KWAZULU-NATAL PROVINCE

EDUCATION
REPUBLIC OF SOUTH AFRICA

CURRICULUM GRADE 10 -12
DIRECTORATE

NCS (CAPS)

LEARNER SUPPORT DOCUMENT

GRADE 12

MATHEMATICS STEP AHEAD

2023

This document has been compiled by the KZN FET Mathematics Subject Advisors.

PREFACE

This support document serves to assist Mathematics learners on how to deal with curriculum gaps, it also captures the challenging topics in the Grade 10 – 12 work. It will cover the following:

TABLE OF CONTENTS

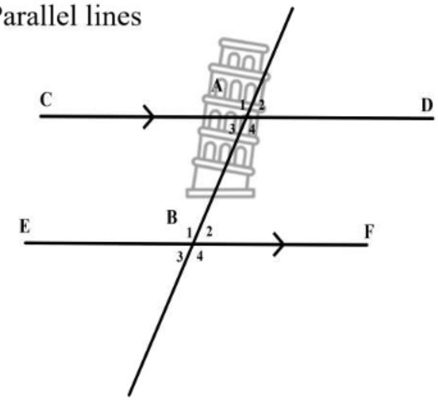
TOPIC: EUCLIDEAN GEOMETRY

LESSON 1: Revision of work from earlier grades

NOTES

- Revise & discuss extensively the following from previous grades:

Parallel lines



Corresponding angles are equal

$\hat{A}_1 = \hat{B}_1$

$\hat{A}_2 = \hat{B}_2$

$\hat{A}_3 = \hat{B}_3$

$\hat{A}_4 = \hat{B}_4$

Alternate angles are equal

$\hat{A}_3 = \hat{B}_2$

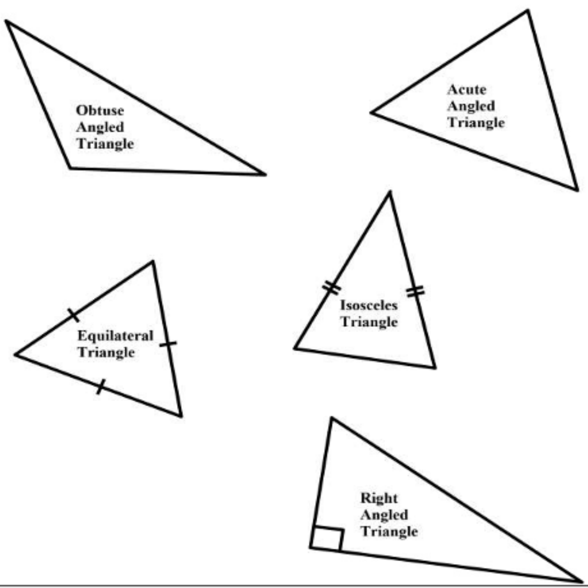
$\hat{A}_4 = \hat{B}_1$

Co-interior angles are supplementary

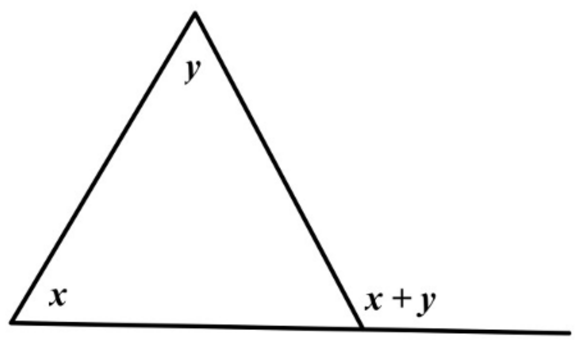
$\hat{A}_3 + \hat{B}_1 = 180°$

$\hat{A}_4 + \hat{B}_2 = 180°$

Types of triangles

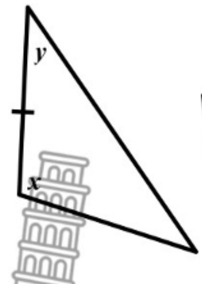


The exterior angle of a triangle equals the sum of its interior opposite angles

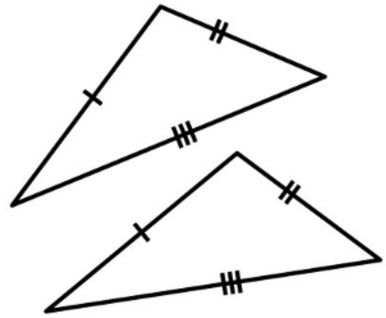


Cases of congruency

Angle / Side / Angle

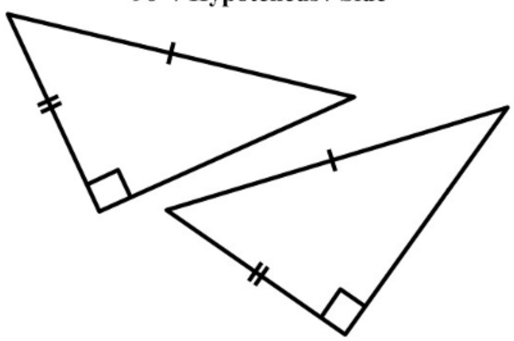


Side / Side / Side

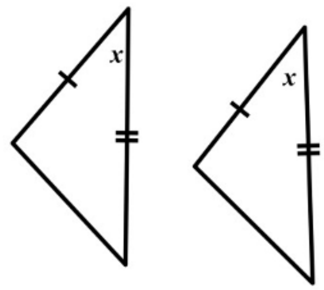


hypotenuse

90° / Hypoteneus / Side

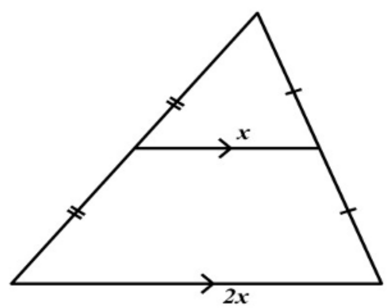


Side / Angle / Side

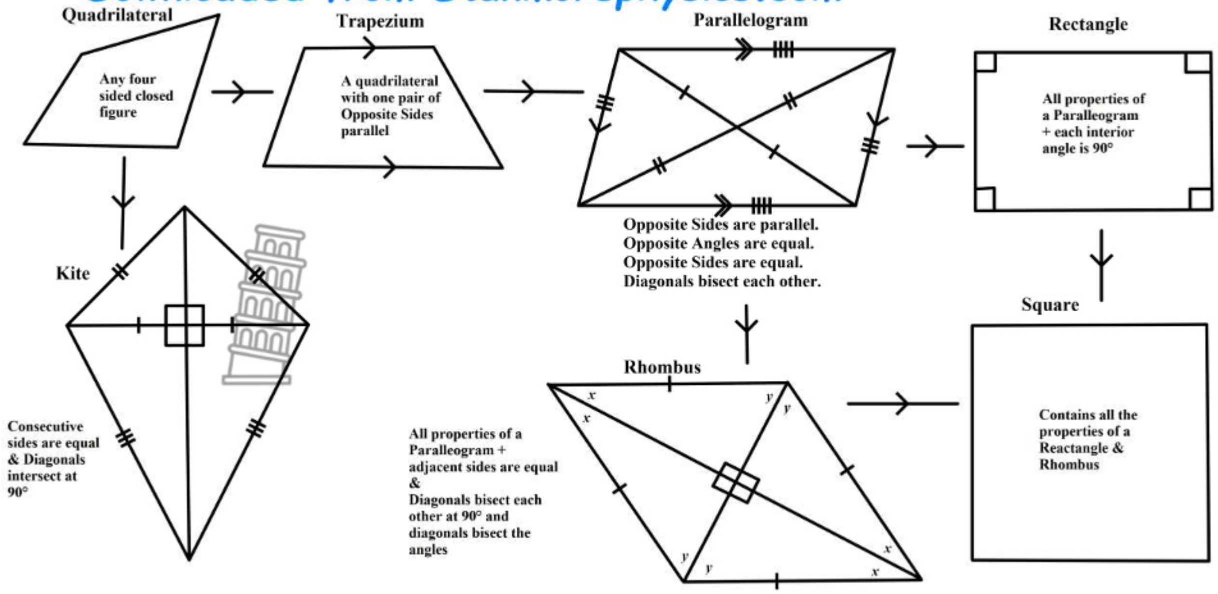


Midpoint Theorem

The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



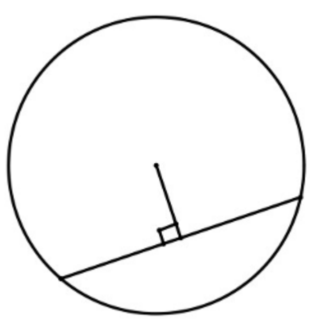
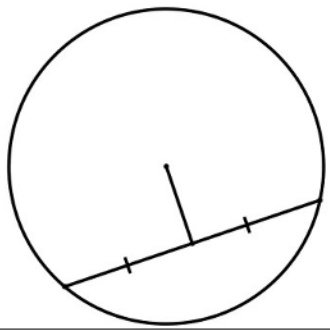
Grade 10 quadrilaterals



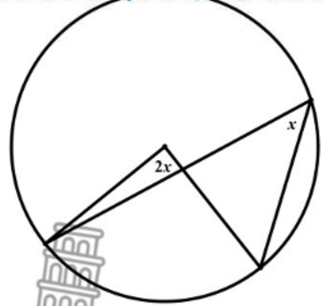
How to prove a quadrilateral is a parallelogram?

1. Prove that the opposite sides are parallel (definition) **<u>or</u>**
2. Prove that the opposite sides are equal. **<u>or</u>**
3. Prove that the opposite angles are equal. **<u>or</u>**
4. Prove that the diagonals bisect each other. **<u>or</u>**
5. Prove that one pair of opposite sides is equal and parallel.

Grade 11 Circle Geometry

No.	ILLUSTRATION	THEOREM OR COROLLARIES (Acceptable Reasons for Formal Proof is in brackets)
1.		The line drawn from the centre of a circle perpendicular to a chord bisects the chord. (line from centre ⊥ to chord)
2.		The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. (line from centre to midpt of chord)

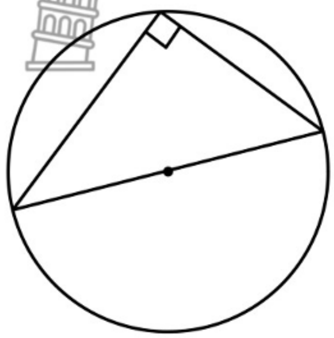
3.



The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

(∠ at centre = 2 ×∠ at circumference)

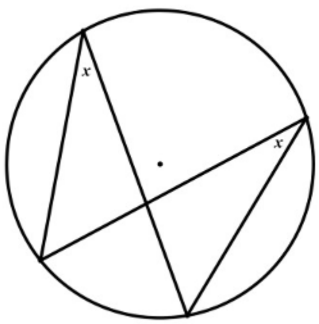
4.



The angle subtended by the diameter at the circumference of the circle is 90°.

(∠s in semi circle **OR**
diameter subtends right angle)

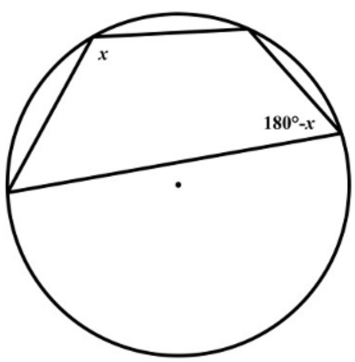
5.



Angles subtended by a chord of the circle, on the same side of the chord, are equal

(∠s in the same seg)

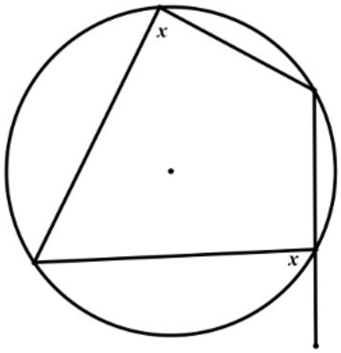
6.



The opposite angles of a cyclic quadrilateral are supplementary

(opp ∠s of cyclic quad)

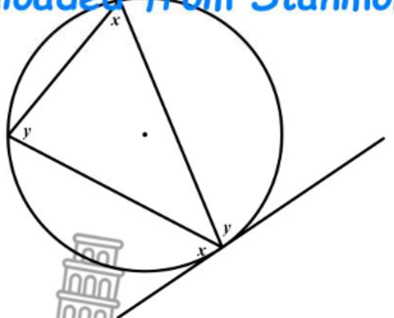
7.



The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

(ext ∠ of cyclic quad)

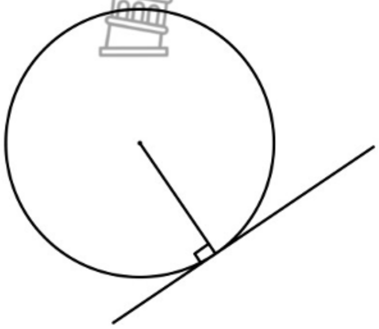
8.



The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

(tan chord theorem)

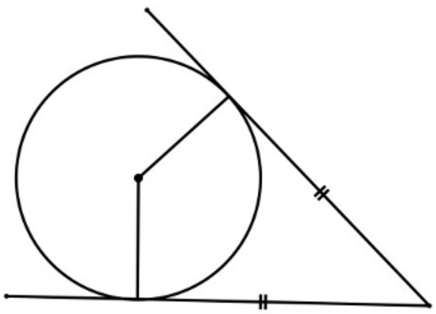
9.



The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.

(tan ⊥ radius **OR**
tan ⊥ diameter)

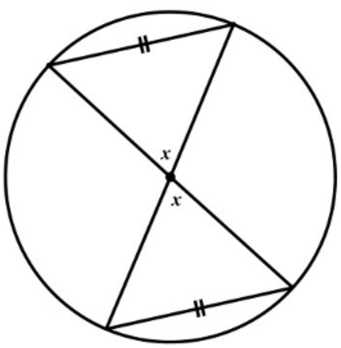
10.



Two tangents drawn to a circle from the same point outside the circle are equal in length

(Tans from common pt **OR**
Tans from same pt)

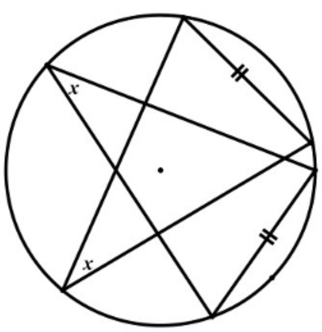
11.



Equal chords subtend equal angles at the centre of the circle.

(equal chords; equal ∠s)

12.



Equal chords subtend equal angles at the circumference of the circle.

(equal chords; equal ∠s)

ACTIVITIES/ASSESSMENTS (numerical geometry)

Activity 1

In the diagram, AC is the diameter of the circle with Centre O. AC and Chord DB intersect at E such that $DE = EB$. Chords AB, BC and AD and radius OD are drawn. $ACB = 33^\circ$.

Determine the size of:

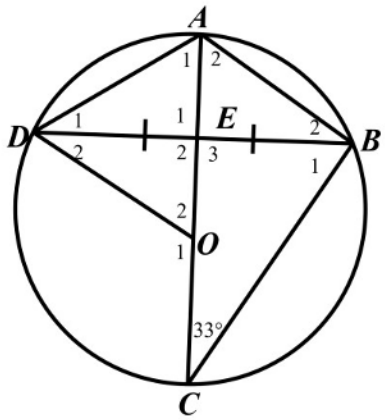
1.1 D_1

1.2 A_1

1.3 O_1

1.4 D_2

1.5 A_2



Activity 2

In the diagram below, BAED is a cyclic quadrilateral with $BA//DE.BE=DE$ and $DBC = 68^\circ$. The tangent to the circle at D meets AB produced to C.

2.1 Calculate, with reasons, the size of:

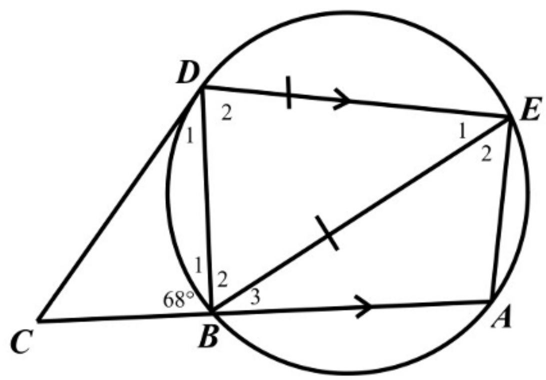
a) DEC

b) A

c) D_2

d) B_2

e) D_1



2.2 Prove that ΔBDC isosceles.

2.3 Prove that DE is a tangent to the circle that passes through the points C, B and D at D.

Activity 3

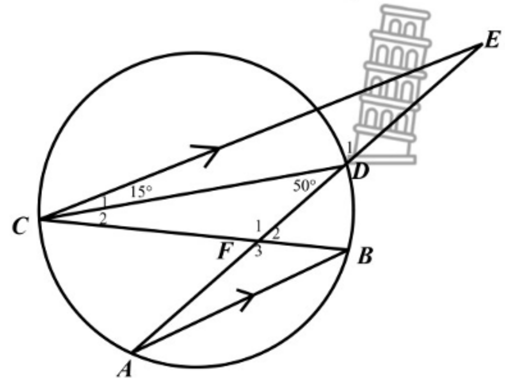
In the diagram, points A, B, and C lies on the circle. CE // AB with E on AD produced chords CB and AD intersect at F. $D_2 = 50^\circ$ and $C_1 = 15^\circ$.

3.1 Calculate, with reasons, the size of:

a) A

b) C_2

3.2 Prove, with reasons that CF is a tangent to the circle passing through points C, D and E.



TOPIC: EUCLIDEAN GEOMETRY

LESSON 2: Revision of work from earlier Grades

NOTES (non-numerical geometry)

1. **Use maximum of three variables to complete diagram.**
2. **How to prove that lines are parallel**
   - Alternate angles equal **or**
   - Corresponding angles equal **or**
   - Co-interior angles supplementary.
3. **How to prove that a quadrilateral is cyclic:**
   - That a pair of opposite angles are supplementary **or** (converse: opp. ∠'s of cyclic quad)
   - The exterior angle is equal to the interior opposite angle **or** (converse: ext. ∠ of a cyclic quad)
   - The angles in the same segment are equal. (converse: angles in the same segment.)
4. **How to prove that a chord is a diameter:**
   - That the angle subtended by the chord on the circumference is a right angle. (converse: angle in a semi-circle)
5. **How to prove that a line is a tangent :**
   - That the angle between the line and a chord is equal to the angle subtended by the chord in the alternate segment. (converse tan- chord theorem)
   - That the line is perpendicular to the radius at point of contact on circle. (converse: tan ⊥ radius)

Activity 1

In the sketch TA and TB are tangents to the circle with Centre O. TQP is a secant and chord AC is parallel to QP. QP cuts AB and BC in H and K respectively.

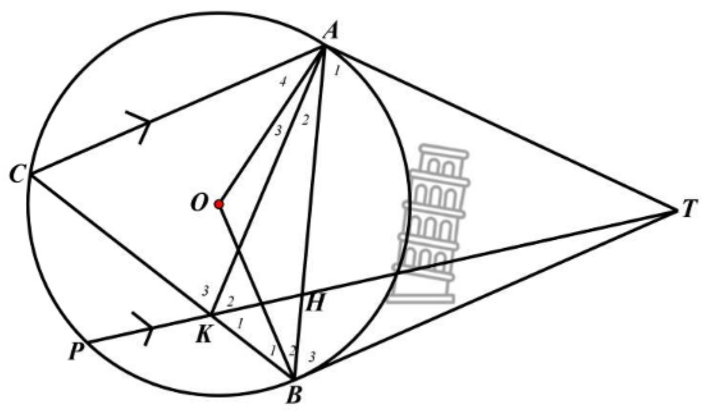
Prove that:

1.1 AOBT is a cyclic quadrilateral

1.2 $K_1 = A_1$

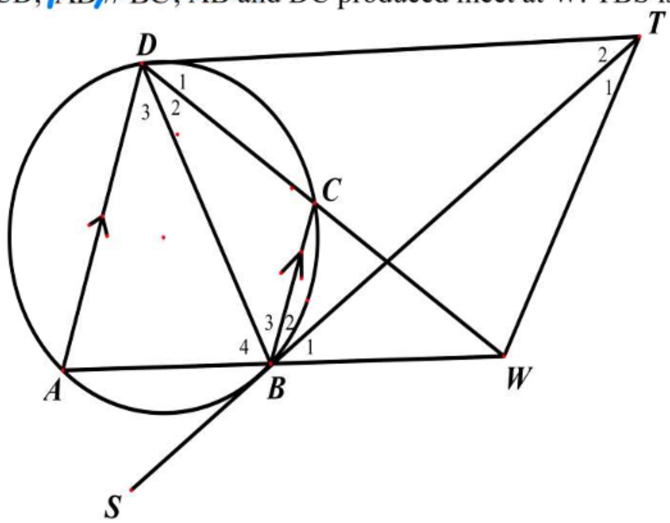
1.3 AKBT is a cyclic quadrilateral

1.4 TA is a tangent to the circle AHK.



Activity 2

In the diagram TD is a tangent to the circle ABCD, AD // BC, AB and DC produced meet at W. TBS is a straight line. $B_1 = B_2$



Prove that:

2.1 BWTD is a cyclic quadrilateral

2.2 TBS is a tangent to the circle ABCD

2.3 TW // BC

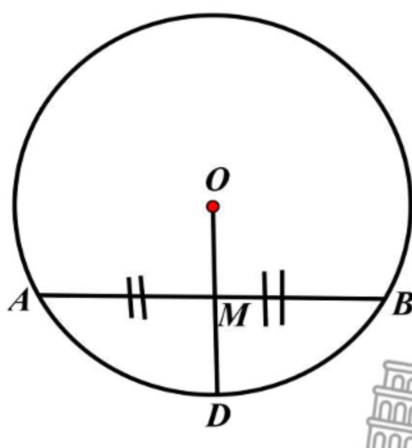
TOPIC: EUCLIDEAN GEOMETRY
LESSON 3: REVISION

NOTES

- **Join radii to form right angled triangle**
- **Use Pythagoras theorem to calculate the required side**
- **Radii are equal in length**

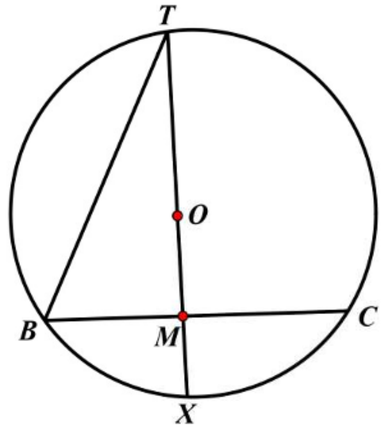
Activity 1

AB is a chord of a circle with Centre O and AB = 24 cm. M is the midpoint of AB. If MD = 8 cm, calculate the radius of the circle.



Activity 2

TOMX is a diameter of the circle with Centre O and chord $BC = 20$ cm. if $TOMX \perp BC$ and OM=2MX,



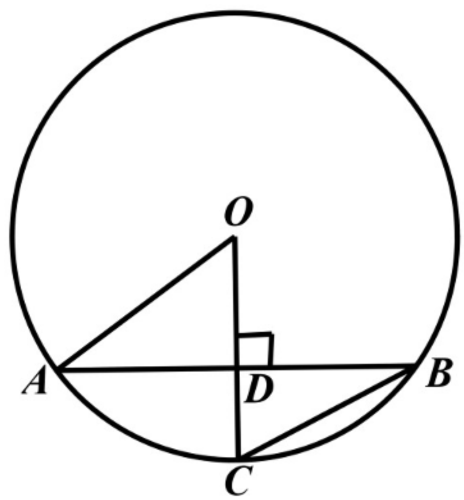
Calculate with reasons:

2.1 TB

2.2 The radius of the circle MTB

Activity 3

In the accompanying figure, AB is a chord of the circle with Centre O. OD is drawn perpendicular to AB and meets the circle in C.



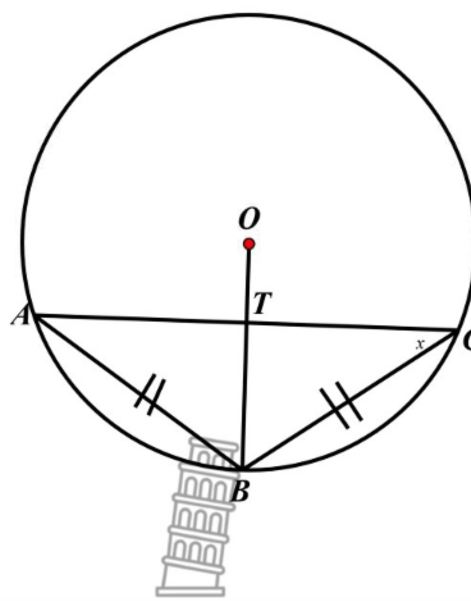
3.1 Prove that $CB^2 = 2OC.DC$

3.3 If it is further given that OA = 9 units and OD = 1 unit, calculate the length of BC.

Activity 4

In the figure O is the Centre of the circle;

AB = BC and $C = x$



4.1 Determine ABO in terms of x

4.2 Prove that OB bisects ABC.

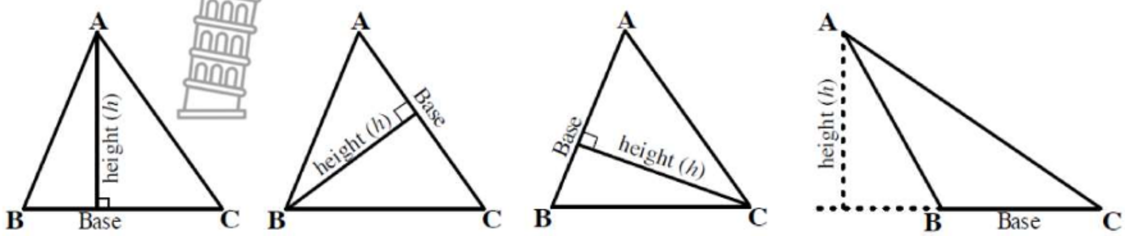
4.3 If OT: TB = 1: 3 and OA = r, determine AC in terms of r.

TOPIC: EUCLIDEAN GEOMETRY

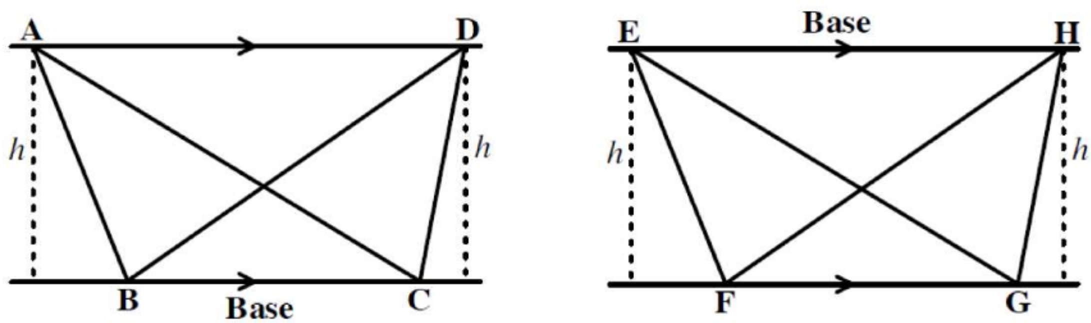
LESSONE 4: RATIO AND PROPORTION

NOTES

- Area
    - ✓ Triangles which share a common vertex have the same height.



- Triangles on the same base between parallel lines are equal in area.

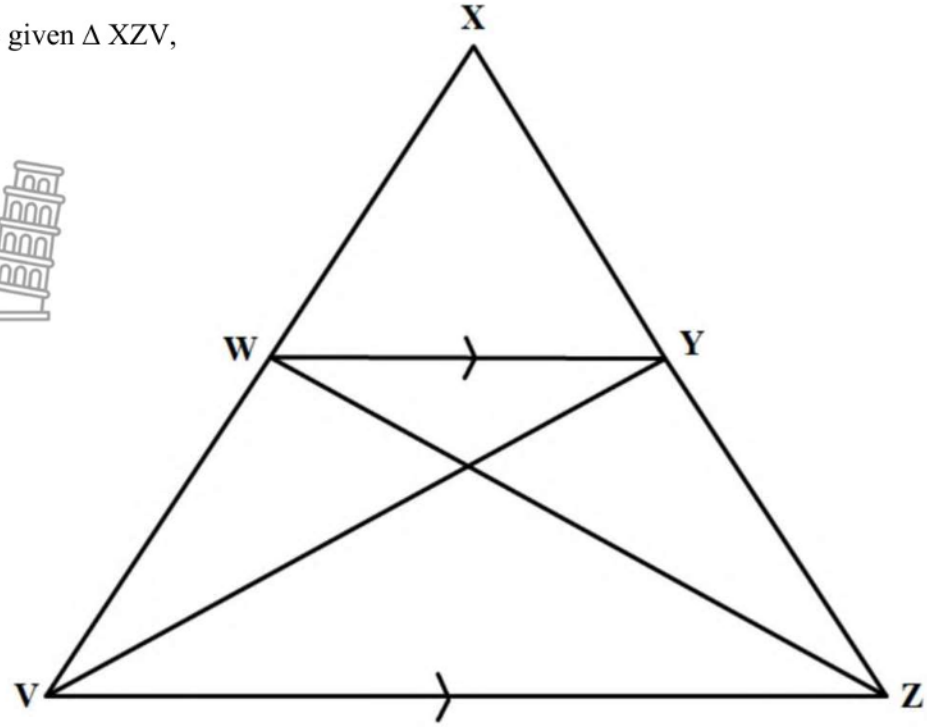


- Proceed to do this short investigation:

LEARNER ACTIVITY 1
question 1

Day 1 – Classwork Activity

In the diagram you are given Δ XZV, with WY//VZ



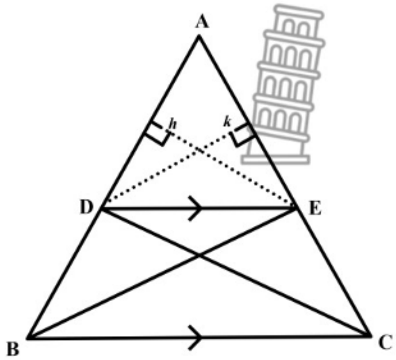
Consider the diagram above and answer the questions that follow:

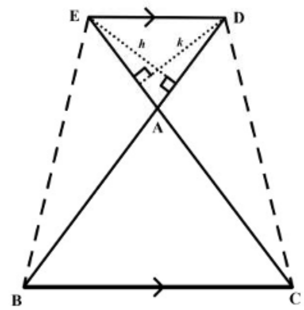
1.1	Use the diagram above and construct:	
	1.1.1	Height "h" perpendicular to base XV of Δ XVY
	1.1.2	Height "k" perpendicular to base XZ of Δ XWZ
1.2	Using height "h" write down expressions for the following:	
	1.2.1	Area of Δ XWY = ……………………
	1.2.2	Area of Δ VWY = ……………………
	1.2.3	$\frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ VWY}$ = ……………………………….
	Using height "k" write down expressions for the following:	
	1.2.4	Area of Δ XWY = ……………………
	1.2.5	Area of Δ WYZ = ……………………
	1.2.6	$\frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ WYZ}$ = ……………………………….
1.3	Considering your answers in 1.2.3, 1.2.6, give a reason why the following statement can be made: $\frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ VWY} = \frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ WYZ}$	
1.4	Hence or otherwise, simplify $\frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ VWY} = \frac{\text{Area of } \Delta\ XWY}{\text{Area of } \Delta\ WYZ}$	
1.5	Considering Δ XVZ with line WY parallel to base VZ, and your answer in 1.4, make a conjecture.	
1.6	Suggest a possible reason why the conjecture in 1.5 will not work if WY was not parallel to VZ.	

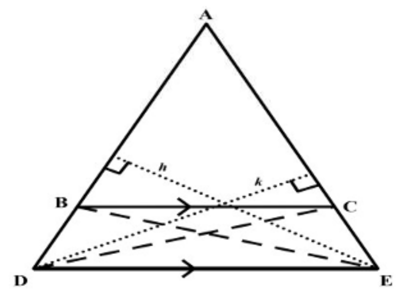
- Formal Proof of the Ratio & Proportionality Theorem

THEOREM: The line drawn parallel to one side of a triangle divides the other two sides proportionally

Given : ΔABC, D lies on AB and E lies on AC. And DE // BC.







R.T.P : $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Proof : Join D to C and E to B. Draw altitudes *h* and *k* relative to bases AD and AE

$$\frac{\text{Area } \Delta\text{ ADE}}{\text{Area } \Delta\text{ BDE}} = \frac{\frac{1}{2}\text{AD}\times h}{\frac{1}{2}\text{DB}\times h} = \frac{\text{AD}}{\text{DB}} \quad \text{(same height)}$$

$$\frac{\text{Area } \Delta\text{ ADE}}{\text{Area } \Delta\text{ CED}} = \frac{\frac{1}{2}\text{AE}\times k}{\frac{1}{2}\text{EC}\times k} = \frac{\text{AE}}{\text{EC}} \quad \text{(same height)}$$

but Area Δ BDE = Area Δ CED (same base & between // lines)

$$\therefore \quad \frac{\text{Area } \Delta\text{ ADE}}{\text{Area } \Delta\text{ BDE}} = \frac{\text{Area } \Delta\text{ ADE}}{\text{Area } \Delta\text{ CED}}$$

$$\therefore \quad \frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}}$$

Converse of the Ratio & Proportionality Theorem

The formal proof is not tested but the application of this theorem is!

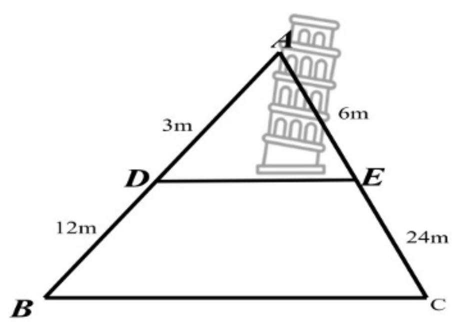
CONVERSE: If a line cuts two sides of a triangle proportionally, then that line is parallel to the third side

Example 1

In ΔABC, AD= 3m, DB =12m, AE = 6m

and

EC =24m. prove that DE || BC



Solution

$\frac{DB}{AD}=\frac{12}{3}=4,\quad \frac{EC}{AE}=\frac{24}{6}=4,\quad \frac{DB}{AD}=\frac{EC}{AE}=4$

$\therefore DE \parallel BC$

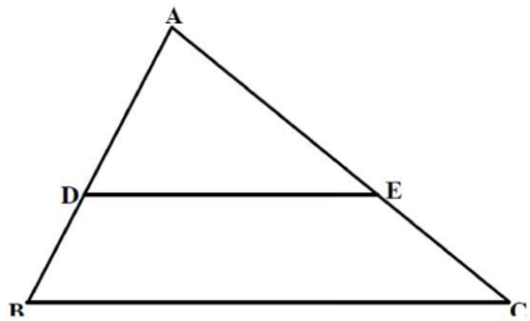
Line divides sides Δ prop

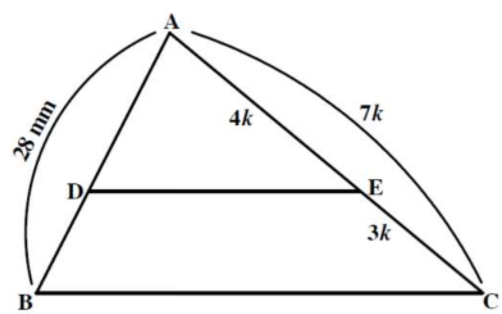
Whenever you use this theorem the reason you must give is:

Line divides sides of Δ proportionally OR prop theorem; name || lines

EXAMPLE 2

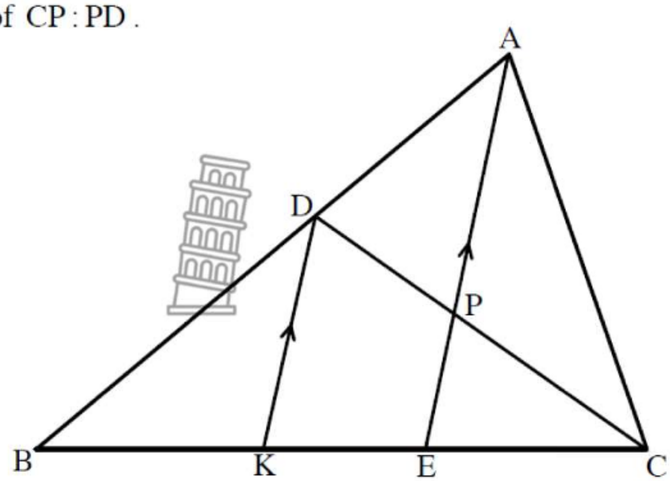
In ΔABC, $DE \parallel BC$, $AB = 28$ mm and $AE:EC = 4:3$. Determine the length of BD.



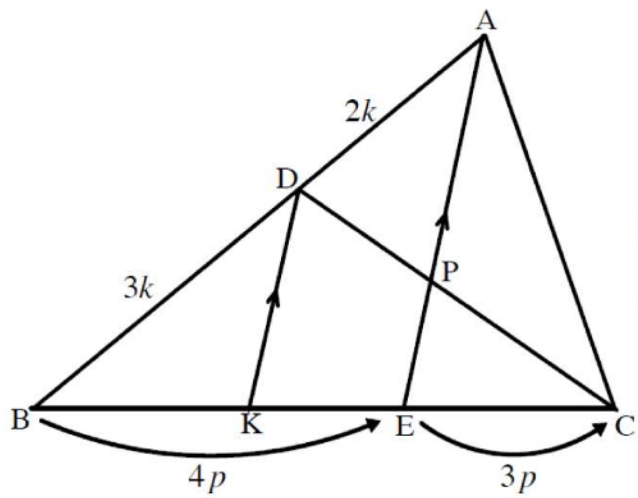


Statement	Reason
$\frac{BD}{28mm}=\frac{3k}{7k}$ $\therefore BD=\frac{3}{7}\times 28mm$ $\therefore BD=12mm$	Line ∥ one side of ΔABC

D and E are points on sides AB and BC respectively of ΔABC such that $AD : DB = 2 : 3$ and $BE = \frac{4}{3}EC$. If DK||AE and AE and CD intersect at P, find the ratio of $CP : PD$.



Solution



$BE = \frac{4}{3}EC$

$\therefore \frac{BE}{EC} = \frac{4}{3}$

Statement	Reason
$\frac{CP}{PD} = \frac{3p}{EK}$	Line ‖ one side of ΔCDK
Now $\frac{EK}{4p} = \frac{AD}{AB}$	Line ‖ one side of ΔABE
$\therefore \frac{EK}{4p} = \frac{2k}{5k}$ $\therefore EK = \frac{2k}{5k} \times 4p$ $\therefore EK = \frac{2}{5} \times 4p$ $\therefore EK = \frac{8p}{5}$ $\therefore \frac{CP}{PD} = \frac{3p}{\frac{8p}{5}}$ $\therefore \frac{CP}{PD} = 3p \times \frac{5}{8p}$ $\therefore \frac{CP}{PD} = \frac{15}{8}$	

The educator must use the guideline below to cover activities over the three days.

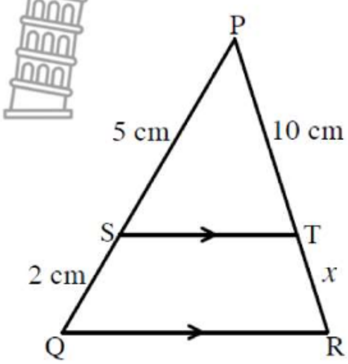
DAY 1 Activities 2.2.1 – 2.2.2

DAY 2 Activities 2.2.3 – 2.2.5

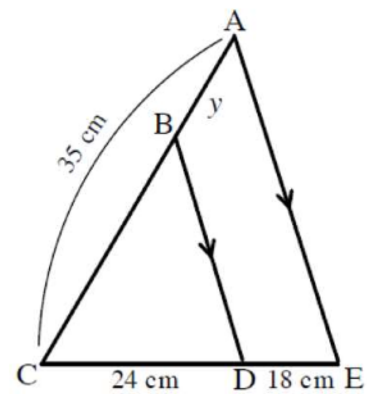
DAY 3 Activities 2.2.6 – 2.2.7

Activity 1

1.1. Find, giving reasons the value of x



1.2 In ΔACE, BD||AE. Calculate the value of y.



1.3.

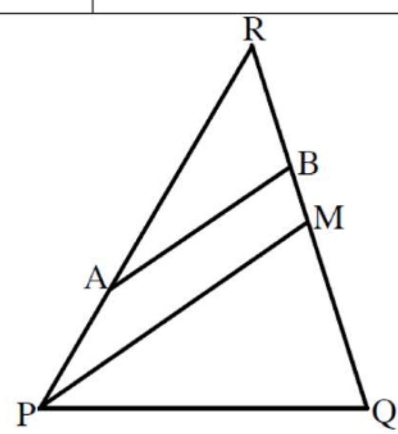
Consider the diagram alongside:

$\frac{RB}{RQ} = \frac{1}{3}$, PA : AR = 1 : 2 and PM//AB

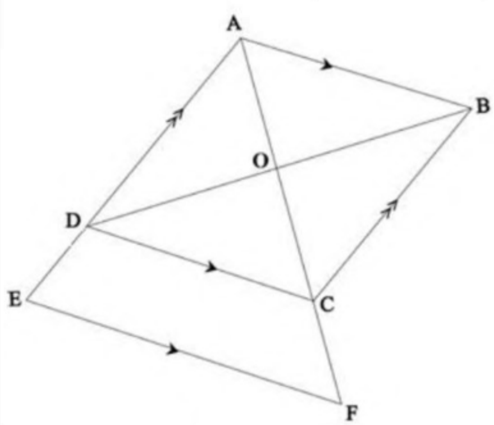
1.3.1 Write down the values for RA:RP and RB : BQ.

1.3.2 Determine BM : BR.

1.3.3 Prove that RM = MQ



1.4. In the diagram below, ABCD is a parallelogram. AD and AC are produced to E and F respectively so that EF ∥ DC. AF and DB intersect at O.
AD = 12 units; DE = 3 units; DC = 14 units; CF = 5 units.



Calculate, giving reasons, the length of:

1.4.1 AC

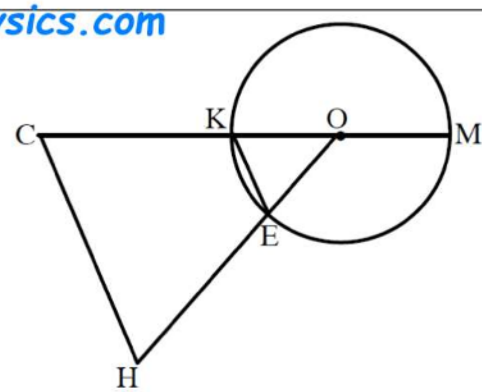
1.4.2 AO

1.4.3 EF

1.4.4 Prove that : $\frac{\text{area of } \Delta ADO}{\text{area of } \Delta AEF} = \frac{8}{25}$

1.5. In the diagram below, KM is a diameter of the circle Centre O. OK = r. OC = $4r$ and $\hat{H} = \hat{C}$.

Prove that EK // HC



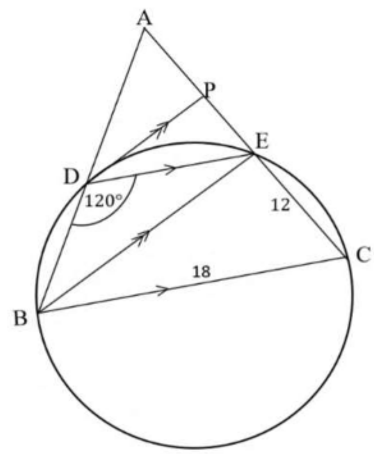
1.6 In Δ ABC in the diagram alongside, D is a point on AB such that AD : DB = 5 : 4. P and E are points on AC such that DE ∥ BC and DP ∥ BE. BC is NOT a diameter of the circle.

Given: BDE=120° , EC=12 units and BC=18 units.

Determine, with reasons:

1.6.1 The length of AE

1.6.2 $\dfrac{\text{area of } \Delta AEB}{\text{area of } \Delta ECB}$



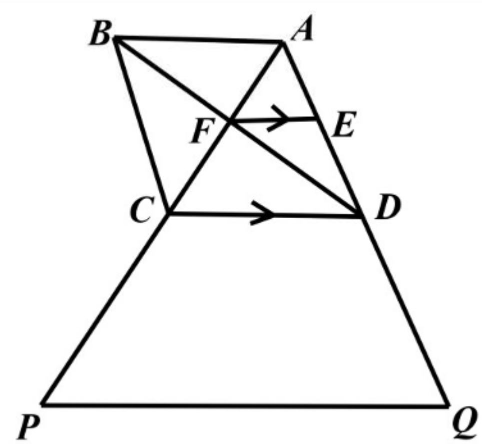
Activity 2

ABCD is parallelogram with diagonals intersect at F. FE is drawn parallel to CD. AC is produced to P such that PC = 2AC and AD is produced to Q such that DQ = 2AD.

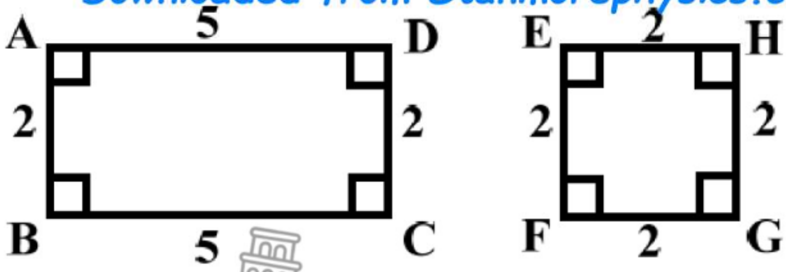
2.1 Show that E is the midpoint of AD. (2)

2.2 Prove that PQ // FE. (3)

2.3 If PQ is 60 cm, calculate the length of EF (5)



TOPIC: EUCLIDEAN GEOMETRY
LESSON 5: SIMILAR TRAINGLES
NOTES
Two polygons are similar if they have the same shape but not necessarily the same size. Two conditions must **both** be satisfied for two polygons to be similar: (a) The corresponding angles must be equal. (b) The ratio of the corresponding sides must be in the same proportion.

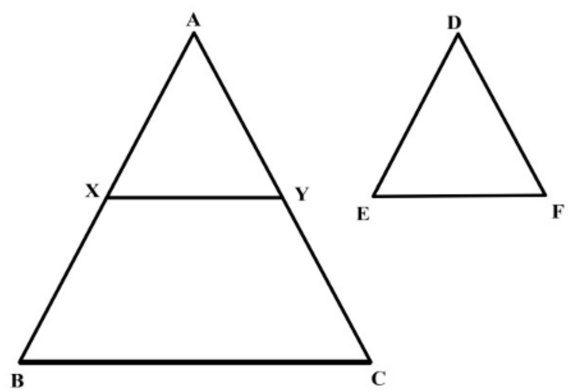


Considering Similar Triangles highlight….
With similar triangles, **only one of the two** conditions needs to be true in order for the two triangles to be similar. This is proved in the theorem below:

THEOREM : Equiangular Triangles are similar

Given Δ ABC and ΔDEF with $\widehat{A} = \widehat{D}$; $\widehat{B} = \widehat{E}$; $\widehat{C} = \widehat{F}$

R.T.P : $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$



Proof : Mark off X on AB and Y on AC such
That AX = DE and AY = DF. Construct XY

In Δ AXY and ΔDEF

1. $AX = DE$ (construction)
2. $AY = DF$ (construction)
3. $\hat{A} = \hat{D}$ (given)

$\therefore \Delta AXY \equiv \Delta DEF$ (SAS)

now $A\hat{X}Y = \hat{E}$ but $\hat{E} = \hat{B}$ (given)

$\therefore \quad A\hat{X}Y = \hat{B}$

$\Rightarrow XY \parallel BC$ (correponding $\angle$'s =)

now $\frac{AB}{AX} = \frac{AC}{AY}$ (line // one side of Δ)

but $AX = DE$ and $AY = DF$ (construction)

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

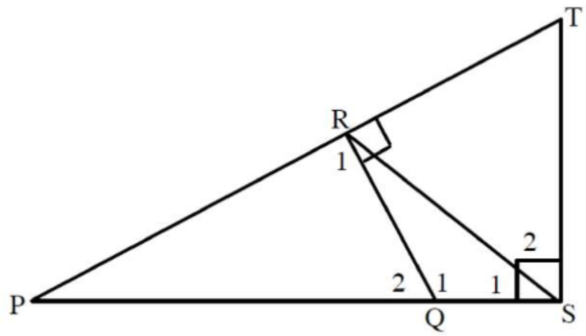
similarly by marking off equal lengths on BA and BC

it can be shown that : $\frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \quad \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

CONVERSE: If the corresponding sides are in the same proportion, then the corresponding angles of the two triangles will be equal

The following Example is to be discussed by the Educator in the lesson:



EXAMPLE 1

In ΔPST, $TS \perp PS$ and $RQ \perp PT$. Prove:

(a) $\Delta PRQ ||| \Delta PST$

(b) $RQ : PQ = ST : PT$

(c) $PR \,.\, PT = PQ \,.\, PS$

Match the corresponding angles of ΔPRQ and ΔPST as follows and then prove the pairs of angles equal.

$\hat{P}$ —— $\hat{P}$

$\hat{R}_1$ —— $\hat{S}_1 + \hat{S}_2$

$\hat{Q}_2$ ········ $\hat{T}$

Draw solid lines for each pair of corresponding angles that are equal. The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle.

a)

In ΔPRQ and ΔPST:	
(1) $\hat{P} = \hat{P}$	common
(2) $\hat{R}_1 = \hat{S}_1 + \hat{S}_2 = 90^\circ$	given
(3) $\hat{Q}_2 = \hat{T}$	sum of angles of Δ
$\therefore \Delta PRQ ||| \Delta PST$	$\angle, \angle, \angle$

b)

Since $\Delta PRQ ||| \Delta PST$:	
$\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$	corr sides of Δ's in proportion
$\therefore \frac{RQ}{ST} = \frac{PQ}{PT}$	
$\therefore \frac{RQ}{PQ} = \frac{ST}{PT}$	cross multiplication
$\therefore RQ:PQ = ST:PT$	

c)

$\frac{PR}{PS} = \frac{PQ}{PT}$	since $\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$
$\therefore PR \,.\, PT = PQ \,.\, PS$	cross multiplication

DAY 1 Activities 1.1.1 – 1.1.3

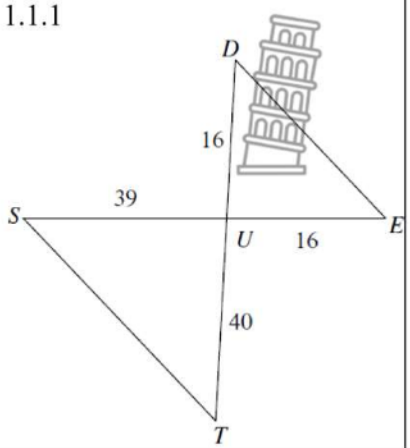
DAY 2 Activities 1.1.4 – 1.1.6

DAY 3 Activities 1.2.1-1.2.3

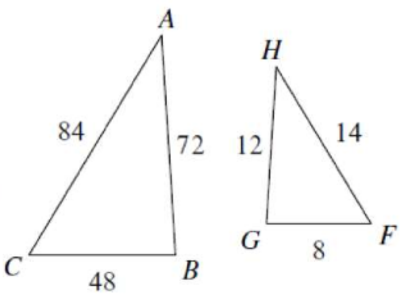
ACTIVITY 1

1.1. In each of the cases below: State if the triangles are Similar OR not and write down the reason:

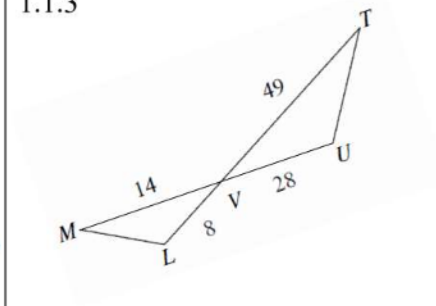
1.1.1



1.1.2

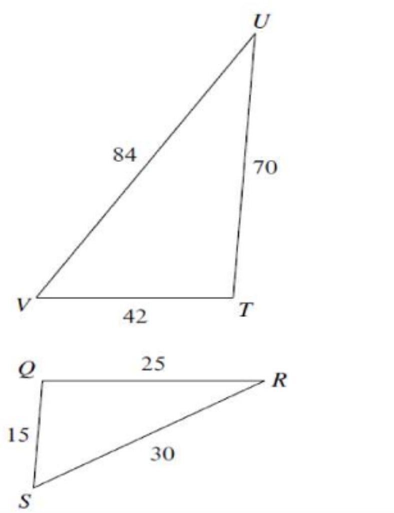


1.1.3

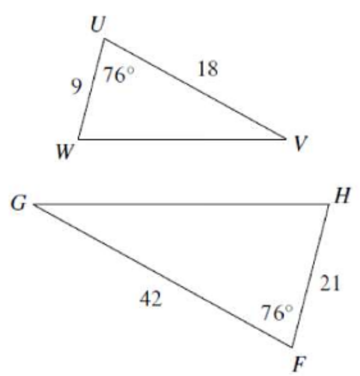


ΔMVL & ΔUVT

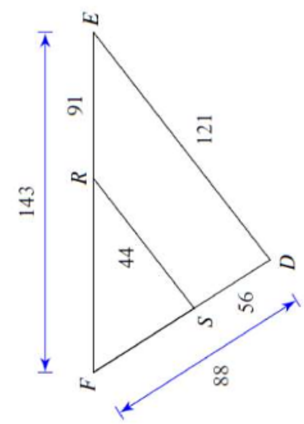
1.1.4



1.1.5



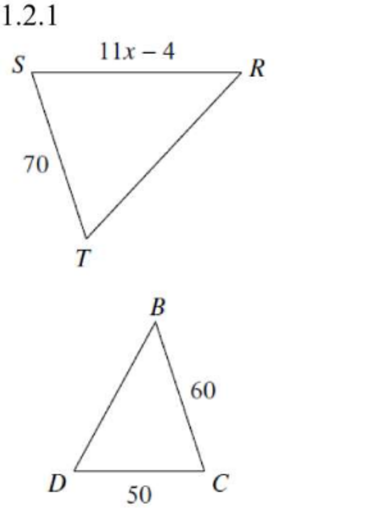
1.1.6



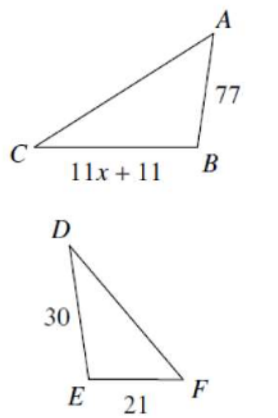
ΔFED& ΔFRS

1.2. Solve for *x*. The triangles in each pair are similar.

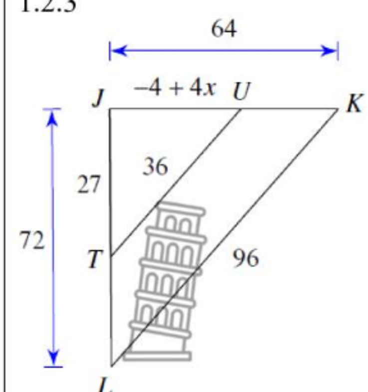
1.2.1



1.2.2



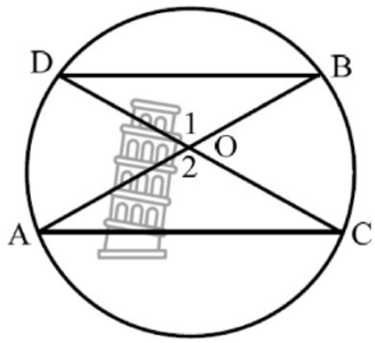
1.2.3



ACTIVITY 2

2.1

A, B, C and D are noncyclic points. DOC and AOB are chords. DB and AC are joined.



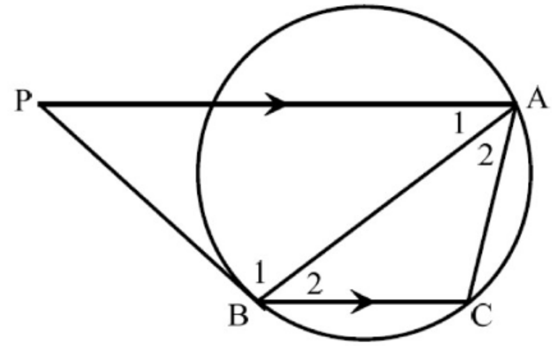
Prove that:

2.1.1 ΔAOC|||ΔDOB

2.1.2 $\frac{OB}{OD} = \frac{OC}{OA}$

2.2

PB is a tangent to circle ABC. PA||BC.



Prove that:

2.2.1 ΔPAB|||ΔABC

2.2.2 PA.AB=AB.BC

2.2.3 $\frac{AP}{BP} = \frac{AB}{AC}$

2.3

In the diagram alongside,

ACBT is a cyclic quadrilateral.

BT is produced to meet tangent AP on D.

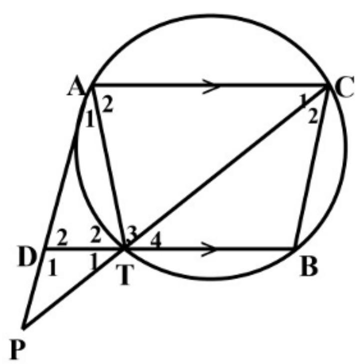
CT is produced to P. AC // DB.

2.3.1 Prove that $PA^2 = PT \cdot PC$

2.3.2 If PA = 6 units, TC = 5 units and PT = x, show that $x^2 + 5x - 36 = 0$.

2.3.3 Calculate the length of PT.

2.3.4 Calculate the length of PD.



2.4 AP is a tangent to the circle at P. CB ∥ DP and CB=DP. CBA is a straight line.

Let $D = x$ and $C_2 = y$.

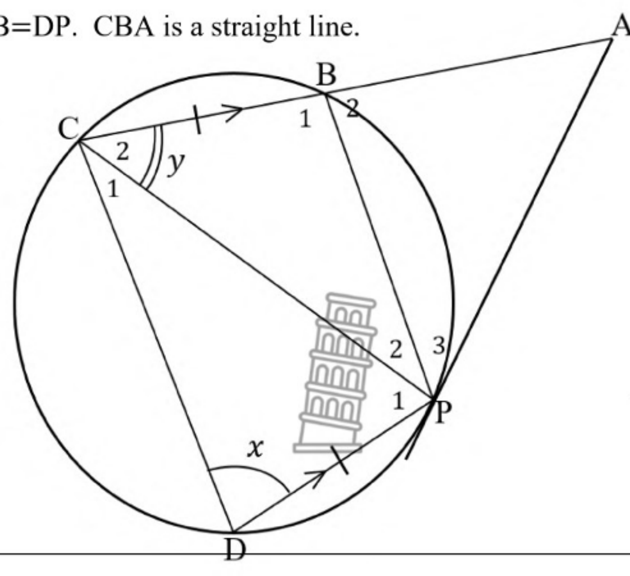
Prove, with reasons that:

2.4.1 ΔAPC///ΔABP

2.4.2 $AP^2 = AB.AC$

2.4.3 ΔAPC///ΔCDP

2.4.4 $AP^2 + PC^2 = AC^2$

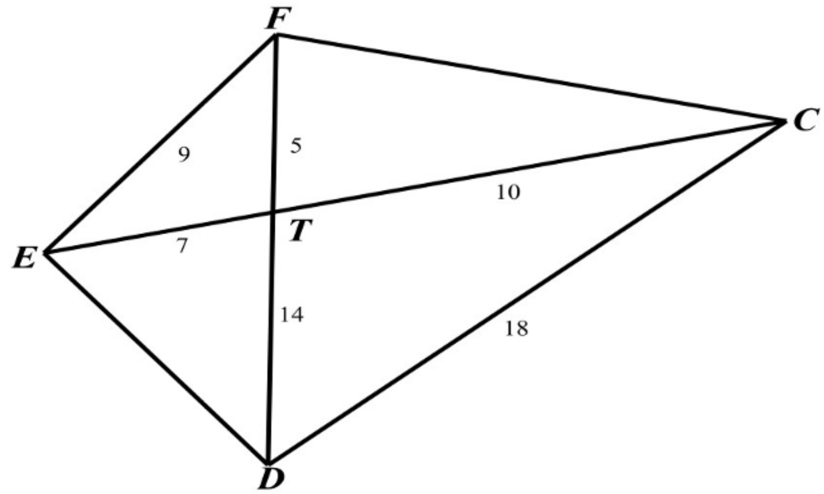


2.5 In the diagram, the diagonal of quadrilateral CDEF intersect at T. EF = 9 units, DC = 19 units, ET = 7 units, TC = 10 units, FT = 5 units and TD =14 units.

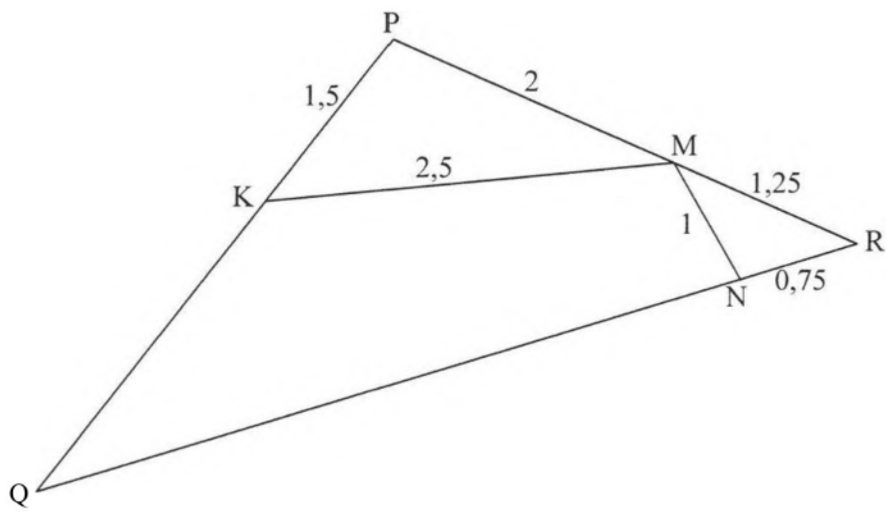
Prove, with reasons, that:

2.5.1 EFD = ECD

2.5.2 DFC = DEC



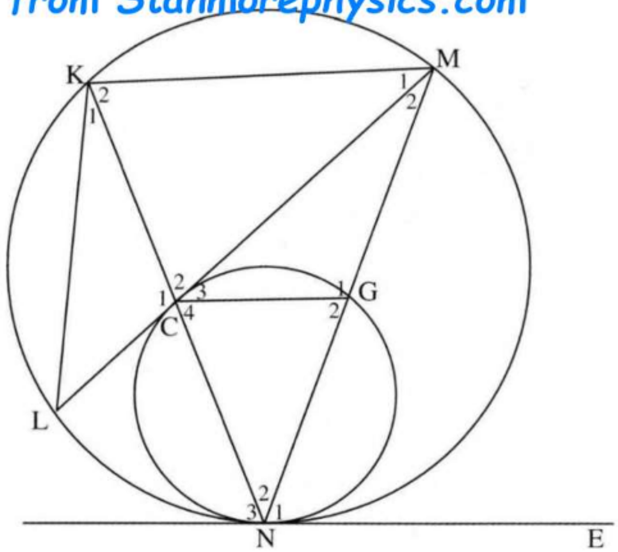
2.6 In the diagram below, K, M and N respectively are points on sides PQ, PR and QR of Δ PQR. 1,5•, PM = 2; KM = 2,5; MN = 1; MR = 1,25 and NR 0,75.



2.6.1. Prove that Δ KPM /// Δ RNM

2.6.2 Determine the length of NQ

2.7 In the diagram below NE is a common tangent to the two circles. NCK and NGM are double chords. Chord LM of the larger circle is a tangent to the smaller circle at point C. KL, KM and CG are drawn.



Prove that :

2.7.1. $\frac{KC}{KN} = \frac{MG}{MN}$

2.7.2 KMGC is a cyclic quadrilateral if CN = NG.

2.7.3 $\Delta MCG ||| \Delta MNC$

2.7.4 $\frac{MC^2}{MN^2} = \frac{KC}{KN}$

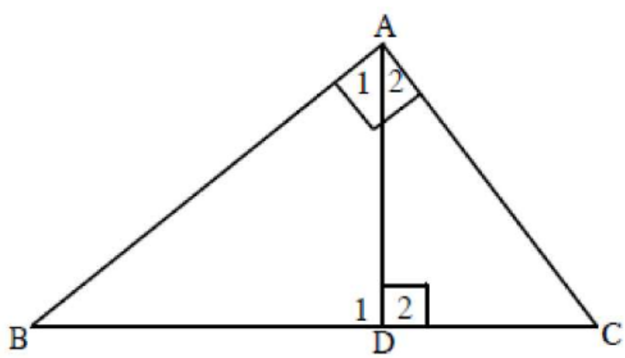
TOPIC: EUCLIDEAN GEOMETRY

LESSON 6:

NOTES

- Discuss the Theorem below The Proof is not for Examination purposes

> The perpendicular drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, divides the triangle into two triangles that are similar to each other and similar to the original triangle.



- See if learners can prove the three triangles similar by inspection and not long formal proofs.
- Develop the lesson further that since $\Delta ABC ||| \Delta DBA ||| \Delta DAC$ …….

Corollaries

$\Delta ABC \vert\vert\vert \Delta DBA$	$\Delta ABC \vert\vert\vert \Delta DAC$	$\Delta DBA \vert\vert\vert \Delta DAC$
$\therefore \frac{AB}{DB} = \frac{BC}{BA} = \frac{AC}{DA}$	$\therefore \frac{AB}{DA} = \frac{BC}{AC} = \frac{AC}{DC}$	$\therefore \frac{DB}{DA} = \frac{BA}{AC} = \frac{DA}{DC}$
$\therefore AB^2 = BD \,.\, BC$	$\therefore AC^2 = CD \,.\, CB$	$\therefore AD^2 = BD \,.\, DC$

- Ultimately by using the corollaries above (and the diagram) one may prove the **Theorem of Pythagoras where: BC^2 □□AB^2 □□AC^2**

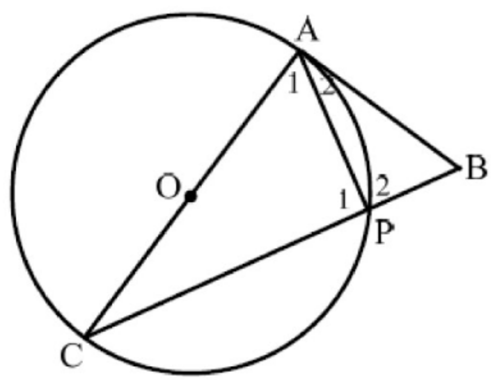
**** NOTE that the PROOF of the Theorem of Pythagoras is not for exam purposes but the APPLICATION of the Theorem is! ******

ACTIVITY 1

1.1

In the figure alongside, O is the Centre of the circle. AC is a diameter. Chord CP is produced to B.

Prove that: $AP^2 = PC. BP$



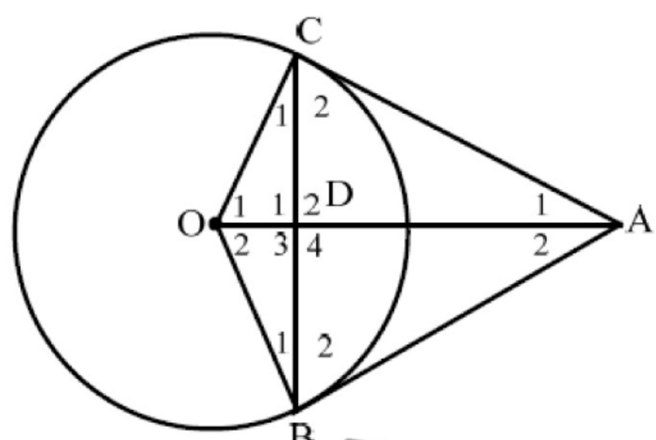
1.2

In the figure alongside, AB and AC are tangents to the circle with Centre O. OD ⊥ BC.

Prove that:

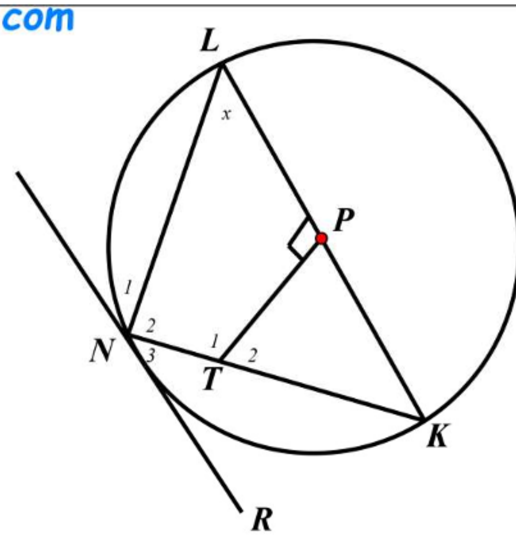
1.2.1 $BD^2 = OD. DA$

1.2.2 $\frac{OC^2}{AC^2} = \frac{OD}{DA}$



1.3

In the diagram, KL is a diameter of the circle with Centre P. RNS is a tangent of the circle at N. T is a point on NK and TP⊥KL. PLN = x



1.3.1 Prove that TPLN is a cyclic quadrilateral

1.3.2 Determine, giving reasons, the size of $\widehat{N}_1$ in terms of x

1.3.3 Prove that:

a) ΔKTP///ΔKLN

b) $KN.KT = 2KT^2 - 2TP^2$

TOPIC: EUCLIDEAN GEOMETRY
LESSON 7: CONSOLIDATION

NOTES

HOW TO GO ABOUT SOLVING A GEOMETRY RIDER

1. What knowledge must you have?

- Know all terminology associated with Euclidean Geometry relevant to the School Curriculum.
- Be able to state ALL Theorems/ Converses of Theorems/ Axioms and Corollaries **AND** be able to draw a rough diagram to describe every statement. Pages 2 to 5 of this supplement indicate the important theorems and corollaries that must be learnt and illustrations that should be remembered.
- Know how to write reasons in abbreviated form for the formal writing of proofs. Approved reasons are found in the Examination Guideline.

2. What approach to use?

- When you see the Diagram (involving a circle) and see the information provided use what we call the **"DOCTOR CAPE TOWN"** Method. That is look for **D**iameter/ **R**adius/ **C**yclic Quadrilaterals/ **P**arallel Lines/ **T**angents in other words **DRCPT** (Doctor Cape Town ☺) This will help you identify all the key aspects in the diagram and make problem solving easier!

- Use Colour Pencils (Maximum of 3 colours). This is particularly important when proving triangles similar.

- Always remember the order of questions is critical. Invariably what is done in a preceding question is vital to solve following questions.

- Remember correct writing of the solution is as important as solving the question itself.

3) That two triangles are similar

Prove: A case of .... The two triangles are equiangular **or** The sides are in proportion

ACTIVITY 1

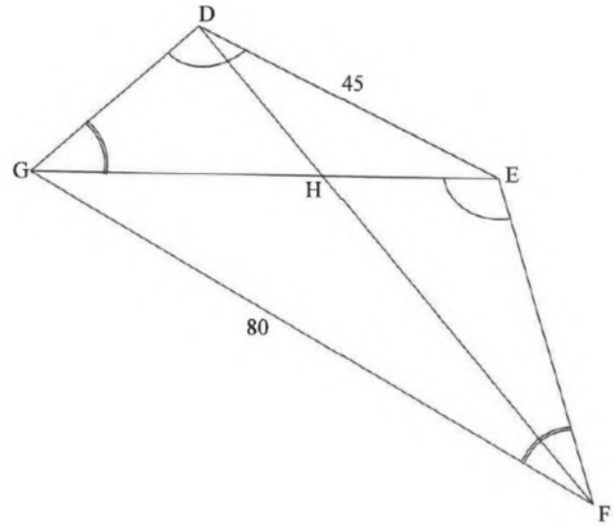
1.1. In the diagram, DEFG is a quadrilateral with DE = 45 and GF = 80. The diagonals GE and DF meet in H. GDE = FEG and DGE = EFG .

1.1.1 Give a reason why $\Delta DEG ||| \Delta EGF$

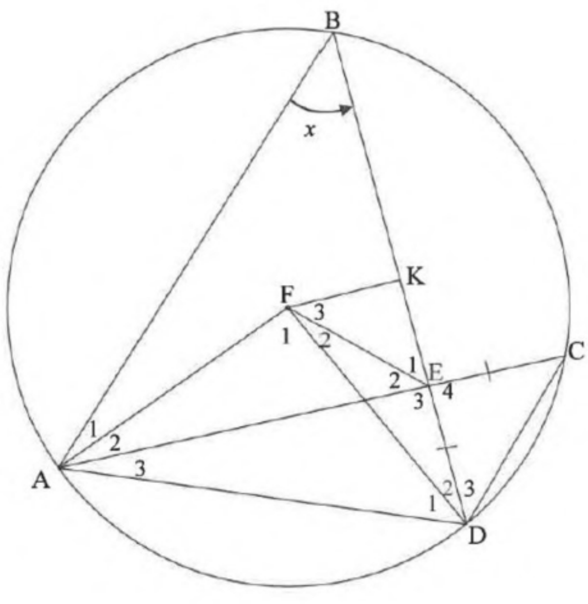
1.1.2 Calculate the length of GE.

1.1.3 Prove that

1.1.4 Hence calculate the length of GH.



1.2. In the diagram, the circle with Centre F is drawn. Points A, B, C and D lie on the circle. Chords AC and BD intersect at E such that EC = ED. K is the midpoint of chord BD. FK, AB, CD, AF, FE and FD are drawn. Let B = x .



2.1 Determine with reasons the size of EACH of the following in terms of x.

a) F_1

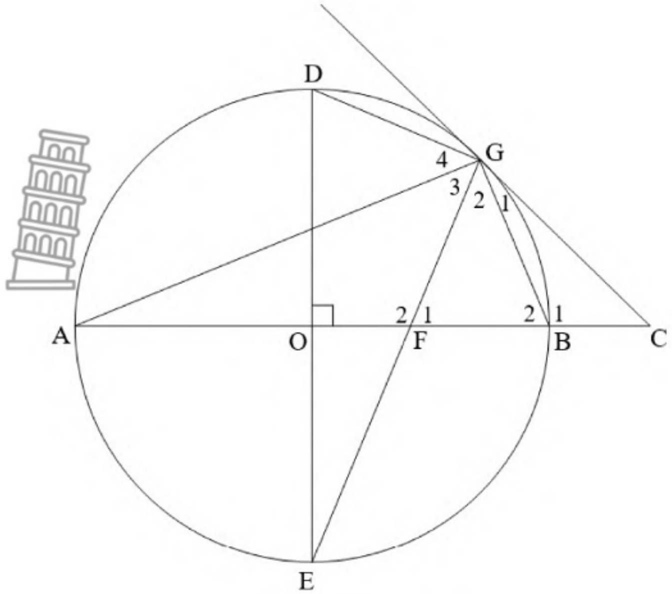
b) C

1.2.2 Prove, with reasons, that AFED is a cyclic quadrilateral.

1.2.3 Prove, with reasons, that $F_3 = x$.

1.2.4 If the area of $\Delta AEB = 6{,}25 \times \Delta DEC$, calculate $\frac{AE}{ED}$

1.3 In the diagram, O is the centre of the circle and CG is a tangent to the circle at G. The straight line from C passing through O cuts the circle at A and B. Diameter DOE is perpendicular to CA. GE and CA intersect at F. Chords DG, BG and AG are drawn.



1.3.1 Prove that:

(a) DGFO is a cyclic quadrilateral

(b) GC = CF

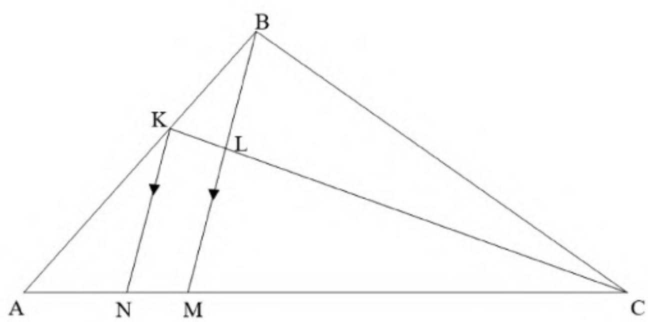
1.3.2 If it is further given that CO = 11 units and DE = 14 units, calculate:

(a) The length of BC

(b) The length of CG

(c) The size of $\hat{E}$.

1.4. In ΔABC in the diagram, K is a point on AB such that AK : KB = 3 : 2. N and M are points on AC such that KN || BM. BM intersects KC at L. AM : MC = 10 : 23.

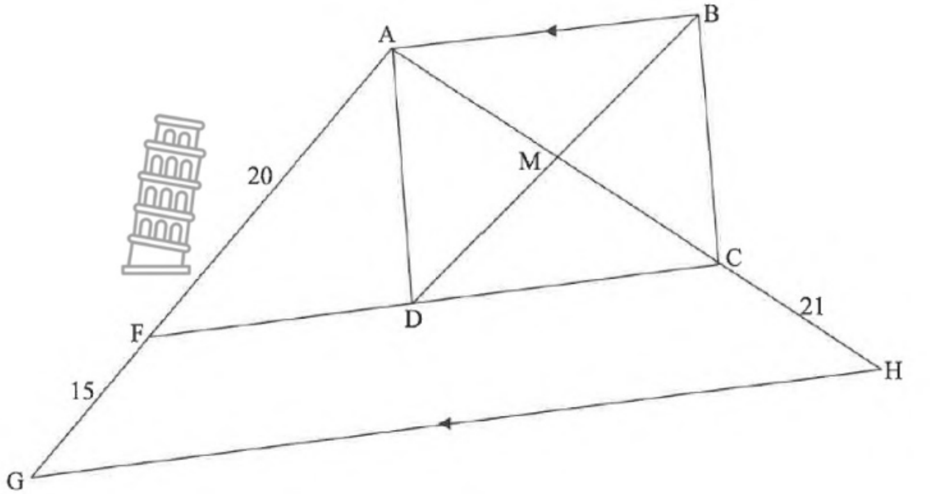


Determine, with reasons, the ratio of:

1.4.1 $\frac{AN}{AM}$

1.4.2 $\frac{CL}{LK}$

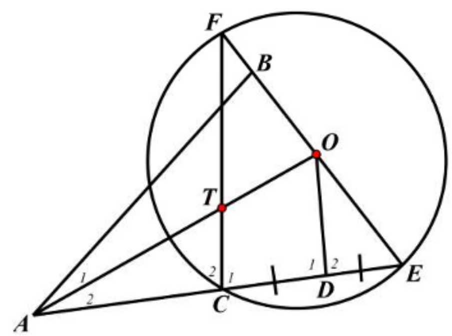
1.5. In the diagram below, ΔAGH is drawn. F and C are points on AG and AH respectively such that AF = 20 units, FG = 15 units and CH = 21 units. D is a point on FC such that ABCD is a rectangle with AB parallel to GH. The diagonals of ABCD intersect at M, a point on AH.



1.5.1 Explain why FC // GH

1.5.2 Calculate, with reasons, the length of DM.

1.6 In the diagram, FBOE is diameter of a circle with Centre O. Chord EC produced meets line BA at A, outside the circle. D is the midpoint of CE. OD and FC are drawn. AFBC is a cyclic quadrilateral.



1.6.1 Prove, giving reasons that:

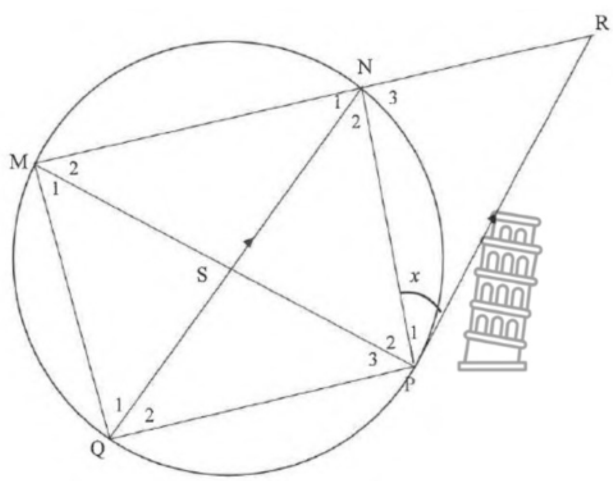
a) FC // OD

b) $D\hat{O}E = B\hat{A}E$

c) $AB \times OF = AE \times OD$

1.6.2 If it is further given that $AT = 3\ OT$, prove that $5CE^2 = 2BE.FE$

1.7 Chord QN bisect   and intersects chord MP at S. The tangent at P meets MN produced at R such that QN // PR. Let $\hat{P}_1 = x$.



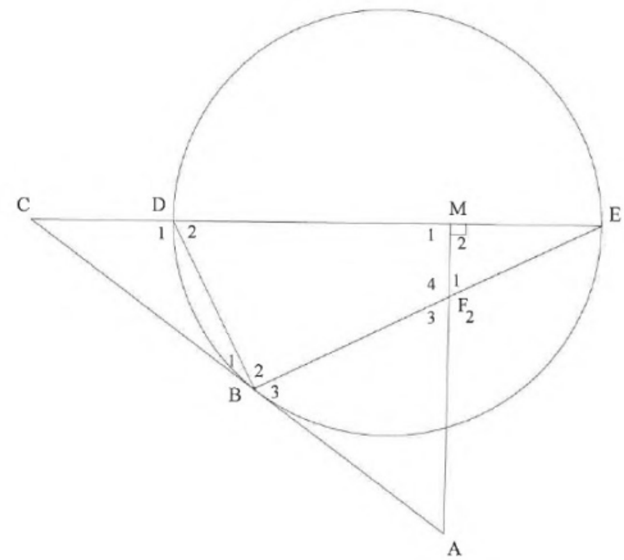
1.7.1 Determine the following angle in terms of x. Give reasons

a) N_2

b) Q_2

1.7.2 Prove, giving reasons, that $\frac{MN}{NR} = \frac{MS}{SQ}$

1.8. In the diagram, a circle passes through D,B and E. Diameter ED of the circle is produced to C and AC is a tangent to the circle at B. M is a point on DE such that AM ⊥ DE. AM and chord BE intersects at F.



1.8.1 Prove giving reasons that:

a) FBDM is a cyclic quadrilateral

b) $B_3 = F_1$

c) ΔCDB /// ΔCBE

1.8.2 If it further given CD = 2 units and DE = 6 units, calculate the length of:

a) BC

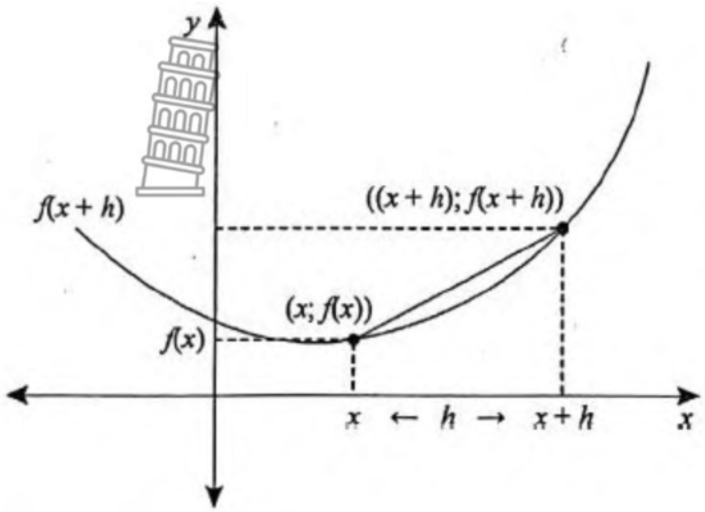
b) DB

TOPIC: DIFFERENTIAL CALCULUS

LESSON 1: GRADIENT OF THE TANGENT

NOTES

- Use the sketch below to compare the gradient of the straight line and the gradient of the curve between the two points where the two graphs meet.

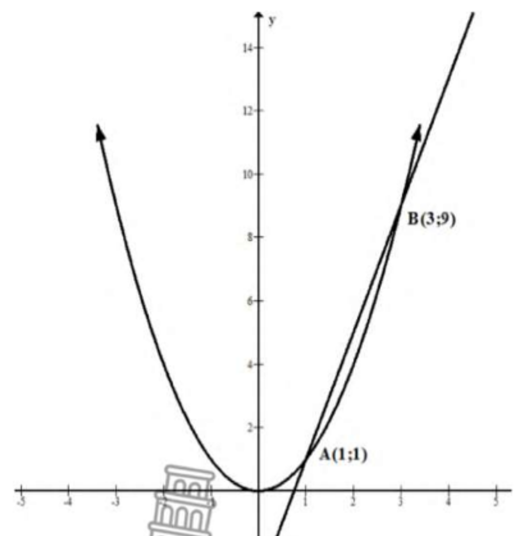


- The straight line has a constant gradient, but a curve does not have a constant gradient. In that instance, it will be concluded that curve has an **average gradient** between those two given points.
- The average gradient of the curve between two points is almost the same as the gradient of the straight line connecting those two points, therefore, the gradient formula has to be used.
- Given the curve above as defined by $y = f(x)$:

$$\text{Average gradient } = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x+h) - f(x)}{x+h-x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

Example 1

Given the graph alongside defined by $f(x) = x^2$, find the average gradient (rate of change) of the curve between $\text{A}(1;1)$ and $\text{B}(3;9)$.



Solution

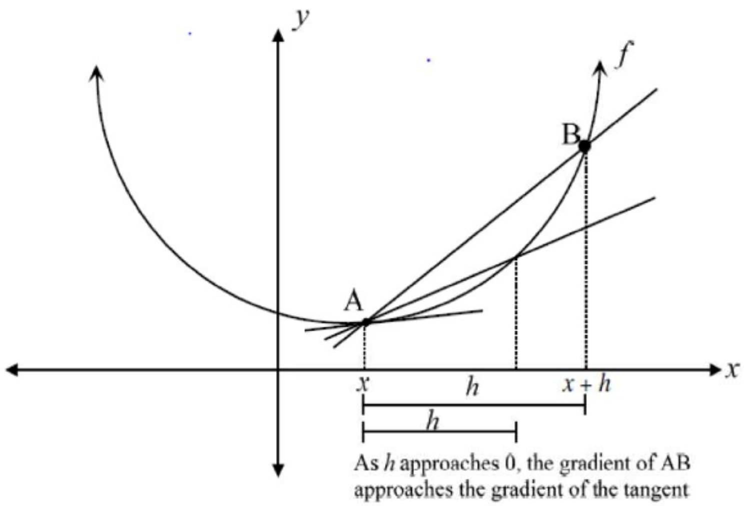
$f(x+h) = 9$

$f(x) = 1$

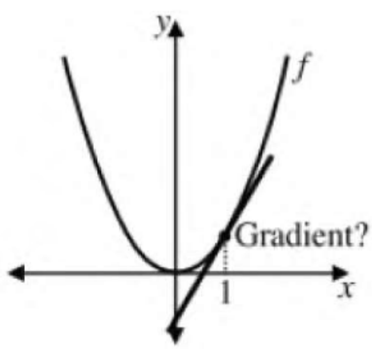
$h = 3 - 1 = 2$

$$\therefore \text{Average gradient } = \frac{f(x+h) - f(x)}{h}$$
$$= \frac{9-1}{2}$$
$$= 4$$

- In the diagram below, the gradient of the tangent at A is also called the gradient of the graph of f at the point A.
- As point B approaches A, the value of h, the distance between the x value of the points tends towards zero. The gradient of AB will, therefore, change and approach the gradient of the tangent line at A.



- In reference to the sketch above:
  - As h decreases (gets smaller and smaller), the line AB gets closer and closer to becoming the line that is the tangent to the curve at A.
  - In other words, as h gets closer and closer to 0 (tends to 0), the value of $f(x+h)$ gets closer and closer to $f(x)$.
  - As this happens, the gradient of straight line AB gets closer to the gradient of the tangent to the curve at A.



  - We can represent this scenario in the form of **limit**.
    ie; $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ where $=\dfrac{f(x+h)-f(x)}{h}$ represents the average gradient of f between x and $x+h$.
  - We call this formula as the formula for calculation of the gradient of the tangent to a curve at the point A.

Example 2

Determine the gradient of the tangent to the curve defined by $f(x)=x^2-x-6$ at $x=2$.

Solution

$f(x) = x^2 - x - 6$

$f(x+h) = (x+h)^2 - (x+h) - 6$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - (x+h) - 6 - (x^2 - x - 6)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{(x^2 - (x+h) - 6 - (x^2 - x - 6)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x - h - 6 - x^2 + x + 6}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2xh + h^2 - h}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{h(2x + h - 1)}{h}$$

$$f'(x) = \lim_{h\to 0} 2x + h - 1$$

$f'(x) = 2x - 1$

$f'(2) = 2(2) - 1$

$f'(2) = 3$

ACTIVITIES/ASSESSMENTS

1. Class work

1.1 Determine the average gradient as x increases from 1 to 3 on the curve defined by $f(x) = 4x^2 - 3$.

1.2 Determine the gradient of the tangent to the curve defined by $f(x) = x^2 + 3x + 2$ at the point $(-1; 0)$.

2. Home work

2.1 Determine the average gradient of the curve defined by $f(x) = -x^2 + 2$ between $x = 1$ and $x = 3$.

2.2 Determine the gradient of the tangent to the curve defined by $f(x) = x^2 + 5x + 4$ at $x = -2$

TOPIC: DIFFERENTIAL CALCULUS

LESSON 2: FIRST PRINCIPLE

NOTES

- The formula that is used to determine the derivative from First Principles is:

  $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

  $f'(x)$ is the gradient at any point.

  To determine $f(x+h)$ substitute $(x+h)$ where there is x in the given function.

  Work out $f(x+h)$ separately before substituting it into the derivative formula.

  Examples

1. Given $f(x) = x^2$

   1.1. Determine $f'(x)$ from first principles.

1.2. Hence determine $f'(2)$

1.3. What does the above answer mean?

2. Determine the derivative of each of the following functions from first principles.

2.1. $f(x)=4x+2$

2.2. $f(x)=x^2+2x-3$

2.3. $f(x)=2x^3$

2.4. $f(x)=\dfrac{2}{x}$

2.5. $f(x)=5$ (You can find the derivative of a constant from first principles)

Solutions

Example 1

1.1 $f(x)=x^2$

$f(x+h)=(x+h)^2$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{(x+h)^2-(x^2)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{x^2+2xh+h^2-(x^2)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{x^2+2xh+h^2-x^2}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{2xh+h^2}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{h(2x+h)}{h}$

$f'(x)=\lim_{h\to 0}2x+h$

$f'(x)=2x$

1.2 $f'(2)=2(2)$

$f'(2)=4$

1.3 The gradient at point $x=2$ is 4.

2.2. $f(x)=x^2+2x-3$

$f(x+h)=(x+h)^2+2(x+h)-3$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{(x+h)^2+2(x+h)-3-(x^2+2x-3)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{x^2+2xh+h^2+2x+2h-3-(x^2+2x-3)}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{x^2+2xh+h^2+2x+2h-3-x^2-2x+3}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{2xh+h^2+2h}{h}$

$f'(x)=\lim_{h\to 0}\dfrac{h(2x+h+2)}{h}$

$f'(x)=\lim_{h\to 0}2x+h+2$

$f'(x)=2x+2$

Example 2

2.1. $f(x)=4x+2$

$f(x+h)=4(x+h)+2$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

2.3. $f(x+h)=2(x+h)^3$

$f(x+h)=2(x+h)(x+h)^2$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{4x+4h+2-(4x+2)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{4x+4h+2-4x-2}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{4h}{h}$$

$$f'(x)=\lim_{h\to 0}4$$

$$f'(x)=4$$

$$f'(x)=\lim_{h\to 0}\frac{2(x+h)^3-(2x^3)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3-(2x^3)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3-2x^3}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{h(6x^2+6xh+2h^2)}{h}$$

$$f'(x)=\lim_{h\to 0}6x^2+6xh+2h^2$$

$$f'(x)=6x^2$$

2.4 $f(x)=\dfrac{2}{x}$

$$f(x+h)=\frac{2}{x+h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{2}{x+h}-\left(\frac{2}{x}\right)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}$$

$$f'(x)=\lim_{h\to 0}\left(\frac{2}{x+h}-\frac{2}{x}\right)\div\frac{h}{1}$$

$$f'(x)=\lim_{h\to 0}\left(\frac{2x-2(x+h)}{x(x+h)}\right)\times\frac{1}{h}$$

$$f'(x)=\lim_{h\to 0}\left(\frac{2x-2x-2h)}{x(x+h)}\right)\times\frac{1}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{-2h}{x^2+xh}\times\frac{1}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{-2}{x^2+xh}\times 1$$

$$f'(x)=\frac{-2}{x^2}$$

2.5 $f(x+h)=5$

$f(x)=5x^0$ ($x^0=1$)

The above step is done to determine $f(x+h)$

$\therefore f(x+h)=5(x+h)^0$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{5(x+h)^0-(5)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{5-(5)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{5-5}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{0}{h}$$

$$f'(x)=0$$

ACTIVITIES/ASSESSMENTS

Activity 1

Differentiate each of the following from first principles.

1. $f(x)=9$
2. $f(x)=4x^2$
3. $f(x)=x-3x^2+4$
4. $f(x)=\frac{5}{x}$
5. $f(x)=-x^3$
6. $f(x)=2p-x$
7. $f(x)=-\frac{4}{x}$
8. $f(x)=\frac{-3}{x}$

Activity 2

1. Given the function : $f(x)=\frac{8}{x}$

   1.1 Determine $f'(x)$ from first principles.

   1.2 Calculate $f'(-1)$

   1.3 Explain what your answer in 1.2 means.

TOPIC: DIFFERENTIAL CALCULUS

LESSON 3: RULES OF DIFFERENTIATION

NOTES

- Differentiation rules allow differentiation without using first principles.

Rules of differentiation

Rule 1(Constant rule)

The derivative of a constant is 0

If $g(x)=t$, where t is a constant real number, then $g'(x)=0$

Example :Determine $g'(x)$ if $g'(x)=12$.

$g'(x)=0$ (This is true for any real number)

Rule 2 (Power Rule)

If $g(x)=x^n$, where n is a constant real number, then $g'(x)=nx^{n-1}$.

Example : If $g(x)=x^7$ determine $g'(x)$.

$g(x)=x^7$

$g'(x)=7x^{7-1}$

$g'(x)=7x^6$

Rule 3 (The derivative of a function multiplied by a constant)

If $g(x)=k.f(x)$, where k is a constant real number, then $g'(x)=k.f'(x)$

Example: If $g(x)=-2x^{-2}$ determine $g'(x)$

$g(x)=-2x^{-2}$

$g'(x)=-2(-2x^{-2-1})$

$g'(x)=4x^{-3}$

Rule 4 (The sum/ difference rule)

If $g(x)=f(x)\pm h(x)$, then $g'(x)=f'(x)\pm h'(x)$

Example: If $g(x)=x^4+2x^{-3}$ determine $g'(x)$

$g(x)=x^4+2x^{-3}$

$g'(x)=4x^3+2(-3x^{-3-1})$

$g'(x)=4x^3-6x^{-4}$

NOTATIONS

There are different notations used in differentiation.

- If $f(x)=x^3$ $f'(x)=3x^2$
- $\frac{d}{dx}\left[3x^4\right]=12x^3$
- $D_x\left[\frac{7}{4}x^4\right]=7x^3$
- If $y=3x^4$ $\frac{dy}{dx}=3x^3$

Examples

1. Determine $\frac{dy}{da}$ if $y=\frac{1}{2}x^4+ax^2$

$\frac{dy}{da}=x^2$

2. Determine $D_x\left[-\frac{7}{3}x^3-\pi\right]$

$D_x\left[-\frac{7}{3}x^3-\pi\right]=-7x^2$

Note

From the examples above it is clear that to differentiate, each term on your equation should be in a form of ax^n, no fractions, no surds, no products and no variables in the denominator.

Different cases of differentiation.

1.Fractions if a denominator is a binomial or polynomial

Example

1. Given $f(x) = \frac{x^3 - 1}{x - 1}; x \neq 1$, determine $f'(x)$.

$f(x) = \frac{(x-1)(x^2 + x + 1)}{x - 1}$ (Factorise using difference of two cubes).

$f(x) = x^2 + x + 1$ (Now all terms are in the form of ax^n, you can now differentiate)

$f'(x) = 2x + 1$ (Always put the notation on your final answer)

2. **Fractions when a denominator is a monomial**

Make use of the laws of exponents so that you are able to change a fraction into an exponential form, given that the denominator is a monomial $e.g. \frac{1}{p} = p^{-1}$.

Examples

Differentiate the following with respect to x

1.1 $f(x) = \frac{3}{2x}$ $f(x) = \frac{3}{2}x^{-1}$ $f'(x) = -\frac{3}{2}x^{-2}$ (Remember the notation)	1.2 $f(x) = \frac{3x^3 + \frac{5}{4}x^2 + 2}{x}$ (Expand) $f(x) = \frac{3x^3}{x} + \frac{\frac{5}{4}x^2}{x} + \frac{2}{x}$ (Use laws of exponents) $f(x) = 3x^2 + \frac{5}{4}x + 2x^{-1}$ $f'(x) = 6x + \frac{5}{4} - 2x^{-2}$ (Remember the notation)

3. Surds

Determine $f'(x)$, given the following:

1.1 $f(x) = \sqrt[3]{x^7}$ $f(x) = \sqrt[3]{x^7} = x^{\frac{7}{3}}$ (Use the rule $\sqrt[n]{a^m} = a^{\frac{m}{n}}$) $f'(x) = \frac{7}{3}x^{\frac{4}{3}}$	1.2 $D_x\left[\frac{4}{\sqrt[5]{x^{-3}}}\right]$ (Use the rule $\sqrt[n]{a^m} = a^{\frac{m}{n}}$) $D_x\left[\frac{4}{x^{\frac{-3}{5}}}\right]$ $D_x\left[4x^{\frac{3}{5}}\right] = \frac{12}{5}x^{-\frac{2}{5}}$

4. Products.

Differentiate the following with respect to x .

1. $y=7x^{-3}.\frac{1}{2}x^{5}$ (Determine the product first) $y=\frac{7}{2}x^{-15}$ $\frac{dy}{dx}=-\frac{105}{2}x^{-16}$	2. $f(x)=3x^{-\frac{1}{2}}.\frac{4}{\sqrt{x}}$ (work out surd and the product) $f(x)=3x^{-\frac{1}{2}}.\frac{4}{x^{\frac{1}{2}}}$ $f(x)=3x^{-\frac{1}{2}}.4x^{-\frac{1}{2}}$ $f(x)=12x^{-1}$ $f'(x)=-12x^{-2}$
3. $y=(x^{2}-2x)^{2}$ (work out the product) $y=x^{4}-4x^{3}+4x^{2}$ $\frac{dy}{dx}=4x^{3}-12x^{2}+4$	

5. Subject of the formula.

To determine $\frac{dy}{dx}$; make sure that y is the subject of the formula.

This should be inter-grated with algebra.

1. Determine $\frac{dy}{dx}$ given the following

1. $3-xy=x^{2},\ x\neq 0$ $xy=3-x^{2}$ $y=3x^{-1}-x$	2. $xy=3x^{2}-3y-27$ $xy+3y=3x^{2}-27$ $y(x+3)=3(x^{2}-9)$ $\frac{y(x+3)}{(x+3)}=\frac{3(x+3)(x-3)}{(x+3)}$ $y=3x-9$ $\frac{dy}{dx}=3$
3. $\sqrt{y}=3x-5$ $\left(\sqrt{y}\right)^{2}=(3x-5)^{2}$ $y=9x^{2}-30x+25$ $\frac{dy}{dx}=18x-30$	

ACTIVITIES

Activity 1	**Activity 2**
1.1 Determine $\frac{dy}{dx}$ if $y=x^{3}-4x^{2}+6x+5$	2.1 Determine $\frac{dA}{dr}$ if $A=2\pi r^{2}+2\pi h$
1.2 Determine $\frac{dy}{dx}$, if $y=\frac{\sqrt[3]{x^{2}}-5\sqrt{x}+15x+2}{5\sqrt{x^{3}}}$	2.2 Determine $s'(t)$ if $s=ut+\frac{1}{2}at^{2}$
	2.3 Determine $f'(x)$ if $f(x)=(x-3)^{3}$

1.3 Determine $f'(x)$ if $f(x)=3x^{\frac{3}{2}}-3x^{\frac{1}{2}}+6x^{-\frac{1}{2}}$	2.4 Determine $f'(x)$ if $f(x)=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$
1.4 Determine $\frac{ds}{dt}$ if $s=\frac{3t^2}{4}+\frac{2}{3t}-\frac{9}{\sqrt[3]{t^2}}$	2.5 If $f(x)=\sqrt{x}-\frac{x^2-2x}{x^2}$, determine $f'(x)$.
1.5 Determine $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ if $y=4x^3+3x^2+2x+1$	2.6 Determine $f'(x)$ if $f(x)=\frac{(1-x)^2}{x^2}$
1.6 Determine $f'(x)$ if $f(x)=\frac{x^2+2x+4}{x^2+2x}$	2.7 Determine $\frac{dy}{da}$ if $ax-\frac{y}{a}=\frac{a^2}{4}$
1.7 Determine $\frac{dy}{dx}$ if $\frac{2-x^2}{y}=x$	2.8 Determine $\frac{dy}{dx}$ if $xy=x^2-2x+y+3$
1.8 Determine $f'(x)$ if $f(x)=\frac{8x^3-125}{5-2x}$	2.9 Determine $\frac{dy}{dx}$ if $\sqrt{y}=2x+9$

TOPIC: DIFFERENTIAL CALCULUS

LESSON 4: EQUATION OF A TANGENT

NOTES

From Euclidean geometry, a tangent is a line that touches a curve exactly at one point.

Given that it is a straight line, it is given by an equation $y=mx+c$

Remember that a derivative is a gradient at any point.

Therefore, a gradient of a tangent is a derivative at a given coordinate of x.

There are many cases of a tangent.

DETERMINING THE EQUATION OF A TANGENT

CASE 1:GIVEN THE x-COORDINATE.

Example Determine the equation of a tangent to $f(x)=x^2-6x+5$ at $x=2$ For an equation of a straight line, you need a gradient and one point. To determine the gradient of a tangent you need to differentiate first and substitute the value of x that is given. (Remember, derivative is a gradient)	$f'(x)=2x-6$ (This is the gradient at any point) $f'(2)=2(2)-6$ $f'(2)=-2$ (Gradient of a tangent at $x=2$) $f(2)=(2)^2-6(2)+5$ (substitute $x=2$ to find y-value) $f(2)=-3$ Point $(2;-3)$ and $m_t=-2$ $y=mx+c$ $-3=-2(2)+c$ $c=1$ $\therefore y=-2x+1$

CASE 2: GIVEN THE GRADIENT OF A TANGENT

In this case, remember that to find the gradient of a tangent you differentiated , substituted the x coordinate of the point of intersection of the curve and the tangent, now go back . **Example**	$f\left(\frac{5}{6}\right)=\frac{65}{12}$ $\left(\frac{5}{6};\frac{65}{12}\right)$ $y=mx+c$

Determine the equation of a tangent to the curve $f(x)=3x^2-2x+5$ given that the gradient of a tangent is 3. $f'(x)=6x-2=m_t$ $\therefore 6x-2=3$ $x=\frac{5}{6}$ $f\left(\frac{5}{6}\right)=3\left(\frac{5}{6}\right)^2-2\left(\frac{5}{6}\right)+5$	$\frac{65}{12}=3\left(\frac{5}{6}\right)+c$ $c=\frac{35}{12}$ $\therefore y=3x+\frac{35}{12}$

CASE 3: GIVEN THE y-COORDINATE

Example Determine the equation of a tangent to the curve $f(x)=x^2+x-11$ at a point $(x;1)$ where $x>0$. $x^2+x-11=1$ ($f(x)=1$, from the given point) $x^2+x-12=0$ $(x+4)(x-3)=0$	$x=-4/x=3$ $x>0 \therefore x=3$ $f'(x)=2x+1$ $f'(3)=2(3)+1$ $m_t=7$ $y=mx+c$ $1=7(3)+c$ $c=-20$ $\therefore y=7x-20$

Case 4: Given the y-intercept.

Example Given the curve $f(x)=x^2-2x-8$ and the equation of a tangent $y=mx-9$, determine the value of m, the gradient of a tangent where $x<0$. $f'(x)=2x-2$ $x^2-2x-8=x(2x-2)-9$ $x^2-2x-8=2x^2-2x-9$	$x^2-1=0$ $(x-1)(x+1)=0$ $x=\pm1, \therefore x=-1$ $f'(-1)=2(-1)-2=m_t$ $m_t=-4$ $\therefore y=-4x-9$

DETERMINING THE EQUATION OF A FUNCTION GIVEN EQUATION OF TANGENT

❖ Be able to go back to finding the equation of a curve using all that you have done above. **Example** $g(x)=9x+10$ Is the equation of a tangent to the curve $f(x)=ax^2+bx+6$ at a point $(-2;-8)$ determine the values of a and b. $f'(x)=2ax+b=m_t$, $m_t=9$ at $x=-2$ $\therefore 2a(-2)+b=9$	$a(-2)^2+b(-2)+6=-8$ $4a-2b=-14$ $4a-2(9+4a)=-14$ $-4a=4$ $a=-1$ $b=9+4(-1)$ $b=5$ $\therefore f(x)=-x^2=5x+6$

ACTIVITIES/ASSESSMENTS

1) Determine the equation of the tangent to the curve $y=3x^2-2x+2$ at $x=-4$
2) Determine the equation of the tangent to the curve $y=x^3-x^2-35x-50$ at the point where $x=-3$
3) Calculate the coordinates of the point where the tangent meets the curve again.
4) Calculate the value(s) of p if $y=p-9x$ is a tangent to the curve $y=-x^3+3x-2$

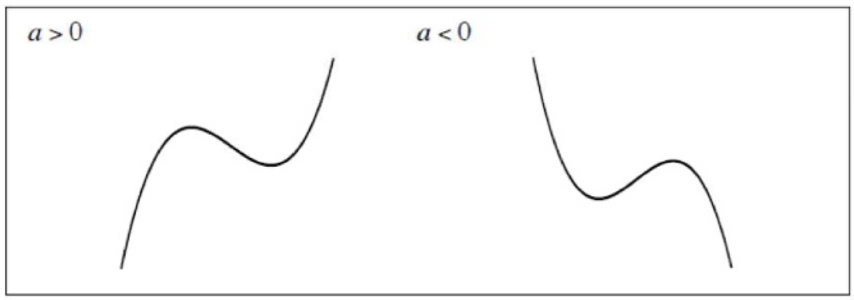
5) Calculate the point on the curve $y = 4\sqrt{x}$ where the gradient of the curve is 1.
6) Given $f(x) = -\frac{x^2}{2} + x$ .A tangent to the graph of f has a gradient of -5 and x intercept $(a\ ;\ 0)$. Determine the value of a.
7) Determine the equation of a tangent to the curve $f(x) = x^3 + 3x^2 - 4x - 2$ at $x = 1$.
8) Determine the gradient of a tangent to the curve $f(x) = 2x^2 - x$ if $y = mx - 8$ is the equation of a tangent to the curve and $x > 0$.
9) Determine the values of p and q if $y = 5x - 7$ is the equation of the tangent to the curve $f(x) = x^3 + px^2 + qx - 2$.

TOPIC: DIFFERENTIAL CALCULUS

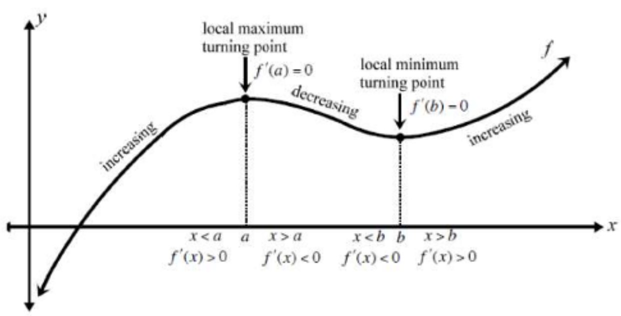
LESSON 5: CUBIC FUNCTION

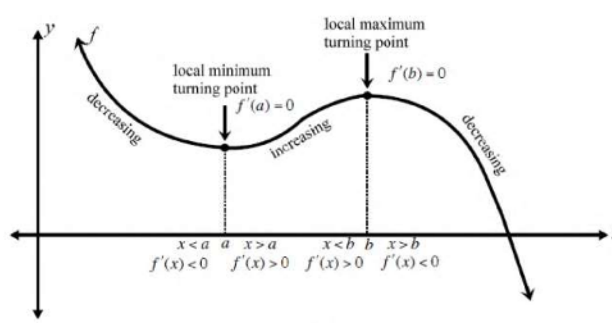
NOTES

- The general form of the cubic function is : $f(x) = ax^3 + bx^2 + cx + d$
- The impact of the value of a on the shape of the cubic function:

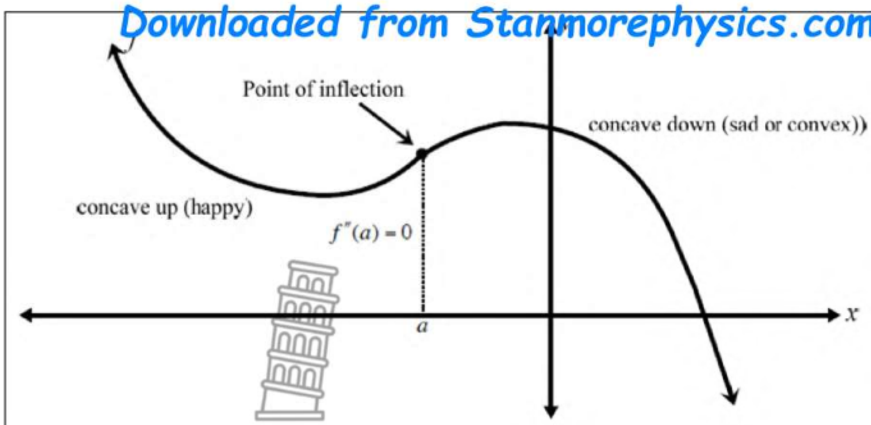


- Cubic Functions have the following important points: Turning (Stationery) point(s), x – intercept(s), y – intercept and an inflection point.

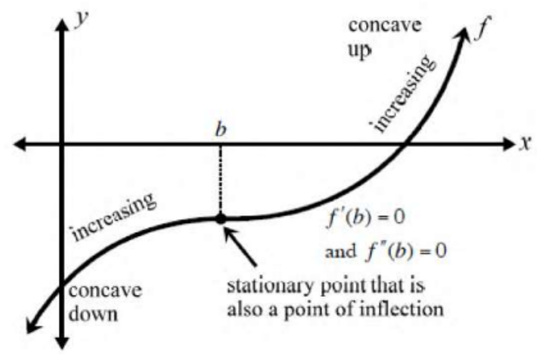




- If the cubic function is defined by $f(x) = ax^3 + bx^2 + cx + d$, then the turning point x – value is obtained by solving for $f'(x) = 0$
. Then the y – value of the turning point is found by substituting the x – value into $f(x)$
- The **inflection point** is the point where the graph changes its concavity. To find the x – value of the inflection point solve for $f''(x) = 0$. Then the y – value of the inflection point is found by substituting the x – value into $f(x)$.



- On rare cases the inflection point and the turning point may be the same point. In these such cases the function may appear like the diagram alongside. Note that in such cases the cubic function will have only one x – intercept.



SKETCHING A CUBIC FUNCTION

Consider the example below to sketch a cubic function:

Sketch the graph of $f(x)=-x^3-4x^2+3x+18$

Step1- Find the intercepts with the axis

Y- intercept: let $x=0$

$$f(0)=-(0)^3-4(0)^2+3(0)+18=18$$

X-intercept let y=0

$$0=-x^3-4x^2+3x+18$$

$$x^3+4x^2-3x^2-18=0$$

$$(x-2)(x^2+6x+9)=0$$

$$(x-2)(x+3)^2=0$$

$$x=2 \quad \text{or} \quad x=-3$$

Step 2:Find the turning points of f (Stationery Points)

$$f(x)=-x^3-4x^2+3x^2+18$$

$$f'(x)=-3x^2-8x+3$$

$$3x^2+8x-3=0$$

$$(3x-1)(x+3)=0$$

$$x=\frac{1}{3} \text{ or } x=-3$$

$$f\left(\frac{1}{3}\right)=-\left(\frac{1}{3}\right)^3-4\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)+18$$

$$f\left(\frac{1}{3}\right)=18\frac{14}{27}=18{,}52$$

$$f(-3)=-(-3)^3-4(-3)^2+3(-3)+18$$

$$f(-3)=0$$

Graph of f has turning points at $\left(\frac{1}{3};18.52\right)$ and $(-3;0)$

Step 3 : Determine the point of inflection of : $f(x)=-x^3-4x^2+3x^2+18$

$$f'(x)=-3x^2-8x+3$$

$$f''(x)=-6x-8$$

$$6x=-8$$

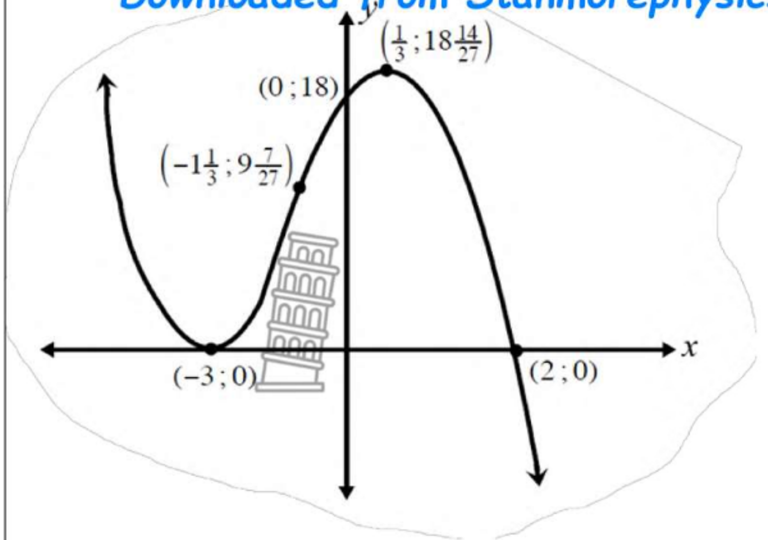
$$x=-\frac{8}{6}=-\frac{4}{3}$$

$$f\left(-\frac{4}{3}\right)=-\left(-\frac{4}{3}\right)^3-4\left(-\frac{4}{3}\right)^2+3\left(-\frac{4}{3}\right)+18$$

$$f\left(-\frac{4}{3}\right)=9\frac{7}{27}$$

Point of inflection is $\left(-1\frac{1}{3};9\frac{7}{27}\right)$

Step 4 : Sketch the graph of : $f(x)=-x^3-4x^2+3x^2+18$



Activities/Assessments:

1. Sketch the graphs of each of the functions below making sure to first find the following:
   i. Find the intercepts with the x – axis and y – axis.
   ii. Find the co-ordinates of the turning points,
   iii. Find the coordinates of the inflection point.

1.1.1 $f(x)=x^3-2x^2-4x+8$

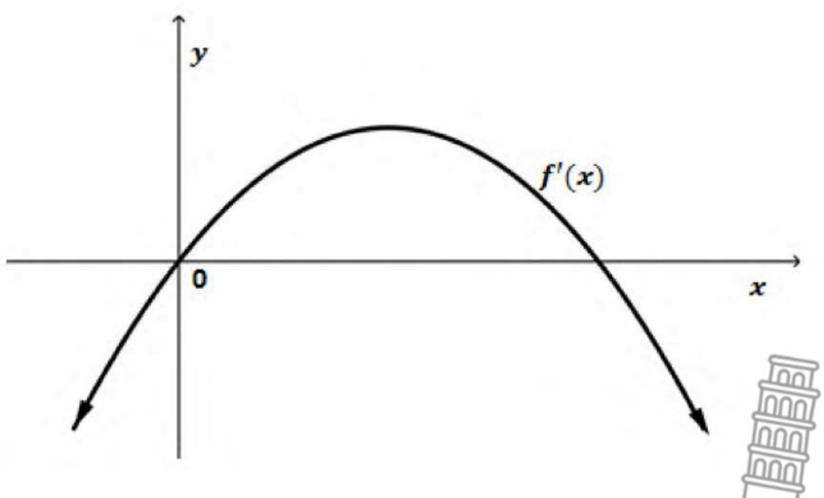
1.1.2 $f(x)=-x^3+6x^2-9x$

1.1.3 $f(x)=-2x^3+x^2-8x-4$

1.1.4 $f(x)=x^3-3x^2+3x-1$

1.1.5 $f(x)=6x^2-x^3$

2. For a certain function $f(x)$, the first derivative is given as $-3x^2+6x$



2.1 Calculate the x - coordinates of the stationary points of $f(x)$

2.2 Determine the value of x at the point where the concavity of $f(x)$

2.3 It is further given that $f(x)=ax^3+bx^2+cx-4$ and $f(2)=0$. Draw a rough sketch of $f(x)$, clearly showing the coordinates of the turning points.

3. Given: $f(x)=ax^3+bx^2+3x+3$ and $g(x)=f''(x)$ where $g(x)=12x+4$.

3.1 Show that $a=2$ and $b=2$

3.2 Prove that f will never decrease for any real value of x.

3.3 Determine the minimum gradient of f

3.4 Explain the concavity of f for all values of x where $g(x)<0$.

TOPIC: DIFFERENTIAL CALCULUS

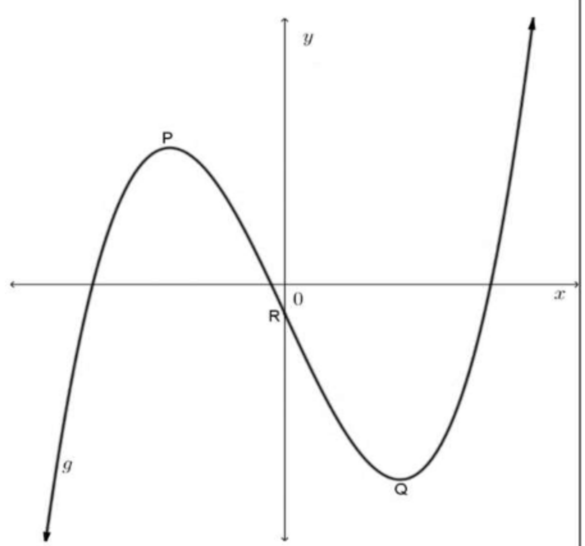
LESSON 6: CUBIC FUNCTION

NOTES

- Recap all previous work covered in graph sketching.

Examples

1. The sketch below shows the graph of $g(x)=x^3-6x-1$. P and Q are the turning points and R the y-intercept of g.



1.1 Determine the coordinates of R.

1.2 Determine the coordinates of the turning points P and Q.

1.3 Calculate the values of x for which g strictly increases as x increases.

1.4 If $h(x)=g'(x)$, determine for which values of x is $h(x)\le 0$.

1.5 Determine the equation of the tangent to g at R.

1.6 Write down the equation of the line perpendicular to the tangent at P.

Discuss the solution below in Full

1.1 $R(0;-1)$

1.2 $g(x) = x^3 - 6x - 1$

$g'(x) = 3x^2 - 6$

$3x^2 - 6 = 0$

$x = \pm\sqrt{2} = \pm 1{,}41$

$g(\pm\sqrt{2}) =$

$y = (-\sqrt{2})^3 - 6(-\sqrt{2}) - 1$ or $y = (\sqrt{2})^3 - 6\sqrt{2} - 1$

$= -2\sqrt{2} + 6\sqrt{2} - 1$ or $= -1 - 4\sqrt{2}$

$= 4{,}66$ $= -6{,}66$

$P(-1{,}41; 4{,}66)$ $Q(1{,}41; -6{,}66)$

1.3 $x < -\sqrt{2}$ or $x > \sqrt{2}$

OR

$x \in (-\infty; -1{,}41)$ or $x \in (1{,}41; \infty)$

1.4 $-\sqrt{2} \le x \le \sqrt{2}$

1.5 $g'(x) = 3x^2 - 6$ and $x = 0$ at R

$m_t = g'(0) = 3(0)^2 - 6 = -6$

$y - y_1 = -6(x - x_1)$

$y + 1 = -6(x - 0)$

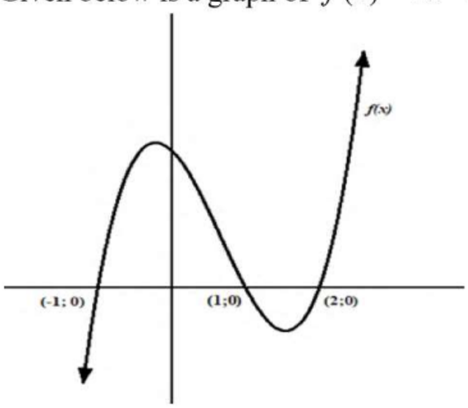
$y = -6x - 1$

1.6 $h(x) = \frac{1}{6}x - 1$

DETERMINING THE EQUATION OF A CUBIC FUNCTION

CASE1 : Given x-intercepts and value of a or other point.

Given below is a graph of $f(x) = 2x^3 + bx^2 + cx + d$. Determine the values of b, c and d.



Use $f(x) = a(x - x_1)(x - x_2)(x - x_3)$

$f(x) = 2(x+1)(x-1)(x-2$

$f(x) = 2(x+1)(x^2 - 3x + 2)$

$f(x) = 2x^3 - 4x^2 - 2x + 4$

$b = -4,\ c = -2, d = 4$

CASE2: Given the turning point.

The function defined by $f(x) = x^3 + ax^2 + bx - 4$ is sketched alongside.

$P(-1; -3)$ is one of the turning points of f. Determine the values of a and b.

$-3=(-1)^3+a(-1)^2+b(-1)-4$

$-3=-1+a-b-4$

$2=a-b$

$f'(x)=3x^2+2ax+b$

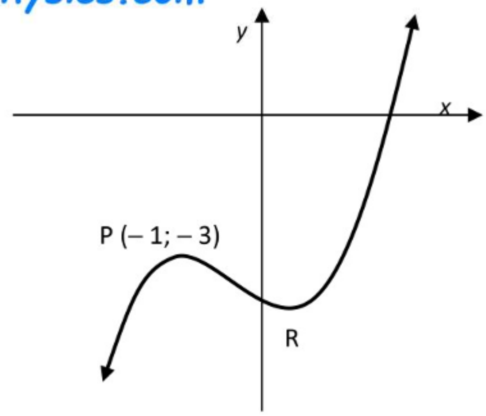
$f'(-1)=0$

$3(-1)^2+2a(-1)+b=0$

$3-2a+b=0$

$a-1=0 \quad a=1$

$\therefore 1-b=2 \therefore b=-1$



CASE3: Given the point of inflection

The graph of $f(x)=ax^3+bx^2+9x$ has the point of inflection at $P(2;2)$ .Find the values of a and b

Use $f''(x)=0$ at $x=2$ and substitute the point into the original function.

$f''(x)=6ax+2b$

$6ax+2b=0$

$6a(2)+2b=0$

$12a+2b=0$

$6a+b=0$...........(1)

$f(2)=a(2)^3+b(2)^2+9(2)=2$

$8a+4b+18=2$

$8a+4b=-16$

$2a+b=-4$

$b=-2a-4$...........(2)

$6a-2a-4=0$

$4a=4$

$a=1$

$b=-2(1)-4$

$b=-6$

CASE4: Given the derivative and a point.

Given that $f'(x)=3x^2+4x-9$ is a derivative of $f(x)=ax^3+bx^2+cx+d$ where $f(0)=18$ is a point on f

Use $f'(x)=3x^2+4x-9$ and $f'(x)=3ax^2+2bx+c$ to equate co-efficients

$3a=3 \qquad 2b=4 \qquad c=-9$

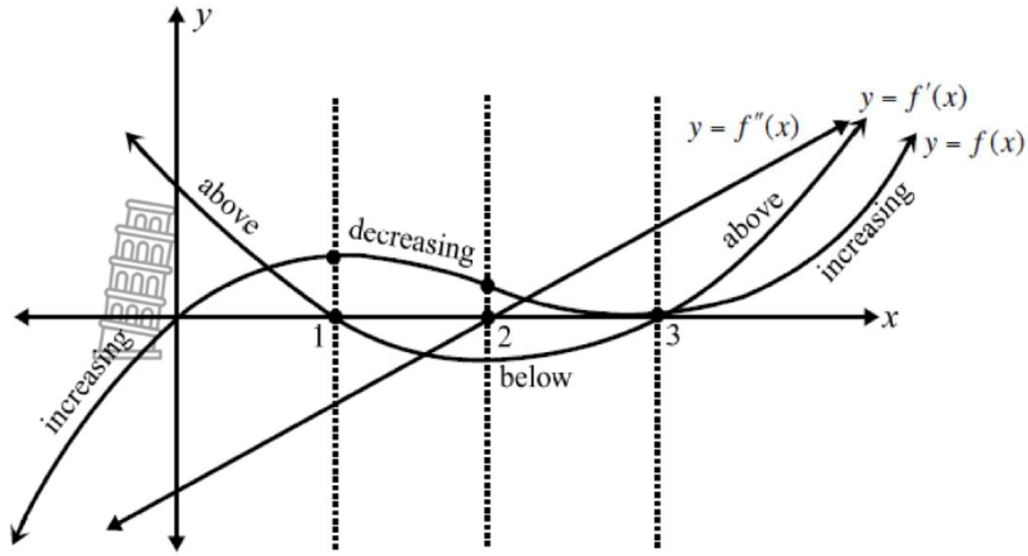
$a=1 \qquad b=2$

$f(x)=x^3+2x^2-9x+d$

$f(0)=(0)^3+2(0)^2-9(0)+d=18$

$d=18$

CASE5: Interprete by comparing $f(x)$, $f'(x)$ and $f''(x)$ graphs / functions



You will be required to know how to equate the coefficients of the above functions to find values of a ,b ,c ,and d (equations)

- $f(x) = ax^3 + bx^2 + cx + d$ cubic function /graph
  e.g. $f(x) = x^3 - 6x^2 + 9x$
- $f'(x) = 3ax^2 + 2bx + c$ first derivative function (parabola)
  eg $f'(x) = 3x^2 - 12x + 9$
- $f''(x) = 6ax + 2b$ 2^{nd} derivative function/ graph
  eg $f''(x) = 6x - 12$

Then equating the coefficients of any pair of equations given above will allow you to find values of a ,b.and c. but as for d you will be given a point or y-intercept of cubic function graph.

Comparing the graphs of $f(x)$, $f'(x)$ and $f''(x)$

Cubic graph ($f(x)$	Parabola ($f'(x)$	Straight line ($f''(x)$)
• If $f'(x) = 0$ Gives x-values turning points cubic graph	If $y = f'(x) = 0$ gives x-intercepts of the parabola	
• If $f''(x) = 0$ Gives the x-value of the point of inflection	If $y = f''(x) = 0$ Gives the x-value of the turning point	If $y = f''(x) = 0$ Gives the x-intercept of the Straight line.
• If $f'(x) > 0$ Cubic graph increases	If $f'(x) > 0$ Parabola is above x-axis	
If $f'(x) < 0$ Cubic graph decreases	If $f'(x) < 0$ Parabola graph is below x-axis	
• $f''(x) > 0$ Cubic graph is concave up above x-axis	If $f''(x) > 0$ Parabola graph is increasing	If $f''(x) > 0$ Straight line is
If $f''(x) < 0$ Cubic graph is concave down below x-axis	If $f''(x) < 0$ Parabola is decreasing	If $f''(x) < 0$ Straight line is

CASE 6 Finding roots in cubic graphs , eg $f(x) = ax^3 + bx^2 + cx + d = k$

Find k for which $f(x) = k$ will have:

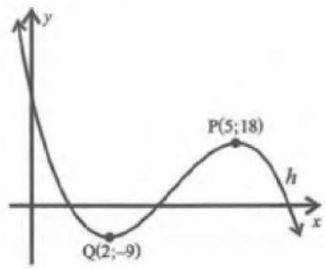
(a) One real root: if k>y at maximum turning point and k<y at minimum turning point

(b) Two unequal real roots: if k=y at maximum turning point and k=y at minimum turning point

(c) Three unequal real roots: if $k \in$ (y at min TP ;y at max TP), in othr words k is between the turning points

Example1

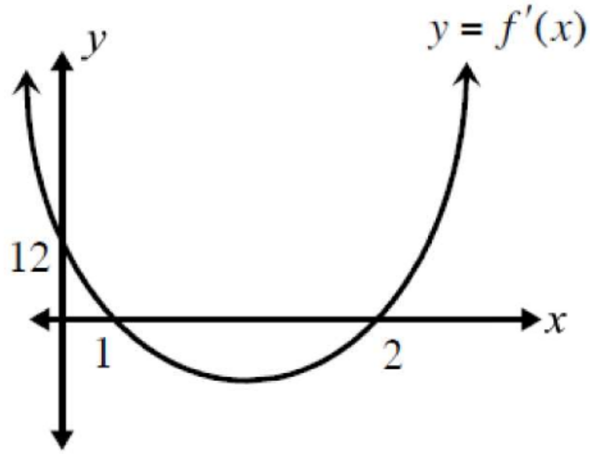
Given the graph of h below. Find k for which $f(x) = k$ will have



(a) one root : answer k>18 and k<-9
(b) two unequal roots : answer k=18 and k=-9
(c) three unequal roots answer $k \in (-9\ ;18)$

Example2

In the diagram ,the graph of $y = f'(x)$ is given where $f(x) = ax^3 + bx^2 + cx$ represents a cubic function.



(a) By referring to the diagram , determine the values of x for which the graph of $f(x)$ has it's stationery points.
Answer: x-intrcepts of a parabola are the x-values of a turning points of the cubic graph.therefore x=1 and x=2

(b) Determine the value of the x-coordinate of the point of inflection of $f(x)$.
Answer: x-coordinate of the turning point of the parabola, $f'(x)$, is the x-coordinate of the point of inflection of the cubic graph, $f(x)$ .therefore $x = \frac{1+2}{2} = \frac{3}{2}$

(c) Determine the values of x for which the graph of the cubic function, $f(x)$, is increasing and decreasing.

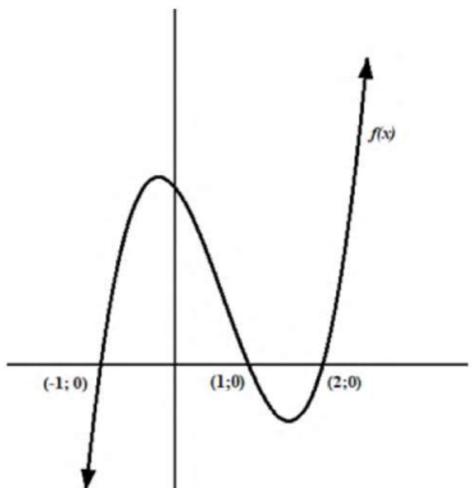
Answer: the cubic function increases if the parabola is above the x-axis , which is $x<1$ or $x>2$.The graph of the cubic function decreases if the parabola is below the x-axis , which is $x \in (1;2)$ ie $1<x<2$

(d) Identify the local minimum and local maximum and give reasons for your answers.

Answer: the parabola is above the x-axis if $x<1$ means that cubic graph increases and it is below the x-axi if $x>1$ means that cubic graph decreases ,therefore $x=1$ is a local maximum of f. The parabola is below the x-axis if $x<2$ means that the cubic graph idecreases and the parabola is above the x-axis if $x>2$ means that the cubic graph increases ,therefore $x=2$ is a local minimum.

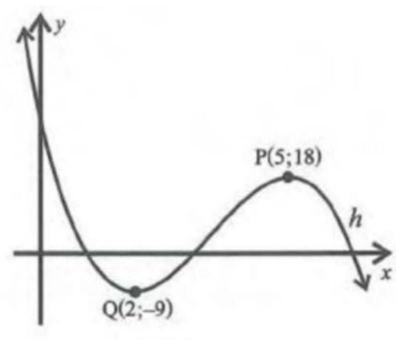
ACTIVITIES/ASSESSMENTS

1. In each of the questions below find the equations of the graphs



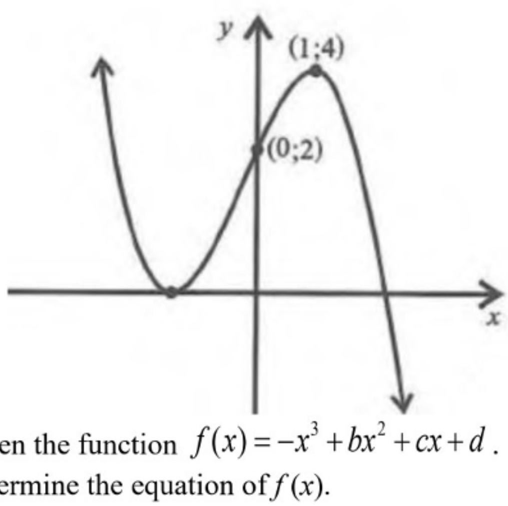
1.1 Given the function $f(x)=2x^3+bx^2+cx+d$. Determine the equation of $f(x)$.

1.2



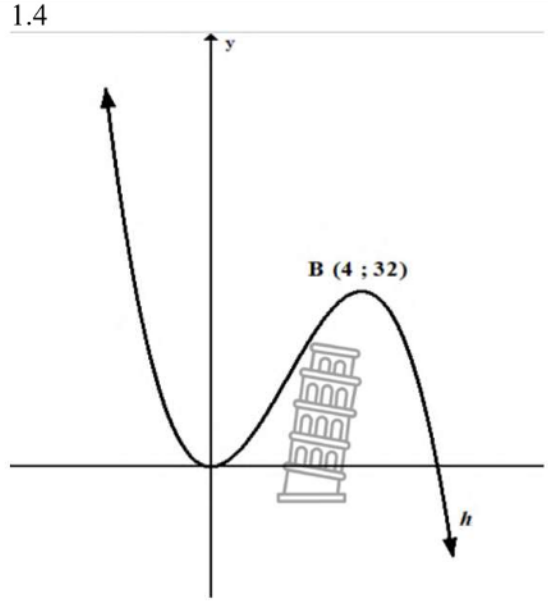
Given the function $h(x)=-2x^3+ax^2+bx+c$. Determine the equation of $h(x)$.

1.3



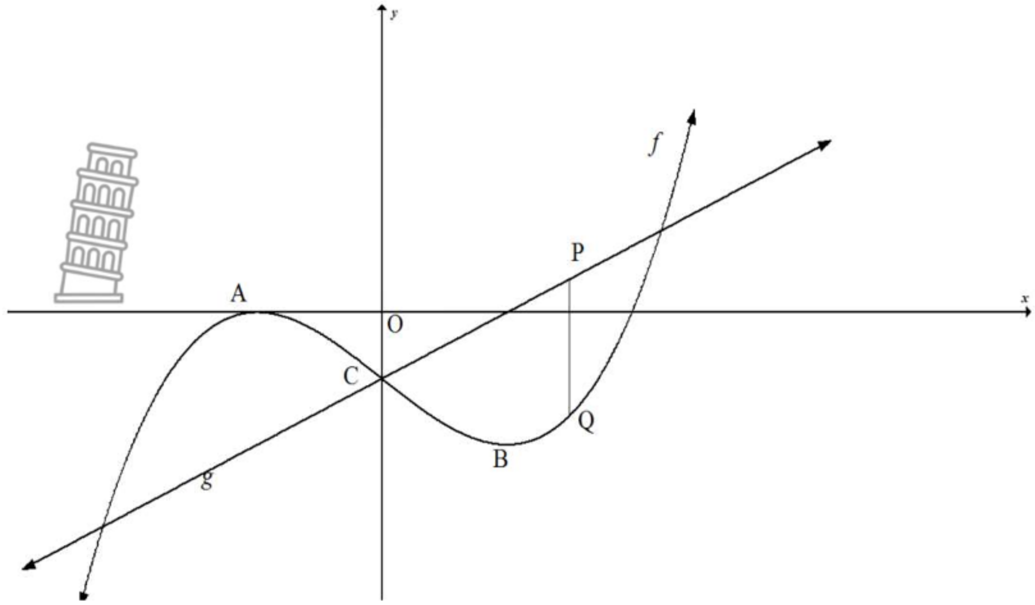
Given the function $f(x)=-x^3+bx^2+cx+d$. Determine the equation of $f(x)$.

1.4



Given the function *f*. Determine the equation of $h(x)$.

2. The graph of a cubic function with equation $f(x) = x^3 - 3x - 2$ and $g(x) = 2x - 2$ is drawn. A and B are the turning points of f. P is a point on g and Q is a point on f such that PQ is perpendicular to the x – axis.



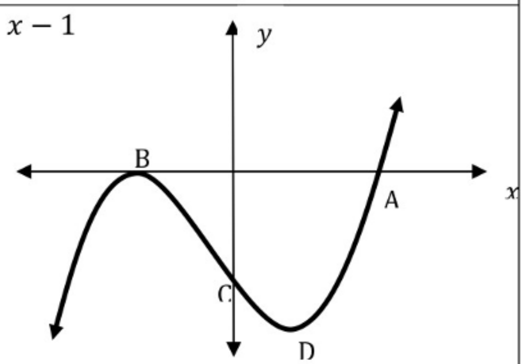
2.1 Calculate the coordinates of A and B.

2.2 If PQ is perpendicular to the x – axis, calculate the maximum length of PQ,

2.3 Determine the values of k for which $f(x) = k$ has only two real roots.

2.4 Determine the values of x for which f is concave up.

3. The diagram below shows the graph of $f(x) = x^3 + x^2 - x - 1$



3.1 Calculate the distance between A and B, the x-intercepts.

3.2 Calculate the coordinates of D, a turning point of f.

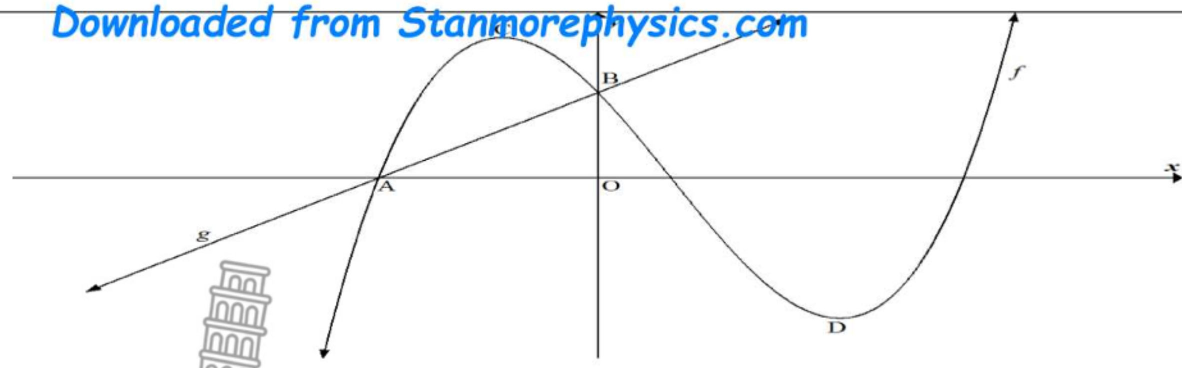
3.3 Show that the concavity of f changes at $x = -\frac{1}{3}$

3.4 For which values of x is:

3.4.1 $f(x) > 0$

3.4.2 $f(x).f'(x) < 0$

4. The graphs of $f(x) = (x+3)(x-1)(x-p)$ and $g(x) = mx + 15$ are sketched below. Both graphs are passing through points A and B. C and D are local maximum and minimum points of f respectively. A is an x - intercept of f and g. B is the y - intercept of f and g.



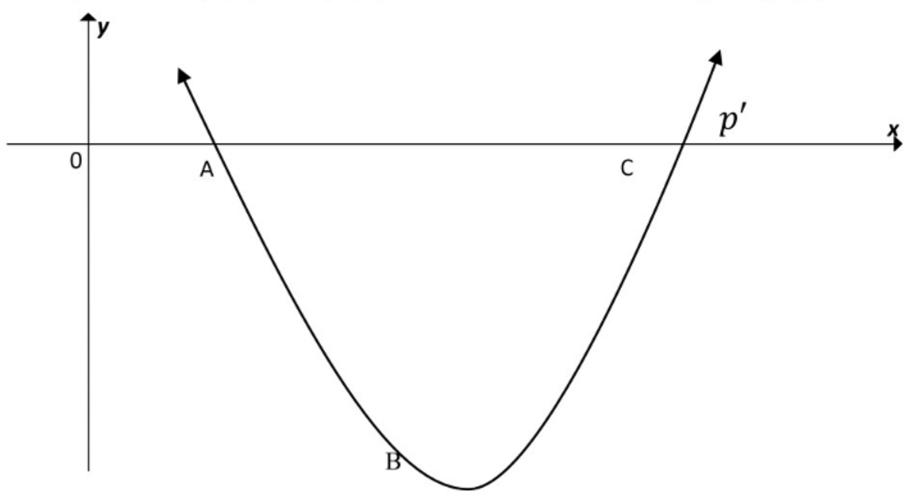
4.1 Explain why the value(s) of $p = 5$ and $m = 5$?

4.2 Determine the coordinates D.

4.3 For which value(s) of x will $f^{/}(x).g(x) \leq 0$?

4.4 For which values of x will f be concave down?

5. The sketch below shows the graph of $p'(x)$ where $p(x) = x^3 + bx^2 + 24x + c$. A(2;0) is an x-intercept of both $p(x)$ and $p'(x)$. C is the other x-intercept of $p'(x)$.



5.1 Show that the numerical value of b is equal to -9. Clearly show all your calculations.

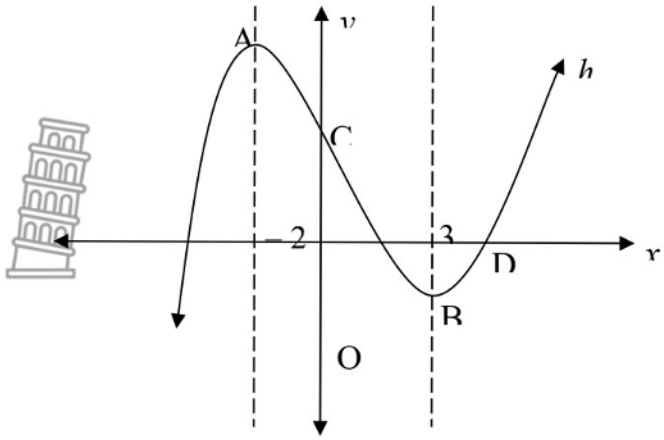
5.2 Calculate the coordinates of C.

5.3 For which value(s) of x will $p(x)$ be increasing?

5.4 Calculate the value(s) of x for which the graph of p is concave up.

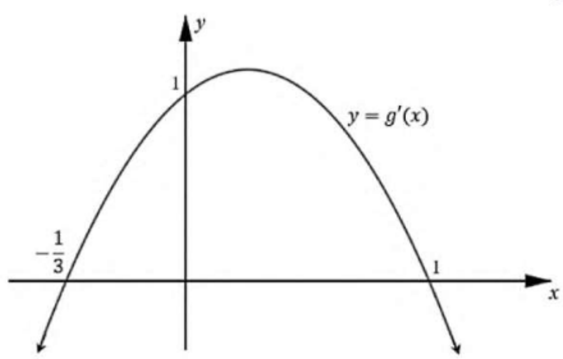
5.5 Sketch a possible graph of $p(x)$. Clearly indicate the x-coordinates of the turning points and the point of inflection.

6. $h(x) = x^3 - \frac{3}{2}x^2 + cx + d$ is sketched below. A and B are the turning points of h at $x = -2$ and $x = 3$ respectively. C is the y – intercept of h. D is the point (4 ; 0).



6.1 Show that $c = -18$ and $d = 32$.
6.2 Calculate the co-ordinates of A.
6.3 Determine the x – value of the point of inflection.
6.4 Write down the interval for which h is concave up.
6.5 If $g(x) = h(-x)$, write down the co-ordinates of the turning point that is the image of A.
6.6 Determine the values of k for which $h(x) = k$ has 2 unequal negative real roots and one positive real root.

7. The diagram below shows the graph of $y = g'(x)$ where $g(x) = ax^3 + bx^2 + cx + d$. The graph $g'(x)$ cuts the y-axis at $(0; 1)$ and the x-axis at $(-\frac{1}{3}; 0)$ and $(1; 0)$.



7.1 Write down the x-coordinate(s) of the stationary point(s) of g.
7.2 Determine the x – coordinate of the point of inflection of g.
7.3 Determine the x values for which g is an increasing function.
7.4 Determine the equation of g'.

TOPIC: CALCULUS

LESSON 7: APPLICATION OF CALCULUS

NOTES

Rules for determing maximum or minimum values

In order to maximize or minimize an object $a(x)$ the following steps must be followed.

Highlights what the question is asking.

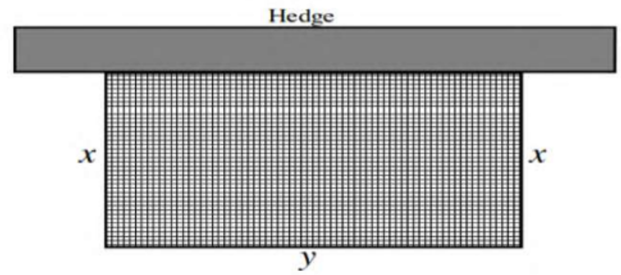
Make an expression for what needs to be maximised or minimised.

If possible, draw a diagram to illustrate the information
If given two variables, eliminate one using the given information and simultaneous equation
Determine derivative if the object is for example $a(x)$ then find $a'(x)$
Let $a'(x)=0$
Solve for x
The value(s) of x obtained will be investigated to establish whether they will yield maximum or minimum values of that given object.

A. Two Dimensional Problems

The following examples to be disussed by the educator in the lesson.

Worked Example 1
A rectangular area is to be enclosed on three sides by a fence and an existing hedge which forms the fourth side as represented by the figure below.



1. Express the length in terms of x if 100 metres of fencing is available.
2. Show that the Area of the rectangle is $A=100x-2x^2$
3. Determine the maximum area of the rectanglar area and its dimensions if 100 metres of fencing is available.
4. The cost of fencing is R20 per metre.Assume that the rectangular area must be 200 m^2 .What are the dimensions such that the cost will be a minimum?

Solution

1. **Since 100 metres of fencing is available and the hedge serves as one of the sides, we know that**

$100=x+x+y$

$\therefore y=100-2x$

2. **We now need to get the Area expression in terms of x before differentiating**

$A=l\times b$
$A(x)=(100-2x).x$
$A(x)=100x-2x^2$

3. **We now need to Apply the golden rule for Maxima and Minima**

$A(x)=100x-2x^2$
$A'(x)=100-4x$
$A'(x)=0$
$100-4x=0$
$4x=100$
$x=25$

Since $A(x)=100x-2x^2$ represents an 'unhappy parabola' with a maximum value
It is therefore clear that the maximum occurs at $x=25$.
Therefore, the maximum Area can be calculated by substituting this value of $x=25$ into the original equation.

$A(x) = 100x - 2x^2$

$A(25) = 100(25) - 2(25)^2 = 1250m^2$

The dimensions of the rectangular area that produces a maximum area is

Length = 50m and breadth = 25 m

4. **Cost =** $20x + 20x + 20y$

Cost = $40x + 20y$

We are given that Area = 200

$\therefore xy = 200$

$\therefore y = \dfrac{200}{x}$

We now substitute $y = \dfrac{200}{x}$ into the cost equation (This is done to eliminate the second variable)

$C(x) = 40x + 20\left(\dfrac{200}{x}\right)$

$\therefore C(x) = 40x + \dfrac{4000}{x}$

$\therefore C(x) = 40x + 4000x^{-1}$

In order to minimise cost proceed as follows:

$\therefore 0 = 40x^2 - 4000$

$\therefore 0 = x^2 - 100$

$\therefore x^2 = 100$

$\therefore x = \pm 10$

Clearly, the value of x cannot be negative in this example.

$\therefore x = 10m$

We can now calculate the value of y:

$y = \dfrac{200}{x}$

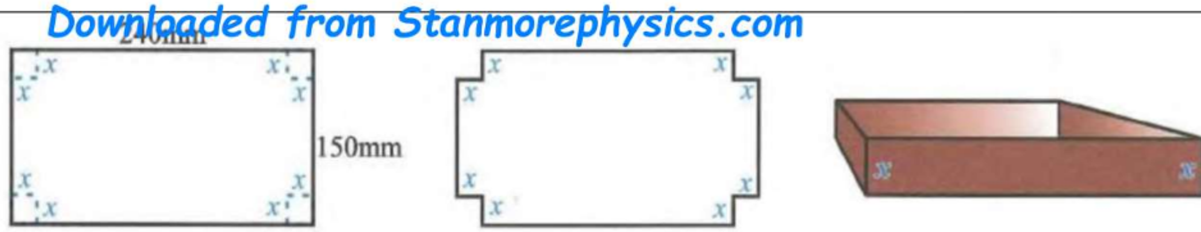
$y = \dfrac{200}{10} = 20m$

B. Three dimensional problems

Teacher will lead discussion on day with the worked example below

Example 1

A sheet of cardboard 240 mm by 150 mm has square edges of side x millimetre cut out. The edges are then folded up to form a box x millimetre high without a lid. Find x so that the volume of the box is a maximum, and hence find the maximum volume.



Solution

The length of the box is $l = 240 - 2x$
The breadth of the box is $b = 150 - 2x$

$V = L \times b \times h = (240-2x)(15-2x)x$
$V = 4x^3 - 780x^2 + 36000x$
$V'(x) = 12x^2 - 1560x + 36000$ …. Derivative of volume

Now equate the derivative to zero.
$\therefore 12x^2 - 1560x + 36000 = 0$
$\therefore x^2 - 130x + 300 = 0$
$\therefore (x-30)(x-100)$
$x = 100$ or $x = 30$

You now need to determine which x value will result in a maximum.
From the equation, it can be seen that the co-efficient of x^3 is positive. This means the graph of volume for this container will have the shape:

This means that the lowest x value of your solution will be a maximum.
$\therefore$ maximum at $x = 30$

To calculate the maximum volume substitute $x = 30$ into the $V(x)$ and not $V'(x)$.

$\therefore$ Max. Volume: $V(30) = 486000 \ mm^3$
The educator must use the guideline below to cover activities over the two days.

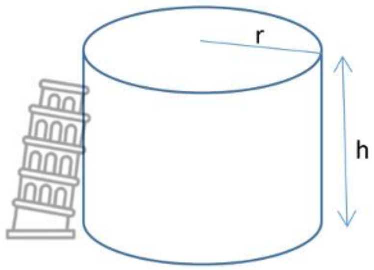
1.1.1 Express the perimeter in terms of x and y.

1.1.2 Show that $y = \dfrac{30 - x(2+\pi)}{2}$

1.1.3 Find the value of x which will maximise the area of the window.
(Leave answer in terms of π)

Activity 1.2

A solid cylinder is cast from 10 litres of molten metal. The cylinder is then covered with a layer of rust-proof paint.
(h and r are in cm and 1litre =1000cm^3)



1.2.1 Determine the height of the cylinder in terms of π and r.

1.2.2 Show that the total surface area is $A = 2\pi r^2 + \frac{20\,000}{r}$.

1.2.3 Determine for which value of r the cost of the paint will be a minimum

Activity 1.3

A local farmer has 500 boxes of peaches for sale, at R6 per box. For every week that he does not sell all the boxes, th cost per box increases by 10c, and 5 boxes spoil, and cannot be sold.

1.3.1 Show that the expression for income is calculated as:

$I(x) = 3000 + 20x - \frac{1}{2}x^2$, where $I(x)$ is the income generated after x weeks.

1.3.2 What is the shortest number of weeks that he can sell all the boxes of peaches, if he receives R3 182 in total as income?

1.3.3 How long should he have kept the peaches in order to maximise his income, and what would the maximum income have been?

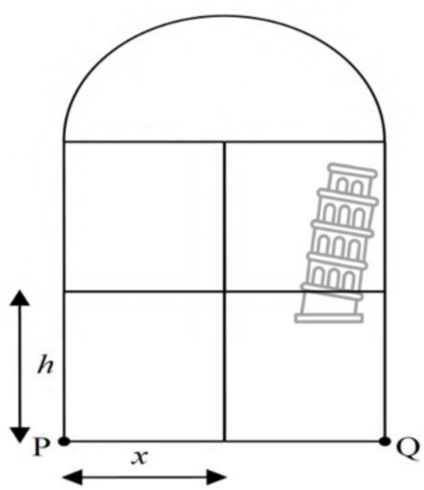
Activity 1.4

A chapel window consists of four equal rectangles and a semi-circle. The length of the metal that is being used for the frame is 36 metres.

1.4.1 Prove that the area for the frame is given by

$$A = 24x - 4x^2 - \frac{\pi x^2}{6}$$

1.4.2 Determine the length of the base PQ for a maxi area of the window.



Activity 1.5

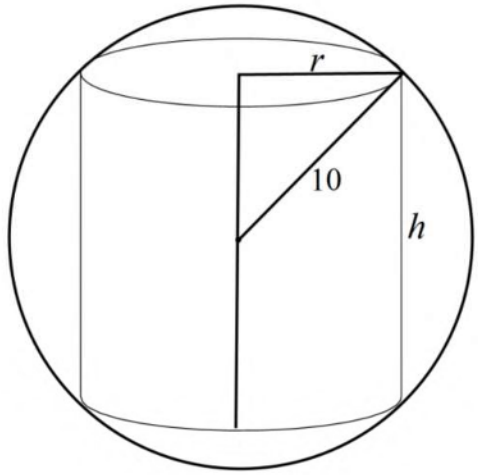
A cylinder with radius r fits neatly into a sphere with radius 10 units.

$\text{volume}_{(\text{cylinder})} = \pi r^2 h$

1.5.1 Show that the volume of the cylinder in terms of h is

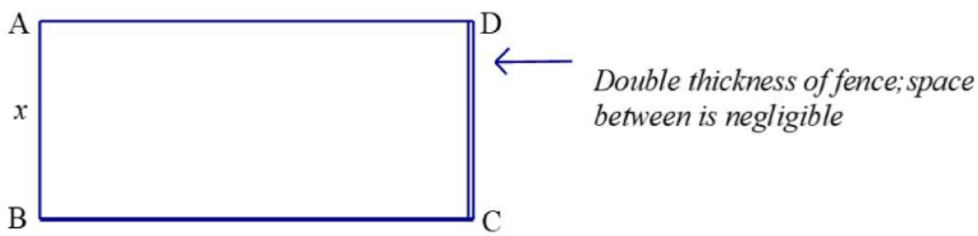
$$V = 100\pi h - \frac{\pi h^3}{4}$$

1.5.2 Calculate the height of the cylinder, correct to 2 decimals, so that the volume is a maximum



Activity 1.6

A farmer makes a rectangular enclosure for his sheep from a roll of fencing which is 100 meters long. He puts a double thickness of fencing along one side to stop the sheep breaking through to his crops in the next field. A plan of his fence is shown below.



The length of AB is x meters. The total length of the fencing available is 100m.

1.6.1 Show that the area of land which can be enclosed in this way is given by the formula

$$A(x) = 50x - \frac{3}{2}x^2$$

1.6.2 Determine the length of AB and AD which should be used to provide an enclosure of maximum area.

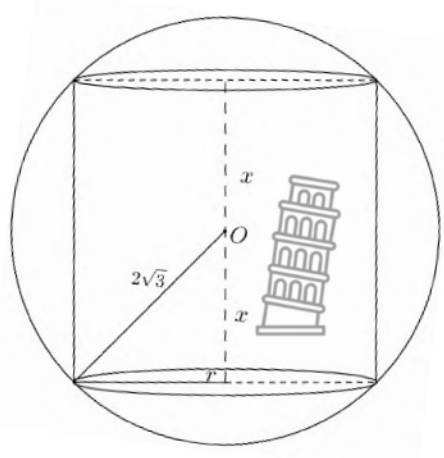
Activity 1.7

The diagram below shows a cylindrical can that fits into circle O with radius $R = 2\sqrt{3}$ units.

1.7.1 Express r in terms of x.

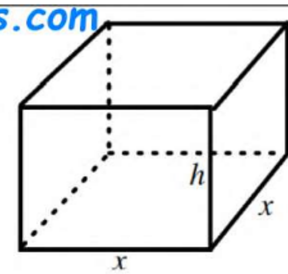
1.7.2 Calculate the value of x for which the volume of the cylinder is a maximum

1.7.3 Calculate the maximum volume of the can in terms of π .



Activity 1.8

A rectangular water tank is to be manufactured. It must contain $32m^3$ of water.
It has a square base (each side equal to x metres). The top of the water tank is open and its height is h metres.



1.8.1 Determine the area (A) of the material that will be used in terms of x and h.

1.8.2 Prove that $A = x^2 + \dfrac{128}{x}$

1.8.3 Find, in terms of x, an expression for C, the total cost of the material, if the material is bought at a price of R10 per m^2.

1.8.4 Determine the values of x and h for which C will be a minimum.

TOPIC: CALCULUS

LESSON 8: APPLICATION OF CALCULUS

NOTES

A. Rates of Change

1.The rate of change of one variable with respect to another can be represented as a derivative.
2. A rate of change can be positive or negative.
For example:
If the volume of air in a balloon is increasing and the radius of the spherical balloon is expanding, then the rate of change of volume with respect to the radius of the spherical balloon is positive. However, if air escapes from the balloon, then the rate of change is negative.

The derivative $f'(x)$ is the instantaneous rate of change of y with respect to x.

It is important to note that:

Whenever you are dealing with a problem where one variable is changing with respect to another you will have to use differentiation.

You will recall that :

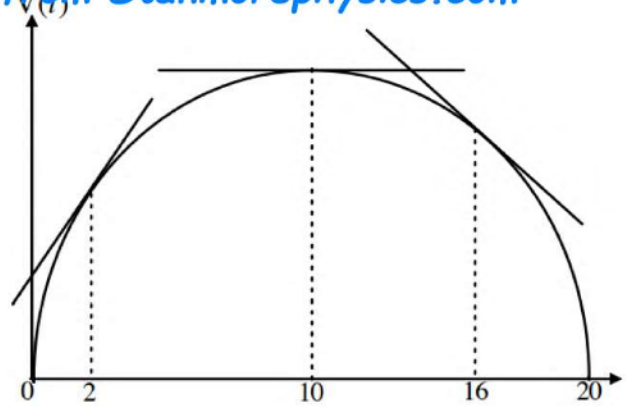
Average Gradient (Average rate of change)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The following worked example will assist in highlighting the above concepts.

Worked Example 1

The volume of water in a horse trough is governed by the equation
$V(t) = 20t - t^2$ for $t \in [0; 20]$ where V is the volume of water in m^3 and t is time in minutes.
The graph of this equation is represented below.



1.1 Determine the volume at $t = 2$.
1.2 Determine the rate of change of volume at $t = 2$.
Explain what is happening by referring to the graph.
1.3 Determine the time at which the volume is a maximum.
1.4 Determine the rate of change of volume at $t =16$.
Explain what is happening by referring to the graph.

Solution

1.1 This is a direct substitution into the equation.

$$V(t) = 20t - t^2$$
$$\therefore V(2) = 20(2) - (2)^2$$
$$\therefore V(2) = 40 - 4$$
$$\therefore V(2) = 36m^3$$

1.2 The rate of change means that we have to find the derivative and then substitute $t = 2$.

$$\therefore V(t) = 20(t) - (t)^2$$
$$\therefore V'(t) = 20 - 2t$$
$$\therefore V'(2) = 20 - 2(2) m^3 / \min$$

The volume of water in the trough is increasing (the derivative is positive).

1.3 For maximum, use the method similar to finding turning points.

$$V'(t) = 20 - 2t$$

At a maximum, $V'(t) = 0$

$$\therefore 0 = 20 - 2t$$
$$\therefore 2t = 20$$
$$\therefore t = 10 \min$$

Notice that the tangent at $t = 10 \min$ is horizontal, which occurs at the maximum turning point on the graph. Here the derivative is zero. The volume is a maximum at $t = 10$.

1.4 The rate of change means that we have to find the derivative and then substitute $t = 16$

$V'(t) = 20 - 2t$

$\therefore V'(16) = 20 - 2(16)$

$\therefore V'(16) = -12m^3 / \min$

The volume of water in the trough is decreasing (the derivative is negative)

The educator must use the guideline below to cover activities over the two days.

Worked Example 2

The volume of soap in a tank is governed by the equation $V = 12 + 10t - t^2$
(Where V is in m^3 and t is in minutes). Determine:
2.1 The initial volume
2.2 The rate at which the volume is increasing when $t = 2\min$.
2.3 The volume at $t = 2\min$.
2.4 The rate at which the volume is increasing at $t = 3\min$.
2.5 The maximum volume in the tank.
2.6 At what time the volume starts to decrease?
2.7 At what time is the tank empty?
2.8 The time at which the volume is decreasing at a rate of $2m^3 / \min$?

Solution

2.1 $V(0) = 12 + 10(0) - (0)^2$
$= 12m^3$
2.2 $V'(2) = 10 - 2(2)$
$= 6m^3$
2.3 $V(2) = 12 + 10(2) - (2)^2$
$= 28m$
2.4 $V'(3) = 10 - 2(3)$
$= 4m^3 / \min$

2.5 For maximum or minimum $V'(t) = 0$
$\therefore 10 - 2t = 0$
$\therefore -2t = -10$
$\therefore t = 5$
$\therefore V_{\max} = 12 + 10(5) - (5)^2$
$= 37m^3$
Notice that maximum volume occurs at $t = 5$.
2.6 Volume is decreasing when $V'(t) < 0$:
$\therefore 10 - 2t < 0$
$\therefore 10 < 2t$
$\therefore 5 < t$
$\therefore$ The volume will start to decrease after 5 minutes.

2.7 The tank will be empty when $V(t) = 0$:

$\therefore -t^2 + 10t + 12 = 0$

$\therefore t^2 - 10t - 12 = 0$

$\therefore t = 11.08\,\text{min}$.

A calculator can be used to obtain the final answer, selection is important here as time cannot be negative.

2.8 Volume is decreasing, therefore the derivative is negative

$V'(t) = -2$

$\therefore 10 - 2t = -2$

$\therefore -2t = -12$

$\therefore t = 6\,\text{min}$

B. Calculus of Motion- Day 2

<u>Some theory on the calculus of motion</u>

Calculus of motion is an extension of the rates of change section.
The following concepts must be explained

- Displacement refers to the distance covered in a certain direction.
- Velocity refers to the speed of an object in a specific direction and is the rate of change (derivative) of displacement.
- Acceleration is the rate of change(derivative) of velocity.

$s(t)$ represents an equation of motion (**height, distance, displacement**) at time t.

$s'(t)$ represents **velocity or speed** at time t.

$s''(t)$ represents **acceleration** at time t.

The educator will lead discussion on the following worked example to explain the above concepts:

Worked Example 3

The displacement of a particle is given by :

$s(t) = 120 - 10t$ where s is in metres and t is in seconds

1.1 Find the displacement when:

1.1.1 $t = 0$

1.1.2 $t = 1$

1.1.3 $t = 2$

1.2 Explain what these values tell you about the motion of the particle?

1.3 Determine the velocity, $v(t)$ of the particle.

1.4 What is the speed of the particle?

Solutions:

1.1.1 $s(0) = 120 - 10(0) = 120m$

1.1.2 $s(1)=120-10(1)=110m$

1.1.3 $s(2)=120-10(2)=100m$

1.2 It can be seen from the above answers that the particle is moving at a constant velocity as the change in displacment is constant every second. As the displacemet is decreasing it can be seen that the particle is moving in the negative direction

1.3 $v(t)=s'(t)=-10m/s$

1.4 The speed of the particle = $10m/s$

The difference between speed and velocity is that speed does not have a direction.

ACTIVITIES/ASSESSMENTS

Activity 2.1

In an experiment, the number of germs in a test tube at any time *t*, in seconds, is given by the equation $g(t)=3t^2+2$

Determine:

2.1.1 The number of germs in the test tube after 4 seconds.

2.1.2 The rate of change in the number of germs after 4 seconds.

2.1.3 The rate of change in the number of germs after 6 seconds.

Activity 2.2

The volume of water in a tank is governed by the equation $V(t)=10t-t^2$ for the interval $t\in[0;10]$ where V represents volume in m^3 and *t* represents the time in minutes. Determine:

2.2.1 The volume after 4 minutes.

2.2.2 The time taken for the volume to reach $9m^3$.

2.2.3 an expression for the rate of change of volume.

2.2.4 The time taken to reach a maximum volume.

2.2.5 The maximum volume.

2.2.6 The time at which the rate of increase is $6m^3$/min.

Activity 2.3 A scientist adds a bactericide into a culture of bacteria. The number of bacteria present *t* hours after the bactericide was introduced is given by the formula: $B(t)=1000+50t-5t^2$ where B(*t*) is the number of bacteria, in millions.

2.3.1 How many bacteria were in the culture when the bactericide was introduced?

2.3.2 Calculate the rate of change of the number of bacteria with respect to time three hours after the bactericide was added.

2.3.3 When does the population of bacteria start to decrease?

2.3.4 When will the whole culture of bacteria be exterminated?

Activity 2.4

The depth (in metres) of water left in a dam *t* hours after a sluice gate has been opened to allow water to drain from the dam, is given by the formula: $d=28-\frac{1}{9}t^2-\frac{1}{27}t$

2.4.1 Calculate the average rate at which the depth changes in the first three hours.

2.4.2 Determine the rate at which the depth changes after exactly two hours.

Activity 2.5

A stone is thrown vertically upwards and its height (in metres) above the ground at time *t* (in seconds) is given by: $h(t)=35-5t^2+30t$

2.5.1 Determine its initial height above the ground.
2.5.2 Determine the initial speed with which it was thrown.
2.5.3 Determine the maximum height above the ground that the stone reached.
2.5.4 How fast was the stone travelling when it reached a height of 60 metres above the ground on the way down?
2.5.5 How fast was the stone travelling when it hit the ground?

Activity 2.6

An object moves from its starting position at point O and its motion is described by the equation $s(t) = 20t - 2t^2$, where s represents displacement in metres and t, the time in seconds.

Determine:

2.6.1 The displacement of the object after 1 second.
2.6.2 The time taken to reach maximum displacement.
2.6.3 The maximum displacement.
2.6.4 The times taken to reach a displacement of 32m from O.

Activity 2.7

A ball is thrown into the air and its height, h, above the ground, after t seconds is $h(t) = -5t^2 + 25t + 4$ metres. Determine:

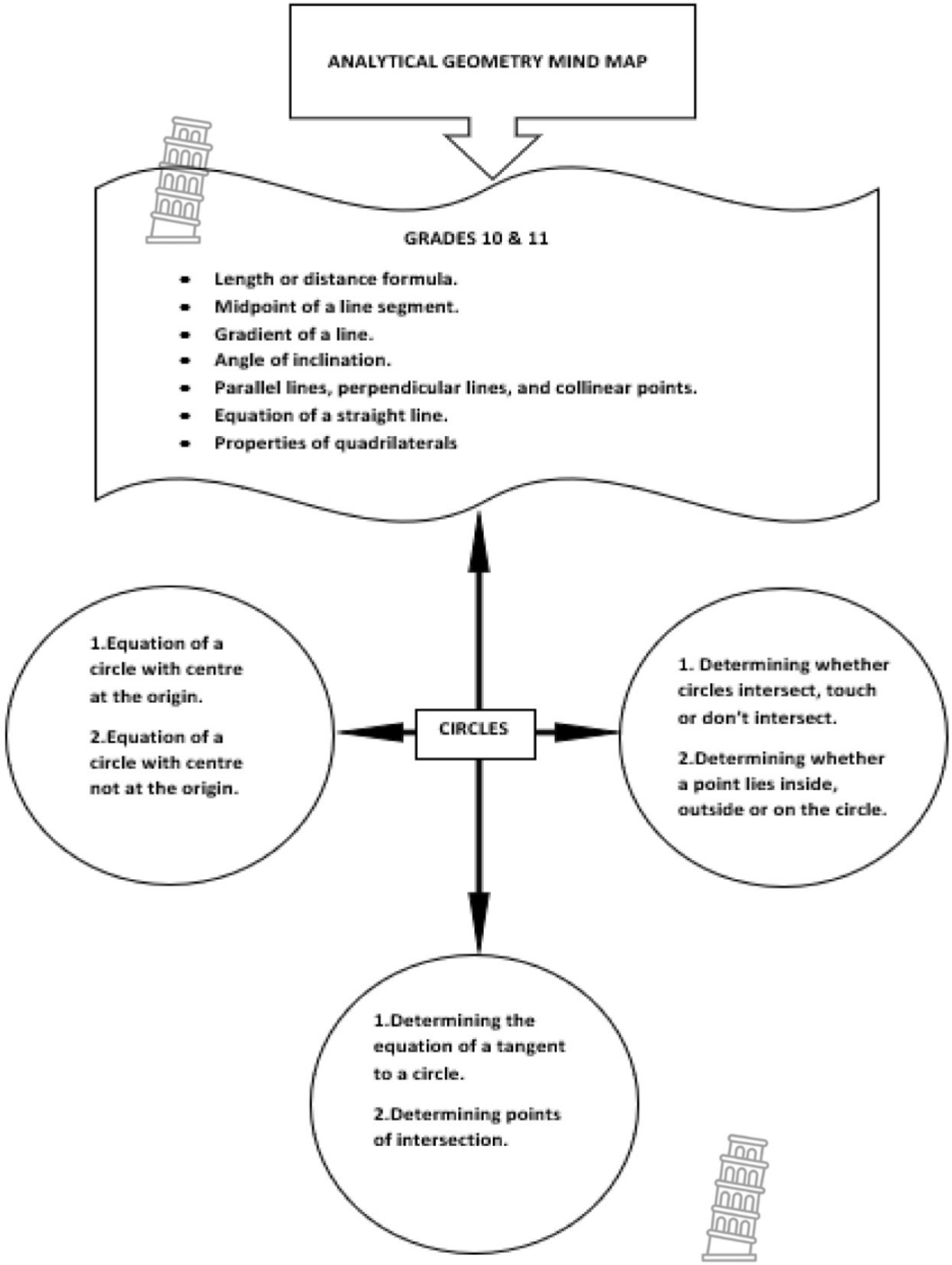
2.7.1 The height of the ball above the ground after 2 seconds.
2.7.2 The velocity of the ball at that moment.
2.7.3 The maximum height of the ball above the ground.
2.7.4 The time taken to reach a downwards velocity of 5 *m/s.*

Activity 2.8

A coin is thrown from the top of a tower. The formula for the height of the coin above the ground in meters (the displacement), at time t in seconds, is given by $s(t) = -4{,}9t^2 - 15t + 250$

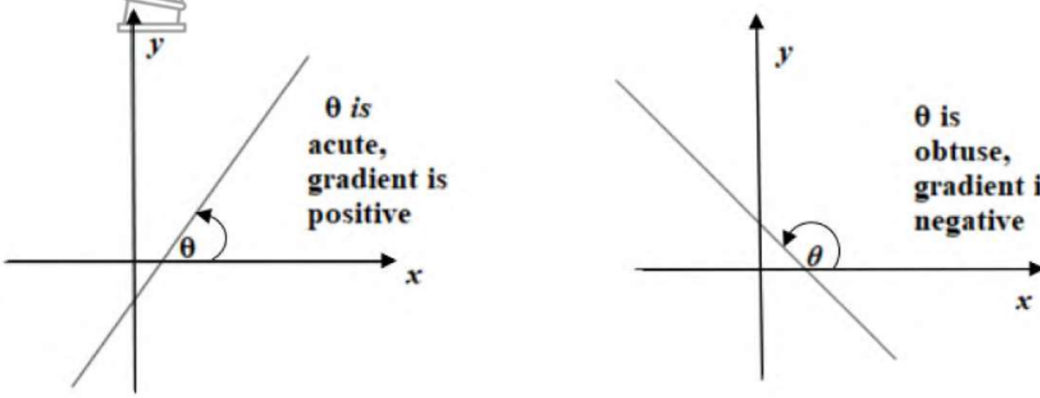
2.8.1 How high is the tower in metres?
2.8.2 What is the average speed between 2 minutes and 4 minutes?
(Round your answer correct to ONE decimal place.)
2.8.3 Determine the speed of the coin after 5 seconds.

TOPIC: ANALYTICAL GEOMETRY
LESSON 2: REVISION OF WORK FROM EARLIER GRADES



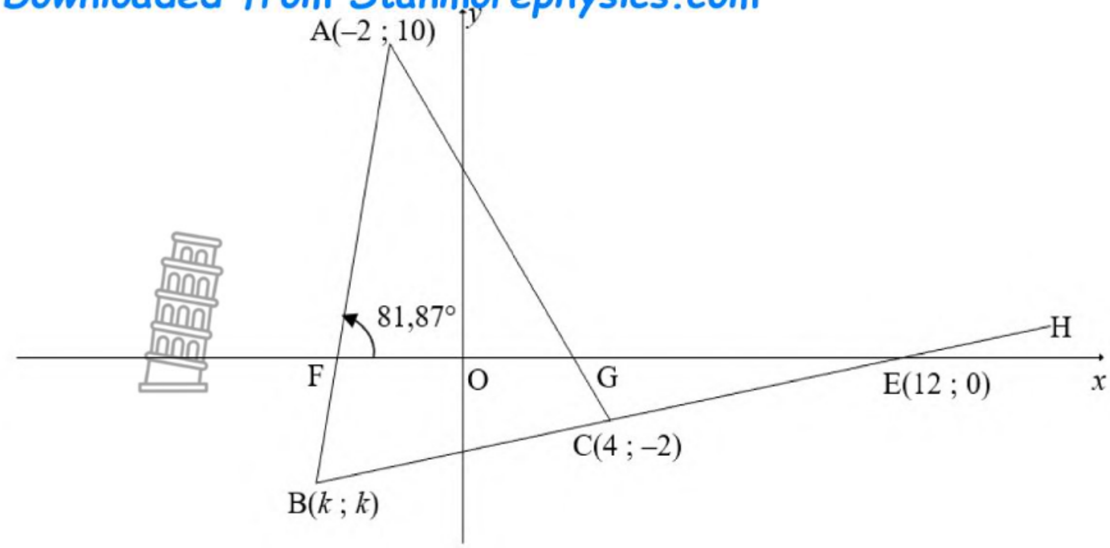
NOTES

1. Length of line AB, with $A(x_A;y_A)$ and $B(x_B;y_B)$	$AB^2=(x_B-x_A)^2+(y_B-y_A)^2$ or $AB=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}$
2. Formula for any point M, the midpoint of a line segment AB, with $A(x_A;y_A)$ and $B(x_B;y_B)$	$M(x_M;y_M)=M\left(\frac{x_B+x_A}{2};\frac{y_B+y_A}{2}\right)$
3. The gradient of line AB, with $A(x_A;y_A)$ and $B(x_B;y_B)$	$m_{AB}=\frac{y_B-y_A}{x_B-x_A}$
4. If θ is the angle of inclination of line OR then:	$\tan\theta=m_{OR};\theta\in[0°;180°]$



5. (a) Parallel (∥) lines: non-vertical lines with gradients m_1 and m_2 are parallel if their gradients are equal. (b) Perpendicular $(\perp)$ lines: non-vertical lines with gradients m_1 and m_2 respectively are perpendicular if (c)If A, B, and C are collinear then:	(a) $(m_1=m_2)$ (b) $m_1\times m_2=-1$ (c) $m_{AB}=m_{BC}=m_{AC}$
6. Equation of a straight line: (a) $y=mx+c$ (b) $y-y_1=m(x-x_1)$ (c) Vertical line (d) Horizontal line	(a) m is the gradient and c is the y-intercept (Gradient-intercept form) (b) straight line with gradient m passing through the point $(x_1;y_1)$ (c) x = number; gradient is undefined (d) y = number; gradient is zero

Example:
In the diagram, A (–2; 10), B ($k;\ k$) and C (4; –2) are the vertices of ΔABC. Line BC is produced to H and cuts the x-axis at E (12; 0). AB and AC intersect the x-axis at F and G respectively. The angle of inclination of line AB is 81,87°.



1 Calculate the gradient of:

1.1 BE (2)

1.2 AB (2)

2 Determine the equation of BE in the form $y = mx + c$ (2)

3 Calculate the:

3.1 Coordinates of B, where $k < 0$ (2)

3.2 Size of $\hat{A}$ (4)

3.3 Coordinates of the point of intersection of the diagonals of parallelogram ACES, where S is a point in the first quadrant (2)

4 Another point T $(p; p)$, where $p > 0$, is plotted such that $ET = BE = 4\sqrt{17}$ units. Calculate the coordinates of T. (5)

Solution:

1.1 $m_{BE} = m_{CE} = \frac{0-(-2)}{12-4} = \frac{1}{4}$ **OR** $m_{BE} = m_{CE} = \frac{-2-0}{4-12} = \frac{1}{4}$ (Collinear points)

1.2 $m_{AB} = \tan 81{,}87°$

$m_{AB} = 7$

(Relationship of the gradient with the angle of inclination)

2 $y = mx + c$ or $y - y_1 = m(x - x_1)$ (You may substitute any of the collinear points.)

$0 = \frac{1}{4}(12) + c$ or $y - 0 = \frac{1}{4}(x - 12)$

$c = -3$ $\qquad y = \frac{1}{4}x - 3$ (Substituting point E)

$y = \frac{1}{4}x - 3$

OR

OR

(Substituting point C)

$y = mx + c$ or $y - y_1 = m(x - x_1)$

$-2 = \frac{1}{4}(4) + c$ or $y - (-2) = \frac{1}{4}(x - 4)$

$c = -3$ $\quad$ $y = \frac{1}{4}x - 3$

$y = \frac{1}{4}x - 3$

3.1 $m_{BC} = m_{BE}$ (Why?)

$\frac{k-(-2)}{k-4} = \frac{1}{4}$ (Cross multiply)

$4k + 8 = k - 4$

$3k = -12$

$k = -4$

$\therefore B(-4;-4)$

(Are there other methods you thought of using, other than this method?)

3.2 $\hat{A} + 81{,}87° = \tan^{-1}(m_{AC})$ (Exterior angle of Δ AFG equals sum of the two opposite interior angles.)

$\hat{A} = \tan^{-1}(m_{AC}) - 81{,}87°$

$= \tan^{-1}\left(\frac{10-(-2)}{-2-4}\right) - 81{,}87°$

$= \tan^{-1}(-2) - 81{,}87°$

$= 180° - \tan^{-1}(-2) - 81{,}87°$ (Negative gradient implies Obtuse angle)

$= 116{,}57° - 81{,}87°$

$= 34{,}7°$

3.3

$M\left(\frac{12+(-2)}{2};\frac{10+(0)}{2}\right)$ (Midpoint = point of intersection of the diagonals, why?)

Diagonals intersect at the point (5 ; 5)

4. BE = ET

$4\sqrt{17} = \sqrt{(12-p)^2 + (0-p)^2}$ (Distance or length formula)

$\left(4\sqrt{17}\right)^2 = \left(\sqrt{(12-p)^2 + (0-p)^2}\right)^2$ (Squaring both sides for easier algebraic manipulation.)

$272 = 144 - 24p + p^2 + p^2$

$p^2 - 12p - 64 = 0$

$(p-16)(p+4) = 0$

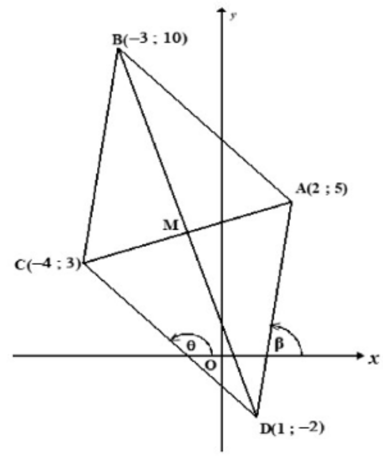
$\therefore p = 16$ or $p = -4$ (n.a.)

$\therefore$ T(16; 16)

ACTIVITIES/ASSESSMENTS: LESSON 1

Classwork/Homework

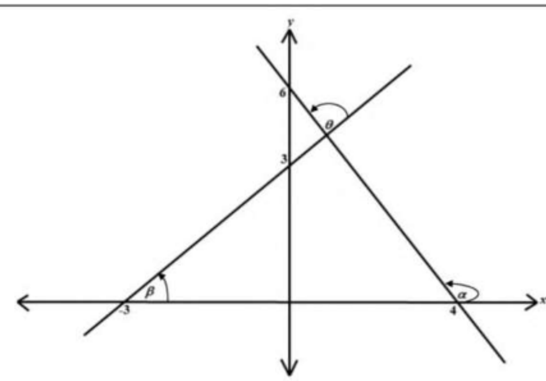
1. ABCD is a quadrilateral with vertices A(2 ; 5), B(−3 ; 10); C(−4 ; 3) and D(1 ;−2)
   - 1.1 Calculate the length of AC. (Leave the answer in simplest surd form.)
   - 1.2 Determine the coordinates of M, the midpoint of AC.
   - 1.3 Show that BD and AC bisect each other perpendicularly.
   - 1.4 Calculate the area of ΔABC.
   - 1.5 Determine the equation of DC.
   - 1.6 Determine θ, the angle of inclination of DC.
   - 1.7 Calculate the size of $A\hat{D}C$



2. A(–1 ; 3) , B(7;1) and C(k ; 2) are points on the Cartesian plane. Calculate the possible value of k if:
   - 2.1 the gradient of BC is 2
   - 2.2 ΔABC is right-angled at B
   - 2.3 C is the midpoint of AB

3. A(1; 4), B(– 2; 2), C(4; 1) and $M(x; y)$ are points on the Cartesian plane. Use analytical methods to determine the coordinates of M so that ABCM, in this order, is a parallelogram.

4. Refer to the diagram below to answer the questions that follow: Determine:
   - 4.1 the size of α
   - 4.2 the size of β
   - 4.4 the size of θ



TOPIC: ANALYTICAL GEOMETRY

LESSON 2: THE EQUATION OF A CIRCLE

NOTES

1. CIRCLE WITH CENTRE AT THE ORIGIN

Suppose P (x; y) is always r units from the origin, (0; 0)
(This implies P (x; y) lies anywhere on the circle)
From the distance formula we know that:

$$OP^2 = (x-0)^2 + (y-0)^2$$

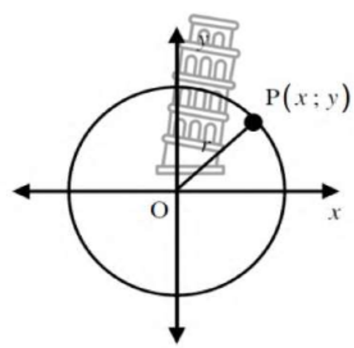
$$\therefore OP^2 = x^2 + y^2$$

But $OP = r$ and therefore $OP^2 = r^2$

$$\therefore x^2 + y^2 = r^2$$

The equation of the circle with centre (0; 0) is therefore

$$x^2 + y^2 = r^2$$



2. CIRCLE NOT CENTRED AT THE ORIGIN

Let P $(x; y)$ be any point on the circle with centre M $(a; b)$ and radius r.

From the distance formula we know that:

$MP^2 = (x-a)^2 + (y-b)^2$

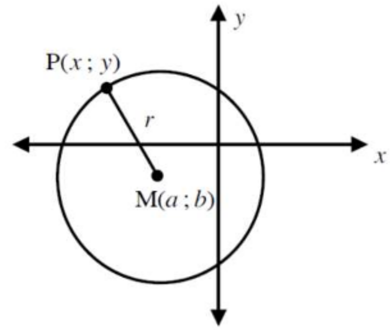
But MP is the radius r, and therefore $MP^2 = r^2$

The equation of a circle with centre $(a;b)$ is:

$(x-a)^2 + (y-b)^2 = r^2$ (This is called the standard form.)

Another form of a circle's equation with any centre is:

$x^2 + cx + y^2 + dy + e = 0$



This form can be manipulated by completing a square to $(x-a)^2 + (y-b)^2 = r^2$ so that the centre and radius of the circle can be determined.

Example 1: Determine the coordinates of the centre and the length of the radius of each of the following circles. a) $x^2 + y^2 = 36$ b) $(x+3)^2 + (y-2)^2 = 32$

Solution: a) Circle centre at (0; 0) and $r = \sqrt{36} = 6$ units

b) Circle centre at (-3; 2) and $r = \sqrt{32} = 4\sqrt{2}$ units

Example 2: Determine the coordinates of the centre and the length of the radius of each of the following circles:

a) $x^2 + y^2 - 6y - 55 = 0$ b) $x^2 + y^2 + 2x + 4y = 11$

Solution: a) This equation is not in standard form; you therefore need to complete a square.

$x^2 + (y^2 - 6y + 9) = 55 + 9 \rightarrow$ complete the square on y

$x^2 + (y-3)^2 = 64$

$\therefore$ Circle centre at $(0;\ 3)$ and $r = \sqrt{64} = 8$ units

b) $(x^2 + 2x + 1) + (y^2 + 4y + 4) = 11 + 1 + 4 \rightarrow$ complete the square on x and y

$(x+1)^2 + (y+2)^2 = 16$

$\therefore$ Circle centre at $(-1;\ -2)$ and $r = \sqrt{16} = 4$ units

Example 3: Determine the equation of the circle with centre M $(-1; 3)$ and A $(-4; -1)$ a point on the circle.

Solution:

$\text{MA} = r$

$\therefore \text{MA}^2 = r^2$

$\therefore \text{MA}^2 = \left(-1-(-4)\right)^2 + \left(3-(-1)\right)^2$

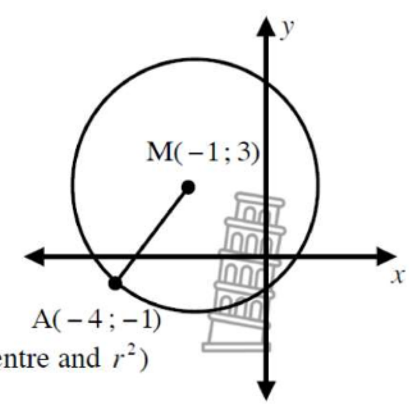
$\therefore r^2 = \left(-1+4\right)^2 + \left(3+1\right)^2$

$\therefore r^2 = 25$

$\left(x-a\right)^2 + \left(y-b\right)^2 = r^2$

$\therefore \left(x-(-1)\right)^2 + \left(y-3\right)^2 = 25$ (substitute the centre and r^2)

$\therefore \left(x+1\right)^2 + \left(y-3\right)^2 = 25$



ACTIVITIES/ASSESSMENTS: LESSON 2

1. Determine the equation of a circle with centre at the origin, and:

1.1 a radius of 4 units

1.2 a radius of $\sqrt{10}$

1.3 passing through the point (–2 ; –4)

1.4 passing through the point ($\sqrt{3}$; 1)

2. Point (–2 ; *b*) lies on the circle $x^2 + y^2 = 13$. Determine the values of *b*.
3. Point (*a*; $\sqrt{5}$) lies on the circle $x^2 + y^2 = 21$. Determine the values of a.
4. P(–5; 12) lies on the circle with centre the origin:

   4.1 Determine the equation of the circle.

   4.2 Determine the coordinates of Q if PQ is the diameter.

   4.3 Show that the point M (0; -13) lies on the circle.

   4.4 Show that $P\hat{M}Q = 90°$

5. Determine the equation of the circles with:

   5.1 Centre (5; –5) and passing through the point (–3; 1).

   5.2 A radius of 5 units and circle centre C (*a; b*), cutting the y-axis at A (0; 1) and B (0; 7) if $a<0$

6. Determine the equation of the circles whose diameter is the line joining the points:

   6.1 A (2; 5) and B (–2; 3)

   6.2 A (–3; 2) and B (1; 6)

7. The equation of a circle with radius $5\sqrt{3}$ units is given by $x^2 + y^2 - 10x + 8y - p = 0$

   7.1 Rewrite the equation in the form $(x-a)^2 + (y-b)^2 = r^2$

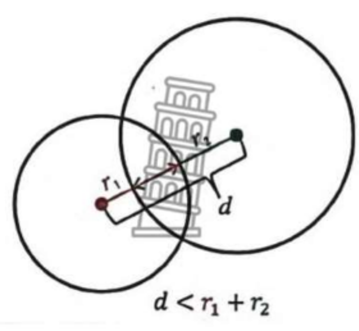
   7.2 Hence, determine the value of *p*.

TOPIC: ANALYTICAL GEOMETRY					
LESSON 3: INTERSECTING CIRCLES					
Term	2	**Week**		**Grade**	12

NOTES

Intersection of circles: Two circles can intersect each other at two points, one point or not at all.

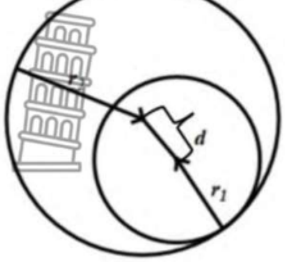
Case 1: Circles intersecting at two points.
If the distance between the centres ***(d)*** of the two circles is less than the sum of the radii (r_1 and r_2), then the circles intersect each other twice.



Case 2: Circles intersecting at one point (touching).

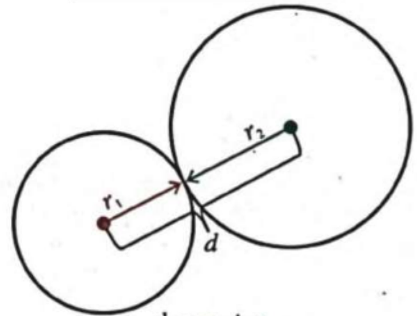
If the distance between the centres ***(d)*** of the two circles is equal to the sum of the radii (r_1 and r_2), then the circles intersect each other once. Note that circles can touch internally or externally.

Touching Internally



$d = r_2 - r_1$

Touching Externally



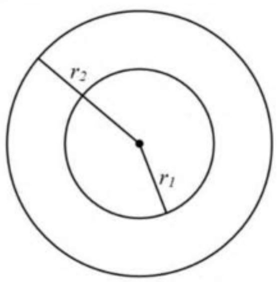
$d = r_1 + r_2$

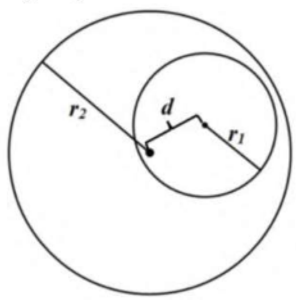
Case 3: Circles that do not intersect.

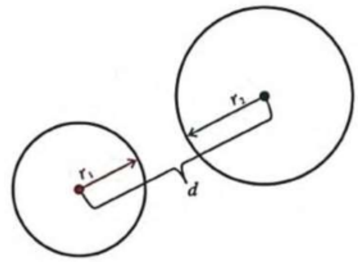
Concentric circles:
Circles with a common centre.
$d = 0$

Smaller circle inside the larger circle but no common centre:
$d < r_2 - r_1$

Circles outside each other:
$d > r_1 + r_2$







Examples:

1. Determine whether the following circles intersect or touch, and if so in what way.

   1.1 $x^2 + y^2 = 4$ and $(x-2)^2 + (y+3)^2 = 9$

   1.2 $(x-2)^2 + (y-1)^2 = 5$ and $(x-4)^2 + (y-5)^2 = 45$

   1.3 $(x-1)^2 + (y-1)^2 = 9$ and $(x-5)^2 + (y-4)^2 = 4$

Solutions:

1.1

$C_1 = (0;0)$; $C_2 = (2;-3)$; $r_1 = 2$; $r_2 = 3$

$\therefore d = \sqrt{(2-0)^2 + (-3-0)^2}$ (distance formula to calculate distance between centres)

$= \sqrt{4+9}$

$= \sqrt{13}$

$\simeq 3,61$ units

$r_1 + r_2 = 5$

$d = r_1 + r_2$

$\therefore$ circles intersect at two points.

1.2

$C_1 = (2;1);\ C_2 = (4;5);\ r_1 = \sqrt{5};\ \ r_2 = \sqrt{45} = 3\sqrt{5}$

$\therefore d = \sqrt{2^2 + 4^2}$ (distance formula to calculate distance between centres)

$= \sqrt{20}$

$= 2\sqrt{5}$ units

$r_1 + r_2 = 5$

$d = r_2 - r_1$

$\therefore$ circles touch internally.

1.3

$C_1 = (1;1);\ C_2 = (5;4);\ r_1 = 3;\ \ r_2 = 2$

$\therefore d = \sqrt{4^2 + 3^2}$ (distance formula to calculate distance between centres)

$= 5$ units

$r_1 + r_2 = 5$

$d = r_1 + r_2$

$\therefore$ circles touch externally.

ACTIVITIES/ASSESSMENTS: LESSON 3

1. Consider the following:

   1.1 A circle with equation $x^2 + y^2 + 8x + 4y - 44 = 0$ is centred at M. Determine the radius and the coordinates of M.

   1.2 A second circle, centred at P, has equation $(x - 4)^2 + (y - 6)^2 = 49$. Calculate the length of MP.

   1.3 Show that the two circles described in 1.1 and 1.2 intersect each other at two points.

   1.4 Determine whether the point $A(-2;3)$ lies on the circle described in 1.1, or not.

2. Two circles with equations $(x-3)^2 + (y+2)^2 = 25$ and $(x-12)^2 + (y-10)^2 = 100$, are centred at A and B respectively..

   2.1 Calculate the distance between the two centres (A and B).

   2.2 In how many points do these two circles intersect? Justify your answer.

TOPIC: ANALYTICAL GEOMETRY

LESSON 4: STRAIGHT LINES & TANGENTS TO CIRCLES

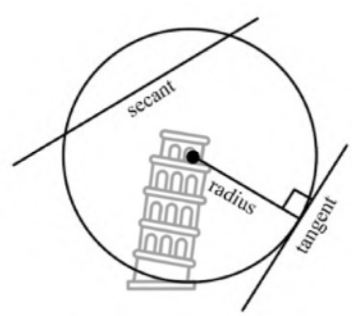
NOTES

A **tangent** is a straight line which touches a graph (in this case a circle) at **one point only.**

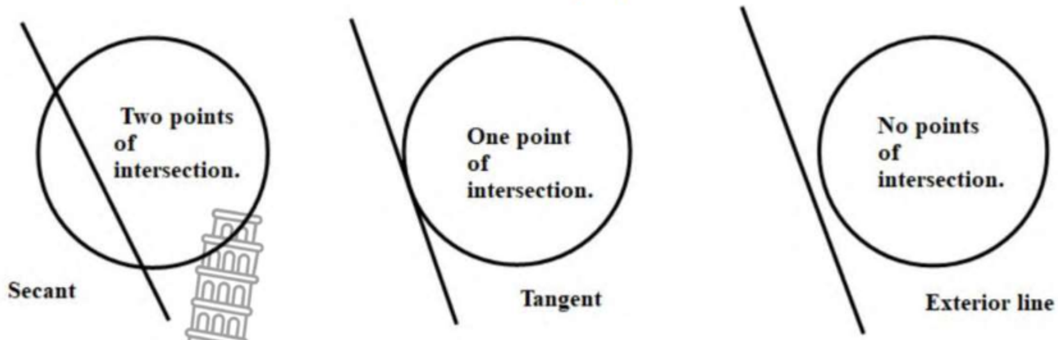
A **secant** is a straight line which cuts a circle at **two distinct points.**

An important principle from Euclidean Geometry is that:

radius $\perp$ tangent



Circle & Line Relationships



Steps to finding the equation of a tangent to a circle when the point of contact is given:

1. Find the gradient of the radius at the given point of contact.
2. Using the gradient of the radius, find the gradient of the tangent using the property $m_{\text{radius}} \times m_{\text{tangent}} = -1$ (this is because tangent is perpendicular to the radius at point of contact).
3. Use the given point of contact and the gradient of the tangent to work out the equation of the tangent.

Examples:

1. Determine the equation of the tangent to the circle $x^2 + y^2 = 8$ at the point A (2; -2)
2. Show that the straight line $y = -x - 4$ is a tangent to the circle $(x-1)^2 + (y+1)^2 = 8$
3. The equation of a circle is $x^2 + y^2 - 8x + 6y = 15$
   (a) Prove that the point A (2; -9) lies on the circumference of the circle.
   (b) Determine the equation of the tangent at point A.
4. Determine the points of intersection between the circle $x^2 + y^2 = 0$ and the line $y = 2x - 1$
   Round your answers to two decimal places.

Solutions:

1. $m_{\text{radius}} = \dfrac{-2-0}{2-0} = -1$ (Substitute the coordinates of A and those of the centre)

   $m_{\text{tangent}} = -\dfrac{1}{m_{\text{radius}}} = 1$

   $y = mx + c$

   $\therefore y = x + c$

   Substitute $A(2;-2)$

   $\therefore -2 = 2 + c$

   $\therefore c = -4$

   $\therefore$ Equation of the tangent is: $y = x - 4$

2. A tangent is a line that touches a circle at one point only. Therefore, it will be enough to show that the two graphs intersect at only one point.

   $(x-1)^2 + (y+1)^2 = 8$.......(1)

   $y = -x - 4$ ....(2)

   Substitute (2) in (1): $(x-1)^2 + (-x-4+1)^2 = 8$

   Simplify: $(x-1)^2 + (-x-3)^2 = 8$

   Remove brackets: $x^2 - 2x + 1 + x^2 + 6x + 9 = 8$

   Add like terms: $2x^2 + 4x + 2 = 0$ (Divide by 2 to make factorising easier)

$$x^2+2x+1=0$$
$$(x+1)(x+1)=0$$
$$x=-1$$

substitute $x=-1$ into (1): $y=-1(-1)-4=-3$

$\therefore$ the straightline $y=-x-4$ is a tangent to the circle $(x-1)^2+(-x-4+1)^2=8$ because the straight line intersects the circle at one only one point (-1;-3)

3. (a) Substitute the point into the circle equation and prove that the LHS = RHS

$LHS = x^2+y^2-8x+6y$

$=(2)^2+(-9)^2-8(2)+6(-9)$

$=4+81-16-54$

$=15=RHS$

(b) Determine the coordinates of the centre, then calculate the gradient of the radius.

$x^2-8x+y^2+6y=15$

Complete the square:

$(x^2-8x+16)+(y^2+6y+9)=15+16+9$

$(x-4)^2+(y+3)^2=40$

Centre: (4;-3)

$\therefore m_{\text{radius}}=\dfrac{-3+9}{4-2}=3$

$m_{\text{tangent}}=-\dfrac{1}{m_{\text{radius}}}=-\dfrac{1}{3}$ (tangent $\perp$ radius)

$y=mx+c$

$y=-\dfrac{1}{3}+c$

Substitute A(2;-9)

$-9=-\dfrac{1}{3}(2)+c$

$\therefore c=-\dfrac{25}{3}$

$\therefore$ Equation of the tangent is: $y=-\dfrac{1}{3}x-\dfrac{25}{3}$

4.

$x^2+y^2=9$.......(1)

$y=2x-1$ ....(2)

$x^2+(2x-1)^2=9$

$x^2+4x^2-4x+1-9=0$

$5x^2-4x-8=0$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=1,73$ or $x=-0,93$.....(3)

$y=2,46$ or $y=-2,86$

$\therefore$ (1,73;2,46) and (−0,93;−2,86)

ACTIVITIES/ASSESSMENTS: LESSON 4
1. Determine the equations of the tangents to $(x-2)^2+(y+3)^2=16$ which are: 1.1 parallel to the y-axis. 1.2 parallel to the x-axis.

2. The straight-line $y = x + 2$ cuts the circle $x^2 + y^2 = 20$ at A and B.

2.1 Determine the coordinates of A and B.

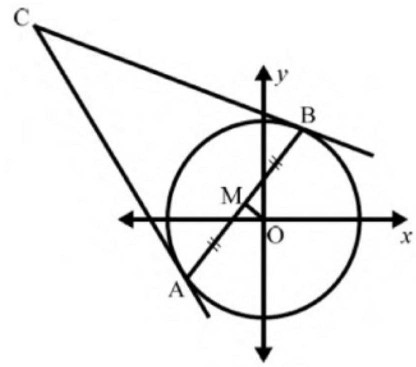
2.2 Determine the length of the chord AB.

2.3 Determine the coordinates of M, the midpoint of the chord AB.

2.4 Show that OM ⊥ AB if O is the origin.

2.5 Determine the equations of the tangents to the circle at A and B.

2.6 Determine the coordinates of C, the point of intersection of the tangents found in 2.5.



TOPIC: ANALYTICAL GEOMETRY

LESSON 5: CONSOLIDATION

NOTES

- Review all content, then give activities to learners.
- Address solutions and common errors and misconception.

ACTIVITIES/ASSESSMENTS

1. In the diagram alongside, the circle centred at N (–1; 3) passes through A (–1; –1) and C. B (–4; 2), C, D and E are joined to form a parallelogram such that BE is parallel to the x-axis. CD is a tangent to the circle at C and CD = 6 units.

1.1 Write down the length of the radius of the circle.

1.2 Calculate the:

1.2.1 Coordinates of C

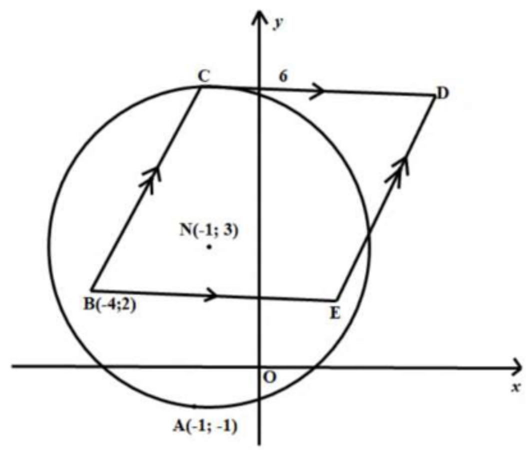
1.2.2 Coordinates of D

1.2.3 Area of ΔBCD

1.3 The circle, centred at N, is reflected about the line $y = x$. M is the centre of the new circle which is formed. The two circles intersect at A and F. Calculate the:

1.3.1 Length of NM

1.3.2 Midpoint of AF



2. $x^2 + y^2 + 8x - 6y = -5$ is the equation of the circle with center M. UE is the tangent to the circle at Q. QMD, DA. AU and UQE are straight lines. DU is parallel to the x- axis.

2.1 Determine the coordinates of M, the center of the circle.

2.2 Calculate the coordinates of Q. If $y < 2$

2.3 Calculate the equation of tangent UE.

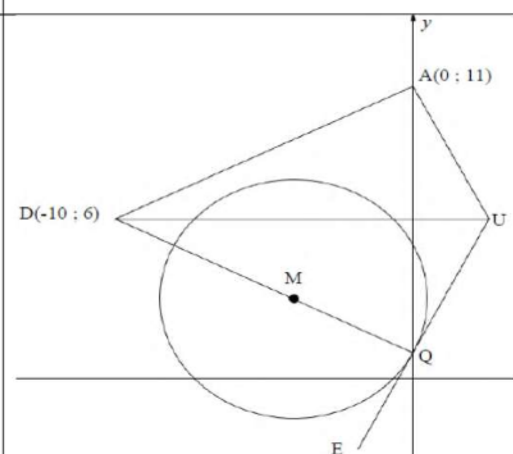
2.4 Write down the equation of DU.

2.5 Calculate the coordinates of U.

2.6 Determine the lengths of AD and AU.

2.7 Prove that QUAD is a cyclic quadrilateral.

2.8 If it is further given that $DA = DQ$, calculate the area of QUAD.

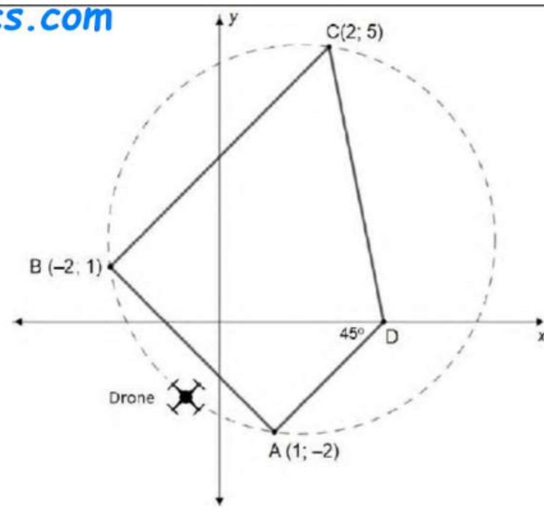


3. A farmer uses a drone to check his fences for ar damage
   - The drone flies in a perfect circle and pass directly above points A, B and C.
   - Pointa A, B and C are on the same horizont plane
   - D lies on the x-axis
   - At present the angle madebetween line AD ar the x-axis is 45°

3.1 Calculate the current coordinates of point D.

3.2 Calculate the size of $\hat{CBA}$

3.3 How far to the right must D be moved along the x-axis so that the drone will fly directly above it on its circular flight path



4. In the diagram below, PRST is a quadrilateral with P (–2; 4), $R\ (-1; -2\frac{1}{3})$ and T (4; 6). PS is drawn and W is a point on the x-axis. PR intersects the x-axis at M and PT || RS. L is the y- intercept of PT.

4.1 Calculate the gradient of PT.

4.2 Determine the equation of RS in the form $y = mx + c$.

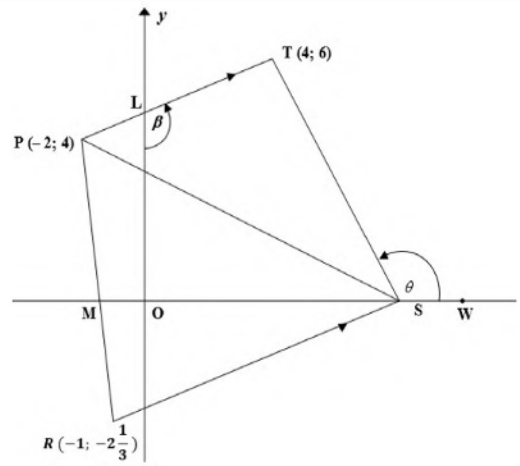
4.3 Show that PT ⊥ TS.

4.4 Calculate the size of:

4.4.1 θ

4.4.2 β, give reasons.

4.5 Calculate the area of LOST.



5 In the diagram, M(3 ; – 5) is the centre of the circle having PN as a diameter. KL is a tangent to the circle at N(7 ; – 2).

5.1 Calculate the coordinates of P.

5.2 Determine the equation of:

5.2.1 The circle in the form $(x-a)^2+(y-b)^2=r^2$

5.2.2 KL in the form of $y = mx + c$

5.3 For which values of k will $y = -\frac{4}{3} + k$ be a secant to the circle?

5.4 Points A(t ; t) and B are not shown on the diagram.
From point A, another tangent is drawn to the circle with centre M at B.

5.4.1 Show that the length of tangent AB is given by $\sqrt{2t^2+4t+9}$.

5.4.2 Determine the minimum length of AB.