.2.2	$x^2 + (y + \frac{3}{2})^2 = \frac{25}{4}$	(4)
		Limpopo/ Sep - 2019	
	3.1	$m_{AB} = \frac{-1}{6}$	(2)
	3.2	$A\hat{B}C = 65.77^0$	(5)
	3.3	M(3 ; 0)	(2)
	3.4	y = -x +3	(3)
	3.5	K(-1 ; 4)	(8)
		May/June -2021	
	4.1	M (6;-4)	(2)
	4.2	$m_{NS} = 2$	(2)
	4.3	$y = -\frac{1}{2}x - 6$	(3)
	4.4	$x^2 + y^2 = 256$	(2)
	4.5	T (0;-1)	
4.6		$\frac{LS}{RS} = \frac{PS}{TS} = \frac{2}{3}$ OR $\frac{LS}{RS} = \frac{QS}{MS} = \frac{2}{3}$	(3)
4.7		Area of PTMQ = 25 units^2	(4)
		May/June 2022	

5.1		$\frac{1}{2}$	(2)
5.2		$y = \frac{1}{2}x - 7$	(3)
5.3	5.3.1	C (-6; -10)	(3)
	5.3.2	Area △BCD = 41,25 units^2	(4)
5.4	5.4.1	K (-6; -4)	(2)
	5.4.2(a)	Perimeter △KEC = 31,42 units	(4)
	5.4.2(b)	$K\hat{C}E = 63{,}43^0$	(3)
		Feb/ March - 2018	
6.1	6.1.1	OK = $\sqrt{108}$ or $6\sqrt{5}$	(1)
	6.1.2	$a^2 + b^2 = 180$ K (-12;-6)	(4)
6.2			
	6.2.1	y = -2x - 3	(3)
	6.2.2	M(-16 ; -8)	(6)
	6.3.3	$(x + 16)^2 + (y + 8)^2 = 20$	(2)
6.3		$6\sqrt{5} < r < 10\sqrt{5}$	(3)
6.4		Proof	(5)
		May/June - 2019	
7.1		Proof	(4)
7.2			
	7.2.1	K(2 ; -4)	(2)
	7.2.2	$y = \frac{4}{3}x + \frac{20}{3}$	(4)
	7.2.3	Area of $\Delta LPK = \frac{260}{3}$ units^2	(7)
7.3		The centre of the two circles lie on the same vertical line x = 2 and the sum of the radii = 10. n = 11 or n = -9	(4)

TRIGONOMETRY ANSWERS

1		
	1.1	OP = 5 units
	1.2	$\frac{7}{5}$
2		
	2.1	$k=-7$
	2.2	$\frac{-24}{25}$
	2.3	$\frac{-24}{7}$
	2.4	$\frac{-7}{25}$
3		
	3.1	$\cos\alpha=\frac{12}{13}$
	3.2	$90°-\alpha$
	3.3	OT = 19,5 units
4		
	4.1	Proof
	4.2	$\frac{1}{\sqrt{1+p^2}}$
	4.3	Proof
5		
	5.1	$\frac{-119}{169}$
	5.2	$\frac{-13}{5}$
6		
	6.1	$\frac{-7}{5}$
	6.2	$\frac{24}{7}$
7		
	7.1	$\frac{21}{221}$
	7.2	$-\frac{7182}{48841}$
8		
	8.1	$\frac{\sqrt{5}}{6}$
	8.2	$\sqrt{\frac{5}{6}}$
9		
	9.1	$\sqrt{1-k^2}$
	9.2	$1-2k^2$
	9.3	$2k\sqrt{1-k^2}$
	9.4	$\frac{-k^2}{\sqrt{1-k^2}}$
10		
	10.1	$\frac{p}{\sqrt{1-p^2}}$
	10.2	$2p\sqrt{1-p^2}$
	10.3	$\frac{\sqrt{2}}{2}(\sqrt{1-p^2}+p)$
	10.4	$\sqrt{\frac{1-\sqrt{1-p^2}}{2}}$
11		$\frac{1}{2}p+\frac{\sqrt{3}}{2}.\sqrt{1-p^2}$
12		
	12.1	$\sqrt{1-p^2}$
	12.2	$p.\sqrt{1-q^2}+q.\sqrt{1-p^2}$
13		
	13.1	$p+q$
	13.2	$p-q$
14		
	14.1	$\frac{\sqrt{3}}{3}$
	14.2	$\frac{\sqrt{2}}{3}$
	14.3	$\frac{\sqrt{2}(\sqrt{3}-1)}{4}$
	14.4	$-\frac{\sqrt{2}(\sqrt{3}-1)}{4}$
	14.5	$\frac{\sqrt[3]{2}}{2}$
	14.6	$\frac{\sqrt{3}}{3}$
15		
	15.1	$\tan x$
	15.2	$\frac{1}{\cos x}$
	15.3	$2\tan x$
	15.4	$\tan x$
	15.5	$\tan x$
16		
	16.1	Proof
	16.2	Proof
	16.3	Proof
	16.4	Proof
	16.5	Proof

17		
	17.1	Proof
	17.2	$x \in [90°, 270°]$
18		
	18.1	Proof
	18.2	$x \in [-180°, -90°, 0°, 90°, 180°]$
19		
	19.1	Proof
	19.2	$-1+\sqrt{2}$
20		
	20.1	$\theta = \pm 66,4° + k.360°, \quad k \in Z$
	20.2	$\theta = 103° + k.180°, \quad k \in Z$ $\theta = 160° + k.180°, \quad k \in Z$
	20.3	$\theta = 30° + k.180°, \quad k \in Z$ $\theta = 210° + k.180°, \quad k \in Z$
	20.4	$\theta = \pm 90° + k.360°, \quad k \in Z$ $\theta = \pm 60° + k.360°, \quad k \in Z$
	20.5	$\theta = 210° + k.360°, \quad k \in Z$ $\theta = 330° + k.360°, \quad k \in Z$ $\theta = 90° + k.360°, \quad k \in Z$
	20.6	$\theta = -(\frac{25}{3})° + k.120°, \quad k \in Z$ $\theta = 295° + k.360°, \quad k \in Z$
	20.7	$\theta = 30° + k.180°, \quad k \in Z$ $\theta = -240° - k.360°, \quad k \in Z$
	20.8	$\theta = 109,47° + k.360°, \quad k \in Z$ $\theta = 250,53° + k.360°, \quad k \in Z$

	TRIGONOMETRIC GRAPHS
1.1	$-1 \le y \le 1$
1.2	720^0
1.3.1	$x = 30^0$ or $x = -150^0$
1.3.2	$-90^0 < x < -60^0$ or $90^0 < x < 120^0$
2.1	720^0
2.2	$p = \frac{1}{2}$; $q = 60^0$
2.3	$-360^0 \le \theta < -240^0$ or $-180^0 \le \theta < -60^0$
3.1	Maximum value = 4
3.2	$-3 \le y \le 5$ or $y \in [-3;5]$
3.3	$h(x) = -4\sin x$
4.1	Book work(graph)
4.2	$2 \le y \le 4$
4.3	$135^0 \le x \le 180^0$ or $225^0 \le x \le 270^0$
5.1	Book work(graph)
5.3	$(38,17^0;0,24)$ and $(-218,17^0;0,24)$
6.1	Book work(graph)
6.2	Book work(graph)
6.3	$-80^0;40^0;60^0$
7.1	AB = 1 unit
7.2	$-\frac{1}{2} \le y \le \frac{7}{2}$
7.3	$x = 90^0$
7.4.1	$30^0 \le x \le 90^0$ or $210^0 \le x \le 240^0$
7.4.2	$-55^0 < x < 125^0$

TOPIC		**2D AND 3D TRIGONOMETRY**	
1			
	1.1	$\hat{S} = 65,6°$	
	1.2		
		1.2.1	$12,75\ unit^2$
		1.2.2	6,47 units
2			
	2.1	Proof	
	2.2	Proof	
	2.3	$x = 2$	
	2.4	$2\sqrt{3}$ units	
3			
	3.1	$x = 54,74°$	
	3.2	LN = 12,62 units	
4			
	4.1	$\hat{E} = 63,4°$	
	4.2	$\hat{F} = 51,8°$	

5		
	5.1	Proof
	5.2	Proof
	5.3	$Area = 6,75\ units^2$

ANSWERS
TOPIC 10: EUCLIDEAN GEOMETRY

1	
1.1	$\hat{L}_2 = 48°$ tan-chord theorem
1.2	$\hat{N}_1 = 50°$ ext. $\angle$ of Δ
1.3	$\hat{M}_1 = 82°$ sum of $\angle$'s in Δ
2	
2.1	$\hat{T}_1 = 90°$ line from centre to midpoint
2.2	$\hat{O}_1 = 100°$ $\angle$ at the centre $\perp$ chord
2.3	$\hat{S} = 50°$ $\angle$ in same segment
2.4	$\hat{P}_1 = 40°$ sum of $\angle$'s in Δ
2.5	NO, $\hat{P}_1 + \hat{T}_1 \neq 180°$
3.	
3.1	$\hat{P} = 50°$ $\angle$ sum of $\angle$'s in Δ
3.2	$Q\hat{R}S = 120°$ opp $\angle$'s of a c.q
3.3	Proof
3.4	Proof
4.	
4.1	$\hat{C}_2 = 28°$ alt $\angle$'s BE || CD
4.2	$\hat{D}_2 = 70°$ ext $\angle$ of a c.q
4.3	$\hat{E}_3 = 42°$ ext $\angle$ of a Δ
4.4	Proof
5.	
5.1	$\hat{R} = 80°$ co-int $\angle$'s of a c.q
5.2	$\hat{P} = 100°$ opp $\angle$'s of a c.q
5.3	$P\hat{Q}W = 36°$ ext $\angle$ of a c.q
5.4	$\hat{U}_2 = 136°$ alt $\angle$'s; QW || RK
6.	
6.1	$HM = 2x + 6$
6.2	$r = 15$ units
7.	
7.1	$\hat{B}_2$ alt $\angle$'s; VC || PQ
	$\hat{A}_1$ tan-chord theorem
	$\hat{B}_1$ $\angle$'s opp = sides
	$\hat{Q}$ corr $\angle$'s; VC || PQ
	$V\hat{B}Q$ alt $\angle$'s; VC || PQ
7.2	Proof
8.	
8.1	$\hat{B}_2 = \hat{P}_1 = x$ alt $\angle$'s; BC || AP
	$\hat{P}_1 = \hat{Q}_1 = x$ tan-chord theorem
	$\hat{P}_1 = \hat{P}_2 = x$ Given bisect
8.2	Proof
8.3	Proof
8.4	Proof
8.5	Proof
9.	
9.1	Proof
9.2	Proof
9.3	Proof
10.	
10.1	$\hat{B}_1 = \hat{C}_1 = x$ tan-chord theorem
	$\hat{C}_1 = \hat{C}_4 = x$ vert opp. $\angle$'s
	$\hat{C}_4 = \hat{P}_2 = x$ $\angle$'s opp = sides
10.2	Proof
10.3	Proof
10.4	Proof
11.	
11.1	Proof
11.2	Proof
11.3	Proof
11.4	Proof
11.5	Proof
12.	
12.1	Proof
12.2	Proof
12.3	Proof
12.4	Proof
13.	
13.1	Proof
13.2	Proof
13.3	Line from centre to midpoint
13.4	Proof
14.	
14.1	Proof
14.2	Proof
14.3	Proof
14.4	Proof
15	

15.1	Proof
15.2	Proof
16.	
16.1	$\frac{NP}{PT}=\frac{4}{1}$ prop theorem, KM || RS
16.2	Proof
16.3	15,75
17.	
17.1	$\frac{AF}{AC}=\frac{4}{9}$ Prop theorem EF || BC
17.2	$\frac{HF}{AF}=\frac{25}{36}$ Prop theorem EF || BC
18.	
18.1	Proof
18.2	Proof
18.3	Proof
18.4	Proof
18.5	Proof
19.	
19.1	$x=36,87$ and $x=-143,13$
20.	
20.1	Proof
20.2	Proof