KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

PINETOWN DISTRICT

TEACHING AND LEARNING SUPPORT – CURRICULUM FET (GRADES 10-12)

Grade 12 Mathematics Teacher Support Document

COMPILED BY PINETOWN DISTRICT MATHEMATICS ADVISORS:
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TOPIC: PATTERNS, SEQUENCES AND SERIES (Lesson 1)		**Weighting**	25 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Revision of Quadratic Number Patterns**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Sequence, common difference, consecutive terms, quadratic sequence				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Linear pattern, quadratic pattern

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Identifying the pattern
- Failing to recall that it is only the second difference that is constant/common in a quadratic sequence

METHODOLOGY

A **sequence** or pattern is an ordered set of numbers or variables.

A **quadratic sequence** is a sequence of numbers in which the second difference between any two consecutive terms is constant

Successive or **consecutive terms** are terms that directly follow one after another in a sequence.

Consider the following example: 1; 2; 4; 7;11; ...

```
1     2     4     7     11
 \   / \   / \   / \   /
  +1    +2    +3    +4
```

The second difference is obtained by taking the difference between consecutive first differences

```
1     2     3     4
 \   / \   / \   /
  +1    +1    +1
```

We notice that the second differences are all equal to 1. Any sequence that has a common second difference is a **quadratic sequence.**

The **common or constant difference** (d) is the difference between any two consecutive terms in a linear sequence.

General Term or n^{th} term of the quadratic sequence is a mathematical expression that describes the sequence and that generates any term in the pattern by substituting different values for n.

If the sequence is quadratic, the $general\ or\ n^{th}$ term is of the form $T_n = an^2 = +bn + c$

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	$a + b + c$		$4a + 2b + c$		$9a + 3b + c$		$16a + 4b + c$
1st difference		$3a + b$		$5a + b$		$7a + b$	
2nd difference			$2a$		$2a$		

In each case, the common second difference is a 2a.

Examples:

1. Determine the second difference between the terms

(a) 1; -3; -9; -17; ...

First difference: -4; -6; -8; ...

Second difference: -2

(b) $t - 2; 4t - 1; 9t; 16t + 1; ...$

$4t - 1 - (t - 2); 9t - (4t - 1); 16t + 1 - 9t; ...$

$3t + 1; 5t + 1;\ 7t + 1; ...$

Second Difference: $2t$

2. Complete the sequenceby filling in the missing term:

(a) 11; 21; 35; ...; 75

53

(b) 3; ...; -13; -27; -45

-3

3. Use the general term to generate the first four terms in each sequence:

(a) $T_n = n^2 + 3n - 1$

$n = 1$: $T_1 = 1^2 + 3(1) - 1 = 3$

$n = 2$: $T_2 = 2^2 + 3(2) - 1 = 9$

$n = 3$: $T_3 = 3^2 + 3(3) - 1 = 17$

$n = 4$: $T_4 = 4^2 + 3(4) - 1 = 27$

ACTIVITIES/ ASSESSMENT

1. Determine the second difference between the terms for the following sequences:

(a) 6; 11; 18; 27; ...
(b) 1; 4; 9; 16; ...
(c) 3; 0; -5; -12; ...
(d) 1; 3; 7; 13; ...
(e) 0; -6; -16; -30; ...
(f) $3a + 1; 12a + 1; 27a + 1; 48a + 1; ...$

2. Complete the sequence by filling in the missing term:

(a) 20; ...; 42; 56; 72
(b) ...; 37; 65; 101
(c) 24; 35; 48; ...; 80
(d) ...; 11; 26; 47

3. Use the general term to generate the first four terms in each sequence:

(a) $T_n = 3n^2 - 2n$

(b) $T_n = -n^2 - 5$

(c) $T_n = -2n^2 + n + 1$

TOPIC: PATTERN, SEQUENCES AND SERIES (Lesson 2)		Weighting	25 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Revision of Quadratic Number Patterns				
RELATED CONCEPTS/ TERMS/VOCABULARY	General Term				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

General term: $T_n = an^2 + bn + c$, common difference

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confusing n and T_n
- Failing to recall that the sequence of first differences of a quadratic sequence is linear.
- Failing to recall that the value of n should be a natural number

METHODOLOGY

The general term (nth term) of a quadratic number pattern is defined by: $T_n = an^2 + bn + c$

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	$a+b+c$		$4a+2b+c$		$9a+3b+c$		$16a+4b+c$
1st difference		$3a+b$		$5a+b$		$7a+b$	
2nd difference			$2a$		$2a$		

Examples:

1. Given the sequence: 5; 12; 23; 38; ...

(a) Write down the next two terms of the sequence.

Find the first differences between the terms:

5 ; 12 ; 23 ; 38
+7 ; +11 ; +15

Find the second differences between the terms:

7 ; 11 ; 15
+4 ; +4

The next two terms will be:

...38 ; 57 ; 80...
+19 ; +23

(b) Determine an equation for the nth term of the sequence

$2a = 4$...2nd difference
$a = 2$

$3a + b = 7$ ... first term of first difference
$3(2) + b = 7$
$b = 1$

$a + b + c = 5$... first term
$2 + 1 + c = 5$
$c = 2$

$\therefore T_n = 2n^2 + n + 2$

2. Consider the following number pattern: 2; 3; 6; 11; ...

(a) Determine the nth term (general term) and hence the value of the 42nd term.

$a + b + c \rightarrow$ 2 ; 3 ; 6 ; 11
$3a + b \rightarrow$ 1 ; 3 ; 5
$2a \rightarrow$ 2 ; 2

$2a = 2$ $\quad$ $3a + b = 1$ $\quad$ $a + b + c = 2$

$a = 1$ $\quad$ $3(1) + b = 1$ $\quad$ $(1) + (-2) + c = 2$

$b = -2$ $\quad$ $c = 2 - 1 + 2 = 3$

$\therefore T_n = 1n^2 - 2n + 3$ and $T_{42} = 1(42)^2 - 2(42) + 3 = 1683$

(b) Determine which term will equal 1091.

$T_n = 1091$ and $T_n = 1n^2 - 2n + 3$

$\therefore n^2 - 2n + 3 = 1091$

$n^2 - 2n - 1088$

$(n - 34)(n + 32) = 0$

$n = 34$ or $n = -32$

$\therefore n = 34$

The 34th term will equal 1091

ACTIVITIES/ ASSESSMENT

1. Given the quadratic sequence: 16; 27; 42; 61; ...

(a) Write down the next two terms of the quadratic sequence
(b) Find the general formula for the quadratic sequence above.

2. Consider the following number pattern: 1; 6; 15; 28; ...

(a) Determine the general term and hence the value of the 20th term.
(b) Determine which term will equal 3160.

$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$ ... arithmetic sum formula to n terms

It is also possible to derive a formula to calculate the sum of a finite arithmetic series consisting of n terms.

$S_n = \frac{n}{2}[2a + (n-1)d]$ can be written as follows:

$S_n = \frac{n}{2}[a + a + (n-1)d]$

The **last** term (l) is the nth term and therefore $l = a + (n-1)d$

$\therefore S_n = \frac{n}{2}(a + l)$

Examples:

1. Find the sum of the first 20 terms of the arithmetic series $3 + 10 + 17 + 24 + \ldots$

$a = 3$ $\quad d = 7$

$S_n = \frac{n}{2}[2a + (n-1)d]$ ... arithmetic series sum formula

$S_{20} = \frac{20}{2}[2(3) + (20-1)7] = 1\,390$

2. Calculate the sum of the following finite arithmetic series: $3 + 5 + 7 + \ldots + 111$

It is necessary to first **determine the number of terms** in the series before being able find the sum of the series.

$a = 3$ $\quad d = 2$ $\quad T_n = 111$

$T_n = a + (n-1)d$ ... general formula

$111 = 3 + (n-1)2$

$111 = 3 + 2n - 2$

$110 = 2n$

$n = 55$, there are 55 terms in the series

To calculate the sum of the series, use either one of the two formulae:

$S_n = \frac{n}{2}(a + l)$ **OR** $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{55} = \frac{55}{2}(3 + 111) = 3\,135$ $\qquad$ $S_{55} = \frac{55}{2}[2(3) + (55-1)2] = 3\,135$

3. Calculate: $\sum_{m=2}^{100}(7-2m)$

Expand so as to identify the type of series:

$\sum_{m=2}^{100}(7-2m) = (7-2(2)) + (7-2(3)) + (7-2(4)) + (7-2(5)) + \cdots + (7-2(100))$

$= 3 + 1 + (-1) + (-3) + \cdots + (-193)$

This is an arithmetic series with $a = 3$ and $d = -2$

Now determine the number of terms: **(Number of terms = Top - Bottom +1)**

$= 100 - 2 + 1 = 99$

To calculate the sum of the series, use either one of the two formulae:

$S_n = \frac{n}{2}(a + l)$ **OR** $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{99} = \frac{99}{2}(3 + (-193))$ $= -9\,405$ $\qquad$ OR $\qquad$ $S_{99} = \frac{99}{2}[2(3) + (99-1)(-2)]$ $= -9\,405$

4. How many terms of the arithmetic sequence 4; 7; 10; 13; .... will add up to 175?

$a = 4$ $\quad d = 3$ $\quad S_n = 175$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$175 = \frac{n}{2}[2(4) + (n-1)3]$

$175 \times 2 = n(8 + 3n - 3)$

$350 = 3n^2 + 5n$

$3n^2 + 5n - 350 = 0$ ... standard form of a quadratic

$(3n + 35)(n - 10) = 0$ ... you can also use a quadratic formula

$n = -\frac{35}{3}$ or $n = 10$

$\therefore n = 10$ $\qquad$ ALWAYS REMEMBER: **n is a natural number.**

5. Determine n if $\sum_{r=1}^{n}(6r-1) = 456$

$\sum_{r=1}^{n}(6r-1) = 456 \quad = (6(1)-1) + (6(2)-1) + (6(3)-1) + \cdots + (6(n)-1) = 456$

$= 5 + 11 + 17 + \cdots + (6n-1) = 456$

This is an arithmetic series with $a = 5$ and $d = 6$

Number of terms $= n - 1 + 1 = n$ and $S_n = 456$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$456 = \frac{n}{2}[2(5) + (n-1)6]$

$912 = n(10 + 6n - 6)$

$912 = 6n^2 + 4n$

$6n^2 + 4n - 912 = 0$

$3n^2 + 2n - 456 = 0$

$(3n + 38)(n - 12) = 0$ ... the quadratic formula may be used

$n = -\frac{38}{3}$ or $n = 12$

$\therefore n = 12$ ... n is always a natural number

ACTIVITIES/ ASSESSMENT
1. Calculate the sum of the arithmetic series $4 + 7 + 10 + \ldots + 901$.
2. Calculate the sum of arithmetic series: $10 + 7 + 4 + \ldots$ to 32 terms
3. Calculate: (a) $\sum_{k=1}^{100}(3k+2)$ (b) $\sum_{m=2}^{50}(5-2m)$
4. Consider the series: $-3 + 1 + 5 + \ldots + 313$ (a) How many terms are there in the series?   (b) What is the sum of the series?
5. (a) How many terms of the arithmetic sequence 3; 11; 19; ... will add up to 1580?   (b) Determine n if: $4 + 13 + 22 + \ldots$ (to n terms) $= 539$
6. Determine m in each of the following arithmetic series: (a) $\sum_{k=1}^{m}(7k+5)=1287$ (b) $\sum_{i=0}^{m}(1-3i)=-671$
7. The third term of an arithmetic sequence is -7 and the 7th term is 9. Determine the sum of the first 51 terms of the sequence.
8. The sum of n terms of an arithmetic series is $5n^2 - 11n$ for all values of n. Determine the common Difference
9. The common difference of an arithmetic series is 3. Calculate the values of n for which the nth term of the series is 93 and the sum of the first n terms is 975.
10. An athlete trains by running 600 metres on the first day, 900 metres on the second, 1200 metres on the third and so forth.   (a) How far does he run on the 15th day?   (b) What is the total distance that he will run in 15 days?   (c) How long will it be before he can run a marathon of 42km?

TOPIC: PATTERNS, SEQUENCES AND SERIES (Lesson 7)		Weighting	25 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Geometric Series				
RELATED CONCEPTS/ TERMS/VOCABULARY	Geometric series, geometric sum formula				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Geometric sequence, common ratio, sum

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Using method of determining common difference for common ratio
- Failing to differentiate between a common ratio and the common difference
- Failing to differentiate between T_n and S_n

METHODOLOGY

A **geometric sequence** is a sequence in which each term is obtained from the last by multiplying by a constant number called **common ratio.**

A **geometric series** is obtained by adding successive terms of a geometric sequence.

3, 6, 12, 24, ... is a **geometric sequence** and the sum of that sequence, $3 + 6 + 12 + 24 + \ldots$ is a **geometric series** (sum of terms denoted by S_n)

If the first term of a geometric sequence is a and the common ratio is r, then the sequence is:

$$a; ar; ar^2; \ldots; ar^{n-2}; ar^{n-1}$$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \quad \ldots (1)$$

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \quad \ldots (2) \quad \text{multiply each term by } r$$

$$S_n - rS_n = a - ar^n \ldots (1)-(2)$$

$$S_n(1-r) = a(1-r^n) \ldots \text{taking out a common factor}$$

$S_n = \frac{a(1-r^n)}{1-r}$ where $r \neq 1$ [The general formula for determining the sum of a geometric series]

Similarly, for (2) – (1) the following formula can be derived:

$S_n = \frac{a(r^n-1)}{r-1}$ where $r \neq 1$

Examples:

1. Find the sum of the first 12 terms of the series: $\frac{2}{3}+2+6+\cdots$

$a=\frac{2}{3}$ $\qquad r=3$

$S_n=\frac{a(r^n-1)}{r-1}$

$S_{11}=\frac{\frac{2}{3}(3^{11}-1)}{3-1}=177146,\dot{6}$

2. Calculate the sum of the following finite series 0, 25 + 0,5 +1+ 2 + ... + 256.

Number of terms is not known. Therefore, it is necessary to **first calculate** the number of terms.

$a=0{,}25$ $\qquad r=2$ $\qquad T_n=256$

$T_n=ar^{n-1}$ ... general term of a geometric sequence.

$256=0{,}25(2)^{n-1}$

$2^{n-1}=\frac{256}{0{,}25}$

$2^{n-1}=1024=2^{10}$

$n-1=10$

$n=11$ $\qquad \therefore$ There are 11 terms in the series.

In order to calculate the sum of the series, either one of the two formulae may be used:

$S_n=\frac{a(r^n-1)}{r-1}$

$S_{11}=\frac{0{,}25(2^{11}-1)}{2-1}=511{,}75$

3. How many terms of the geometric sequence -1; 2; - 4; 8; ... will add up to 349 525?

$a=-1$ $\qquad r=-2$ $\qquad S_n=349\ 525$

$S_n=\frac{a(r^n-1)}{r-1}$

$349\ 525=\frac{-1((-2)^n-1)}{-2-1}$

$-1048575=-1((-2)^n-1)$ ... multiply by – 3 on both sides

$1048575=(-2)^n-1$ ... divide by – 1 on both sides

$1048576=(-2)^n$

$(-2)^{20}=(-2)^n$

$n=20$ ... $\therefore$ The first 20 terms must be added to give 349 525

4. Given: $\sum_{k=2}^{m}\frac{1}{15}(3)^{k-1}=24\frac{1}{5}$ Determine m.

Generate the series by substituting the value of k:

$$0{,}2+0{,}6+1{,}8+\cdots+\tfrac{1}{15}(3)^{m-1}=24\tfrac{1}{5}$$

This is a geometric series with $a=0,2$ and $r=3$

Number of terms $=m-2+1=m-1$ ... $\therefore S_{m-1}=24\frac{1}{5}$

$S_n=\frac{a(r^n-1)}{r-1}$

$S_{m-1}=\frac{a(r^{m-1}-1)}{r-1}$

$24\frac{1}{5}=\frac{0{,}2(3^{m-1}-1)}{3-1}$

$24\frac{1}{5}\times 2=0{,}2(3^{m-1}-1)$

$242=3^{m-1}-1$ ... divide by 0,2 on both sides

$243=3^{m-1}$ ... transpose – 1

$3^5=3^{m-1}$ ... write 243 in exponential form (same bases)

$m-1=5$

$m=6$

ACTIVITIES/ ASSESSMENT

1. Determine the sum of each geometric series:

(a) 1 – 3 + 9 + ... eight terms $\qquad$ (b) -64 + 32 – 16 + ... to 10 terms

(c) $\frac{2}{27}+\frac{2}{9}+\frac{2}{3}+\cdots$ to 9 terms $\qquad$ (d) $6+3+\frac{3}{2}+\frac{3}{4}+\cdots$ to 11 terms

2. Given the Given the geometric series: $-64-32-16-\cdots-\frac{1}{32}$

(a) How many terms are there in the series $\qquad$ (b) What is the sum of the series?

3. Calculate: (a) $\sum_{k=1}^{10}\frac{1}{50}(5)^{k-1}$ (b) $\sum_{m=3}^{11}8\left(\frac{1}{2}\right)^{m-4}$ (c) $\sum_{i=0}^{10}3^{4-i}$

4. How many terms of the geometric sequence 64; 32; 16; ... will add up to $127\frac{1}{2}$?

5. Determine n if: 3 6 + + 12 + ... (to n terms) = 765

6. Determine m in each of the following a series:

(a) $\sum_{k=1}^{m}(2)^{8-k}=255\frac{1}{2}$ (b) $\sum_{p=1}^{m}(-8)\left(\frac{1}{2}\right)^{p-1}=-15\frac{3}{4}$ (c) $\sum_{i=1}^{m}5(1)^{i-2}=500$

7. What is the least value of p for which the series $\sum_{k=1}^{p}\frac{1}{16}(2)^{k-2}>31$?

8. Mr Deeds gives his son R1 on his 1st birthday, R2 on his 2nd birthday, R4 on his 3rd birthday, R8 on his 4th birthday and so forth.

(a) How much will he give his son on his 18th birthday?

(b) Find the total amount of money he will have given him by that stage?

TOPIC: PATTERNS, SEQUENCES AND SERIES (Lesson 8)		Weighting	25 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Problems involving Simultaneous Equations**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Elimination, subject of the formula				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Simultaneous equations, common difference, common ratio

General term: Arithmetic Sequence ($T_n = a + (n-1)d$) and Geometric sequence ($T_n = ar^{n-1}$)

Sum Formula: Arithmetic Series ($S_n = \frac{n}{2}[2a + (n-1)d]$) and Geometric Series ($S_n = \frac{a(r^n-1)}{r-1}$)

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Using method of determining common difference for common ratio
- Failing to differentiate between a common ratio and the common difference
- Failing to differentiate between T_n and S_n

METHODOLOGY

Examples:

1. The 5th term of the arithmetic sequence is 12 and the 14th term is – 33.

(a) Determine the first 3 terms of the sequence.

$T_5 = 12$ $\qquad$ $T_{14} = -33$

$T_5 = a + 4d$ $\qquad$ $T_{14} = a + 13d$

$a + 4d = 12 \ldots (1)$ $\qquad$ $a + 13d = -33 \ldots (2)$

Solve the two equations by **elimination method.** In elimination method you either **add or subtract** the **equations** to get and equation in one variable/unknown.

$(1)-(2)$: $\quad -9d = 45$ ... you can also make a the **subject of the formula:** is the single variable usually on the left of the equal sign

$d = -5$ ... divide by – 9 on both sides

$a + 4(-5) = 12$ ... substitute – 5 into (1)

$a = 12 + 20 = 32$

$T_1 = 32,\quad T_2 = 32 + (-5) = 27,\quad T_3 = 27 + (-5) = 22$

∴ The first three terms of the sequence are 32;27;22

(b) Hence determine the 40th term.

$T_{40} = a + 39d$

$= 32 + 39(-5) = -163$

2. The sum of the first 12 terms of an arithmetic series is 96. The 3rd and 6th terms add up to 12. Determine the first term and the common difference.

$S_{12} = 96$

$S_{12} = \frac{12}{2}[2a + (12-1)d]$

$96 = 6(2a + 11d)$

$16 = 2a + 11d$ ... divide by 6 on both sides ... (1)

$T_3 + T_6 = 12$

$a + 2d + a + 5d = 12$

$2a + 7d = 12$ ... add like terms ... (2)

$(1) - (2)$: $4 = 4d$

$d = 1$ ... common difference

$2a + 7(1) = 12$ ... substitute $d = 1$

$2a = 5$

$a = \frac{5}{2}$ ... first term

3. Determine the first three terms of the geometric sequence of which the 7th term is 1458 and the 4th term is 54.

$T_7 = 1458$ and $T_4 = 54$

$ar^6 = 1458$ ... (1)

$ar^3 = 54$ ... (2)

$(1) \div (2)$: $r^3 = 27$

$r^3 = 3^3$

$r = 3$ ... common ratio

$a(3)^3 = 54$ ... substitute by $r = 3$

$a = 2$ ... divide by 27 (3^3) on both sides

$a = 2$ is the first term

$\therefore T_1 = 2$ $\quad T_2 = 2 \times 3 = 6$ $\quad T_3 = 6 \times 3 = 18$

$\therefore$ The first three terms are: 2; 6; 18

4. The constant ratio of a geometric series is $-2\frac{1}{2}$. The sum of the first four terms is $17\frac{2}{5}$. Calculate the first term.

$r = -\frac{5}{2}$

$S_4 = \frac{a(r^4-1)}{r-1} = 17\frac{2}{5}$

$\frac{a\left(\left(-\frac{5}{2}\right)^4-1\right)}{\left(-\frac{5}{2}\right)-1} = \frac{87}{5}$

$a(38{,}0625) = -60{,}9$

$a = -1{,}6$ ... first term

ACTIVITIES/ ASSESSMENT

1. Determine the first three terms of each of the following arithmetic sequences of which:
   (a) the 3rd term of the sequence is 23 and the 26th term is 230.
   (b) the 5th term of the sequence is 19 and the 15th term is 59.

2. The 15th and 3rd terms of an arithmetic sequence are 100 and 28 respectively. Determine the 100th term.

3. The 13th and 7th terms of an arithmetic sequence are 15 and 51 respectively.
   (a) Which term is equal to -21?
   (b) Show that 66 is not a term of the sequence.

4. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.

5. (a) If $T_7 = -4$ *and* $S_{16} = 24$ of an arithmetic series, determine the first term and the constant difference of the series.
   (b) The fifth term of an arithmetic sequence is 0 and thirteenth term is 12. Determine the sum of the first 21 terms of sequence.
   (c) The 1st term of an arithmetic sequence is 6 and the sum of the first five terms is 250. Calculate the 12th term of the sequence.

5. The first term and the last term of an arithmetic series is 5 and 61 respectively while the sum of all the terms is 957. Determine the number of terms in the series.

6. The sum of the first 10 terms of an arithmetic series is 145 and the sum of its fourth and ninth term is five times the third term. Determine the first term and constant difference.

7. Given is the series $1 + 2 + 3 + 4 + 5 + ... + n$
   (a) Show that $S_n = \frac{n(n+1)}{2}$
   (b) Determine the sum of the first 1001 terms excluding all multiples of 7.

8. The 9th term and the 6th term of a geometric sequence are 80 and 10 respectively.
   (a) Find the first term and the constant ratio.
   (b) Find the number of terms if the last term is 5120.

9. If $T_3 = \frac{15}{16}, T_6 = \frac{5}{18}$ and the last term is $\frac{40}{729}$, calculate the number of terms in the sequence if the sequence is geometric.

10. The sum of the first 4 terms of a geometric series is 15 and the sum of the next 4 terms is 240. Determine the positive constant ratio.

11. The ratio between the sum of the first three terms of a geometric series and the sum of the 4th, 5th and 6th terms of the same series is 8: 27. Determine the constant ratio and the first 2 terms if the third term is 8.

TOPIC: PATTERNS, SEQUENCES AND SERIES (Lesson 9)		Weighting	25 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	The sum to infinity of a Convergent geometric series				
RELATED CONCEPTS/ TERMS/VOCABULARY	Infinity, converge				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Common ratio

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Writing $S_\infty = \frac{a}{r-1}$ instead of $S_\infty = \frac{a}{1-r}$

METHODOLOGY

We have been working with **finite sums**, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms.

If the **sum of a series gets closer and closer to a certain value** as we increase the number of terms in the sum, we say that the series **converges**.

Consider the geometric series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

$S_1 = 0{,}5$ $\quad S_2 = 0{,}75$ $\quad S_3 = 0{,}875$ $\quad S_4 = 0{,}9375$

As we continue to add the terms in this way, it seems that the decimal values are tending towards a value of 1, which is referred to as the **sum to infinity** of the convergent geometric series.

If $-1 < r < 1$ then the infinite series will converge. In a **convergent** geometric series where $-1 < r < 1$, the sum to infinity exists. r is the ratio of the series.

In a convergent geometric series in which $-1 < r < 1$, the sum to infinity is given by the formula:

$$S_\infty = \frac{a}{1-r} \ldots \text{ general formula for sum to infinity where } -1 < r < 1$$

Examples:

1. Given the series $18 + 6 + 2 + \ldots$ Find the sum to infinity if it exists.

$a = 18$ $\qquad r = \frac{1}{3} \ldots -1 < r < 1$

$S_\infty = \frac{18}{1-\frac{1}{3}} = 27$

2. Calculate: $\sum_{n=1}^{\infty} 2.10^{1-n}$

$= 2 + 0{,}2 + 0{,}02 + \cdots$

$a = 2$ $\qquad r = 0{,}1 \ldots \; -1 < r < 1$ $\qquad \therefore S_\infty = \frac{a}{1-r} = \frac{2}{1-0{,}1} = 2{,}\dot{2}$

3. Convert the recurring decimal $2{,}\dot{5}\dot{3}$ into a common fraction.

First write the recurring decimal as a geometric series:

$2{,}\dot{5}\dot{3} = 2{,}535353\ldots = 2 + (0{,}53 + 0{,}0053 + 0{,}000053 + \cdots)$

In the brackets there is an infinite geometric series with $a = 0{,}53$ and $r = 0{,}01$

$S_\infty = \frac{a}{1-r} = \frac{0{,}53}{1-0{,}01} = \frac{53}{99}$

$\therefore 2{,}\dot{5}\dot{3} = 2 + \frac{53}{99} = 2\frac{53}{99}$

4. Consider the infinite geometric series: $p + p(p+1) + p(p+1)^2 + \cdots$

(a) For what values of p will the series converge?

$a = p$ $\qquad r = p + 1$

The series will converge if $-1 < r < 1$

$-1 < p + 1 < 1$

$-2 < p < 0 \ldots$ subtract 1 from all terms

(b) Assuming the series is convergent, calculate the sum to infinity.

$S_\infty = \frac{a}{1-r} = \frac{p}{1-(p+1)} = \frac{p}{-p} = -1$

5. A convergent geometric series has a second term of 8 and a sum to infinity of 36. Determine the possible constant ratio(s).

$T_2 = 8$
$ar = 8 \ldots (1)$
$a = \frac{8}{r} \ldots$ a the subject of the formula

$S_\infty = 36$
$\frac{a}{1-r} = 36 \ldots (2)$
$a = 36(1-r)$
$\frac{8}{r} = 36 - 36r$
$8 = 36r - 36r^2$
$36r^2 - 36r + 8 = 0$
$9r^2 - 9r + 2 = 0 \ldots$ 4 common factor
$(3r-2)(3r-1) = 0$

$r = \frac{2}{3}$ or $r = \frac{1}{3}$

ACTIVITIES/ ASSESSMENT

1. Find the sum of each of the following infinite geometric series:

   (a) $2 + \frac{2}{3} + \frac{2}{9} + \cdots$ (b) $-64 + 32 - 16 + \ldots$

2. Calculate:

   (a) $\sum_{m=1}^{\infty} 8(2)^{-2m}$ (b) $\sum_{m=0}^{\infty} 3\left(-\frac{1}{2}\right)^{m}$

3. Convert each of the following recurring decimals to a common fraction by first writing it as a geometric series.

   (a) $0,\dot{2}\dot{3}$ (b) $4,\dot{2}$ (c) $0,5\dot{4}$

4. For which values of x will the following geometric series converge?

   (a) $2 + \frac{2}{3}(x+1) + \frac{2}{9}(x-1)^2 + \cdots$ (b) $(2-x) + (2-x)^2 + (2-x)^3 + \cdots$

   (c) $\sum_{i=0}^{\infty} 4(3-x)^i$

5. (a) The first term of a geometric series is 124. The sum to infinity is 64. Determine the common ratio and the sequence.

   (b) For a geometric series with $r = 0,22$ and $S_\infty = 20$. Calculate the first term.

6. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the constant ratio between consecutive terms.

7. The sum to infinity of a convergent geometric series is 32 and $r = \frac{1}{2}$. Calculate the difference between the sum to infinity and the sum of the first five terms.

8. A specific tree grows 1,5 m in the 1st year. Its growth each year thereafter is $\frac{2}{3}$ of its growth in the previous year. What is the greatest height it can reach?

TEST 2: PATTERNS, SEQUENCES AND SERIES

DBE PAPERS

MARKS:25 DURATION: 30 MIN.

INSTRUCTIONS:

Answer ALL the questions

QUESTION 1 [12 marks]

1.1 Derive a formula for the sum of the first n terms of an arithmetic sequence if the first term of the sequence is a and the common difference is d. (4)

1.2 Consider an arithmetic which has the second term equal to 8 and the fifth term equal to 10.

1.2.1 Determine the common difference of the sequence (3)

1.2.2 Write down the sum of the first 50 terms of this sequence, using sigma notation. (2)

1.2.3 Determine the sum of the first 50 terms of this sequence. (3)

QUESTION 2 [13 Marks]

2.1 Given the geometric series: $(x+1) + (x-1) + (2x-5) + \cdots$

2.1.1 Calculate the value(s) of x, $(x \neq 1 \text{ or } x \neq -1)$. (4)

2.1.2 If $x = -2$, calculate the sum to infinity of the given series. (3)

2.2 Themba is planning a bicycle trip from Cape Town to Pretoria. The total distance covered during the trip will be 1 500 km. He plans to travel 100 km on the first day. For every following day he plans to cover 94% of the distance he covered the previous day.

2.2.1 What distance will he cover on day 3 of the trip? (2)

2.2.2 On what day of the trip will Themba pass the halfway point? (4)

TOPIC: EUCLIDEAN GEOMETRY (Lesson 1)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Revise lines and angles				
RELATED CONCEPTS/ TERMS/VOCABULARY	Parallel lines, intersect, vertex				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Types of angles

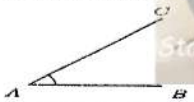
RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Assuming that lines are parallel

METHODOLOGY

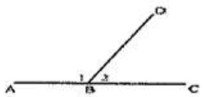
An angle is formed when **two straight lines** meet at a point, also known as a **vertex.**



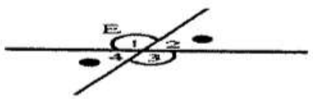
Angles are labelled with a caret on a letter, for example, $\hat{A}$.
Angles can also be labelled according to the **line segments** that make up the angle, for example $C\hat{A}B$ or $B\hat{A}C$.
The "∠" symbol is a short method of writing angle in geometry and is often used in phrases such as "sum of ∠s in Δ".

Types of Angles

1. **Adjacent angles** on a straight line are supplementary.
   Supplementary angles add up to 180°
   Angles that **share a vertex** and a common side.
   ABC is a straight line. ∴ $\hat{B}_1 + \hat{B}_2 = 180°$



2. If two lines intersect, **vertically opposite angles** are equal.
   Two lines intersect if they **cross each** other at a point.
   Vertically opposite angles are **angles opposite each other** when two lines intersect.
   They share a vertex and are equal.



$\hat{E}_1 = \hat{E}_3$ and $\hat{E}_2 = \hat{E}_4$

3. The **angles around a point** add up to 360° (Revolution).

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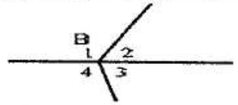
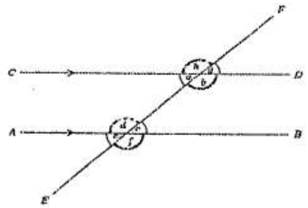


$\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \hat{B}_4 = 360°$

PARALLEL LINES

Parallel lines are always the **same distance apart** and they are **denoted by arrow symbols** as shown below.



$CD \parallel AB$. EF is a **transversal line.**

A transversal line **intersects** two or more parallel lines.

The **properties of the angles** formed by the above intersecting lines.

1. **Corresponding Angles**

   Corresponding angles **lie either both above or both below the lines** and on the **same side of the transversal.**

   If the **lines are parallel,** the **corresponding angles will be equal.**
   $\hat{h} = \hat{d}$, $\hat{a} = \hat{e}$, $\hat{g} = \hat{c}$ and $\hat{b} = \hat{f}$

2. **Alternate Angles**

   Alternate angles **lie on opposite sides of the transversal** and between the lines.
   If the **lines are parallel,** the **alternate angles will be equal.**
   $\hat{a} = \hat{c}$ and $\hat{d} = \hat{b}$

3. **Co-interior Angles**

   Co-interior angles lie on the same side of the transversal between the lines.
   If the **lines are parallel,** the **co-interior angles are supplementary**

If two lines are intersected by a transversal such that **corresponding angles are equal;** or **alternate angles are equal;** or **co-interior angles are supplementary,** then the **two lines are parallel.**

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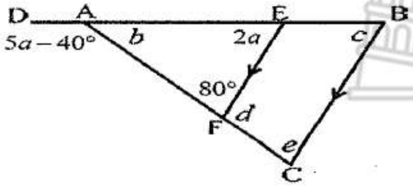
ACTIVITIES/ ASSESSMENT

Exercise 1: Mind Action Series (Pg. 165)

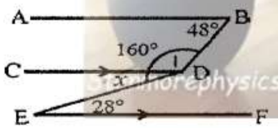
1. In ΔABC, EF∥BC. BA is produced to D.



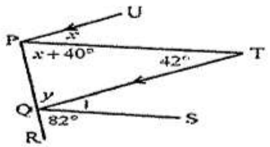
a) Calculate a and hence show that AE = AF.

b) Calculate, with reasons, the value of $b, c, d,$ and e.

2. In the diagram below, CD∥EF, $D\hat{E}F = 28°$, $A\hat{B}D = 48°$ and $B\hat{D}E = 160°$. Prove that AB∥CD.



3. In the diagram below, PU∥ QT , $\hat{T} = 42°$, $R\hat{Q}S = 82°$, $P\hat{Q}T = y$, $U\hat{P}T = x$ and $Q\hat{P}T = x + 40°$



a) Prove that PT∥QS

b) Calculate y

TOPIC: EUCLIDEAN GEOMETRY (Lesson 2)	Weighting	40 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Revise Triangles: Classification, Congruency, Similarity and Midpoint theorem			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Types of triangles

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Using congruency conditions without understanding

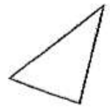
METHODOLOGY

PROPERTIES OF TRIANGLES

A triangle is a three-sided polygon. Triangles can be classified **according to sides** and also be classified **according to angles.**

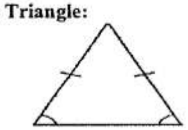
TYPES OF TRIANGLES according to **sides**

a) Scalene Triangle:



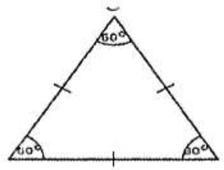
All sides are not equal

((b)) **Isosceles Triangle:**



Two angles are equal.
Angles opposite equal sides are equal.

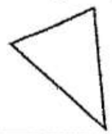
c) Equilateral Triangle:



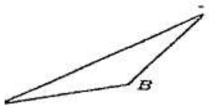
All sides are equal
All angles are equal

TYPES OF TRIANGLES according to **angles**

a) Acute-angled Triangle

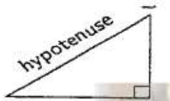


b) Obtuse-angled Triangle



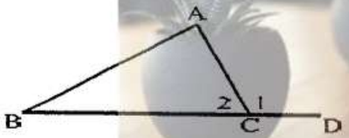
One angle is greater than 90° and the other two angles are acute.

c) Right-angled Triangle



One angle is equal to 90° and the other two are acute.

Exterior angle of a triangle.



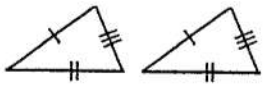
In triangle ABC, BC is produced to D.

$\hat{A} + \hat{C} + \hat{C}_2 = 180°$ (Sum of $\angle s\ of\ a\ \Delta$)

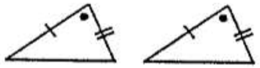
$\hat{C}_1 = \hat{A} + \hat{B}$ (ext. $\angle\ of\ \Delta$)

2. **CONGRUENT TRIANGLES** – 4 Conditions

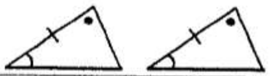
a) If three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent. Side, Side, Side **(S, S, S)**



b) If two sides and the **included** angle of a triangle are equal to the corresponding two sides and **included** angle of another triangle, then the two triangles are congruent. Side, Angle, Side **(S, A, S)**



c) If one side and two angles of a triangle are equal to the corresponding one side and two angles of another triangle, then the two triangles are congruent. Angle, Side, Angle **(A, S, A)**



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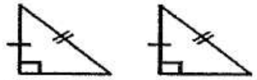
d) If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle, then the two triangles are congruent. 90°, Hypotenuse, Side (R, H, S).



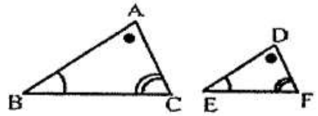
We use ≡ to indicate that triangles are congruent.

NOTE: The order of letters when labelling congruent triangles is very important.

3. **SIMILAR TRIANGLES**

Two triangles are similar if **one triangle is a scaled version of the other**. This means that their **corresponding angles are equal** in measure and the **ratio of their corresponding sides are in proportion.** The two triangles have **the same shape**, but different scales.
Congruent triangles are similar triangles, but **not all similar triangles are congruent.**
We use /// to indicate that two triangles are similar.

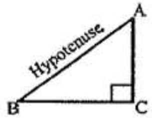
a) If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar. Angle, Angle, Angle (A, A, A)



b) If all three pairs of corresponding sides of two triangles are in proportion, then the triangles are Similar. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. Side, Side, Side (S, S, S)

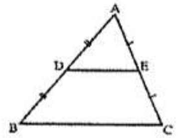
NOTE: The order of letters for similar triangles is very important.
Always label similar triangles in corresponding order.

4. **PYTHAGORAS THEOREM**

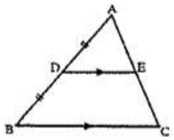


$AB^2 = BC^2 + AC^2$

5. **MIDPOINT THEOREM**



If AD = DB and AE = EC,
Then $DE \parallel BC$ and $DE = \frac{1}{2}BC$



If AD = DB and DE||BC,
then $AE = EC$ and $DE = \frac{1}{2}BC$

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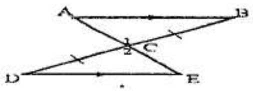
ACTIVITIES/ ASSESSMENT

Exercise 1: Mind Action Series (Pg. 166)

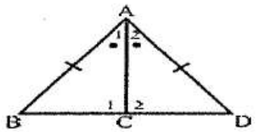
1. AB∥DE and DC = CB



a) Prove that AC = CE

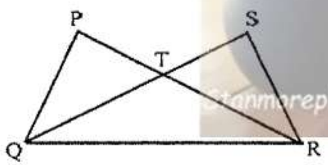
b) AB = DE

2.



Prove that Δ ABC≡ Δ ADC using different conditions of congruency.

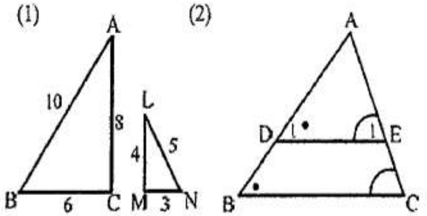
3. In the diagram below, sides PR and QS of triangles PQR and SQR intersect at T. PD = SR and $\hat{P} = \hat{S} = 90°$.



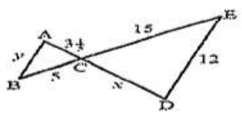
Prove that $\Delta PQR \equiv \Delta SRQ$

4. Show that the following triangles are similar:

(1) (2)



5. If $\Delta ABC///\Delta DEC$, calculate x and y.



TEST 1: LINES, ANGLES AND TRIANGLES

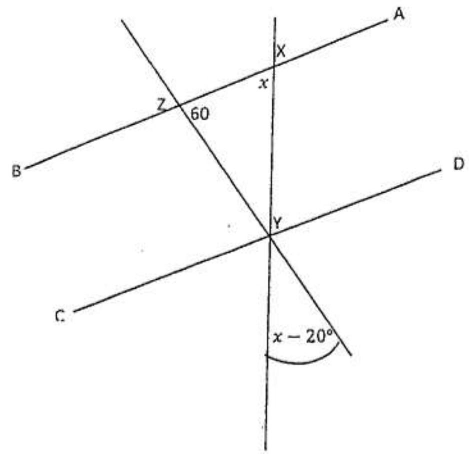
MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [16 Marks]

1.1 In the diagram below, AB and DC are two parallel lines cut by two transversal lines at X, Y and Z respectively.



1.1.1 Determine giving reasons, the value of x in the diagram: (6)

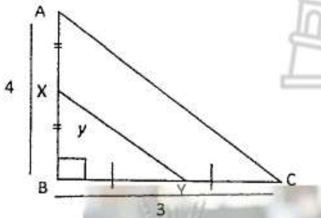
1.1.2 Name one pair of co-interior angles (1)

1.1.3 Name one pair of alternate angles (1)

1.1.4 Complete: If two parallel lines are cut by a transversal, then the co-interior angles are ....(1)

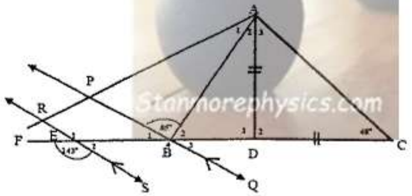
1.1.5 Complete: The size of angle XYD = Reason .... (2)

1.2 Determine with reasons, the value of y (XY) in the diagram below. Given that AB = 4 units and BC = 3 units. X and Y are the midpoints of AB and BC respectively. (5)



QUESTION 2 [9 Marks]

In the diagram below, AD = CD and PQ||RS. AR and FC are straight lines. RS and FC intersect at E also PQ intersects FC at B.



1.1 Determine the sizes of the following angles, giving appropriate reasons:

2.1.1 $\hat{D}_1$ (2)

2.1.2 $\hat{B}_1$ (2)

2.1.3 $\hat{A}_2$ (2)

2.2 Show that $R\hat{E}F = \hat{B}_3$ (3)

TOPIC: EUCLIDEAN GEOMETRY (Lesson 3)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Circle Geometry				
RELATED CONCEPTS/ TERMS/VOCABULARY	Circle Geometry Theorems				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Grade 11 theorems and application of theorems

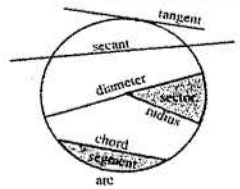
RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

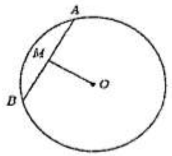
Failing to write acceptable reasons and also confusing reasons

METHODOLOGY

Radius: A line from the centre to any point on the circumference of the circle.
Diameter: A line passing through the centre of the circle. It is double the length of the radius.
Chord: A line from the circumference to the circumference.
Tangent: A line touching the circle at only one point.
Secant: A line passing through two points on the circle.



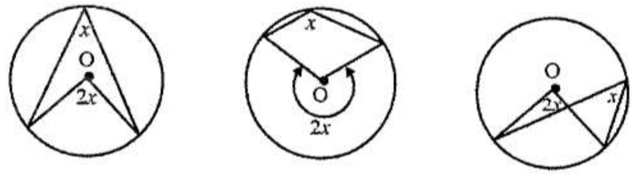
1.



If O is the Centre and $OM \perp AB$, then $AM = MB$

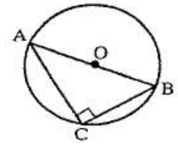
If O is the Centre and $AM = MB$, then $OM \perp AB$

2.



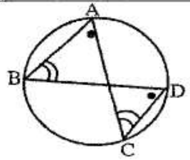
If an arc subtends an angle at the centre of a circle and at the circumference, then the **angle at the centre is twice the size of the angle at the circumference.**

3.



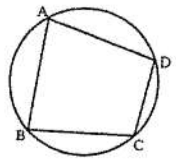
If AOC is a diameter, then $\hat{C} = 90°$
(∠ in semi-circle)

4.



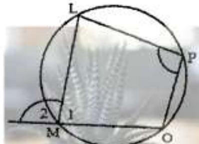
If A, B, C and D are concyclic (lie on a circle) and if AB, AC, BD and CD are chords of the circle, then $\hat{A} = \hat{D}$ and $\hat{B} = \hat{C}$ (∠ in same segment) or (line/arc subtends equal angles)

5.



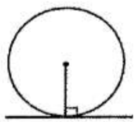
If ABCD is cyclic, then $\hat{A} + \hat{C} = 180°$
And $\hat{B} + \hat{C} = 180°$ ((opp ∠s of cyclic quad)

6.



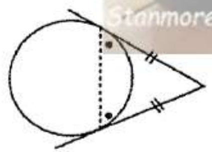
If LMOP is cyclic then $\hat{M}_2 = \hat{P}$
(Ext ∠ cyclic quad)

7.



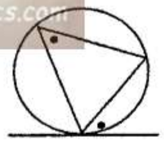
tan ⊥ rad

8.



Tangents from the same point

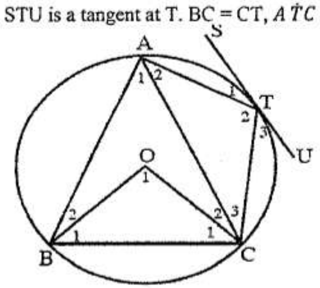
9.



tan-chord

ACTIVITIES/ ASSESSMENT

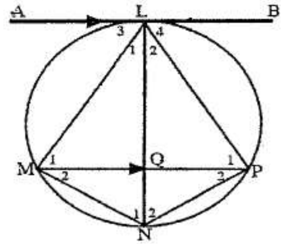
1. O is the centre of the circle. STU is a tangent at T. BC = CT, $A\hat{T}C = 105°$ and $C\hat{T}U = 40°$.



Calculate the size of:

(a) $\hat{A}_2$ (b) $\hat{B}_1 + \hat{B}_2$ (c) $\hat{A}_1$ (d) $\hat{C}_2$

2. ALB is a tangent to circle LMNP. ALB||MP.



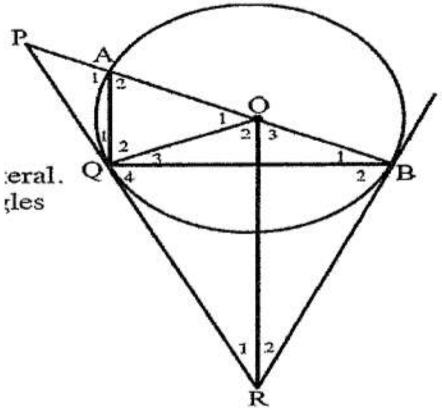
Prove that:

(a) LM = LP

(b) LN bisects MN̂P

(c) LM is a tangent to circle MNQ

3. In the figure below, RQ and RB are tangents at the points Q and B respectively to the circle with centre O. The radius BO produced meets the circle at A and RQ produced at P.



Let $\hat{Q}_1 = x$

(a) Prove that RBOQ is a cyclic quadrilateral.

(b) Name, with reasons, **FOUR** other angles in the figure which are equal to x.

(c) Express $\hat{P}$ in terms of x.

TOPIC: EUCLIDEAN GEOMETRY (Lesson 4)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Ratio and Proportion				
RELATED CONCEPTS/ TERMS/VOCABULARY	Area of triangles, Ratio, Base				
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					
Ratio, fraction, area of triangle, parallel lines					
RESOURCES					

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to recognize triangles with the same base and or same height

METHODOLOGY

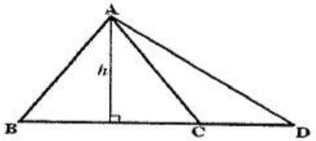
Ratio is a comparison between two quantities of the same kind and has no units. It is usually **written as a fraction** and is usually given in its **simplest** form. e. g. $\frac{1}{2}$ or 1: 2 pronounced as 1 to 2. A ratio gives no indication of actual length. **Do not convert** a ratio to a decimal.

If **two or more ratios are equal** to each other, then we say that they are in the **same proportion. Proportion** describes the equality of ratios.

The **height or altitude** of a triangle is always relative to the chosen **base**.
The **area of the triangles** can be calculated by using the formula: Area $\Delta ABC = \frac{1}{2} base \times height$

1. Triangles with **equal heights** have the ratio of their areas proportional to the ratio of their bases.

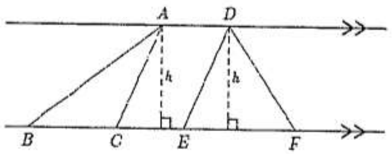
- Two triangles which share a **common vertex** have a **common height**.



Area of $\Delta ABC = \frac{1}{2} BC \times h$
Area of $\Delta ACD = \frac{1}{2} CD \times h$

$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta ACD} = \frac{\frac{1}{2}BC \times h}{\frac{1}{2}CD \times h} = \frac{BC}{CD}$

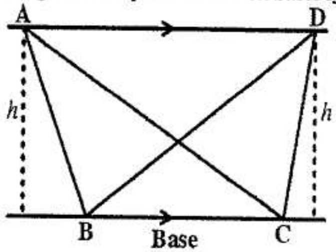
- Two triangles **between the same parallel lines** have the **same height.**



Area of $\Delta ABC = \frac{1}{2} BC \times h$
Area of $\Delta DEF = \frac{1}{2} EF \times h$

$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF} = \frac{\frac{1}{2}BC \times h}{\frac{1}{2}EF \times h} = \frac{BC}{EF}$

2. Triangles with **equal or common bases lying between parallel lines** have the same area.



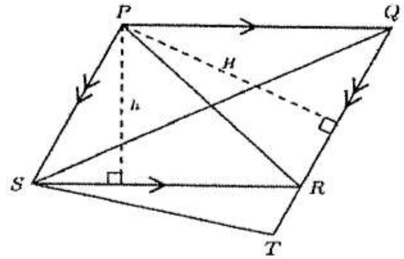
Area of $\Delta ABC = \frac{1}{2} BC \times h$

Area of $\Delta DBC = \frac{1}{2} BC \times h$

Rea of ΔABC = Area of ΔDBC

Example:

Given parallelogram PQRS with QR produced to T. RS = 45 cm, QR = 30 cm and h = 10 cm.



1. Calculate H.

2. If TR: RD = 1: 4, show that

$\frac{Area\ of\ \Delta STR}{Area\ of\ \Delta PRQ} = \frac{1}{3}$

Solution:

1. Use the formula for area of a parallelogram to calculate H.

Area $PQRS = SR \times h$
$= 45 \times 10 = 450cm^2$

Area $PQRS = QR \times H$
$450 = 30 \times H$
$H = 15\ cm$

2. Use proportionality:
$\frac{TR}{TQ} = \frac{1}{4}$ ... given that $TR: RD = 1: 4$

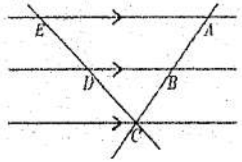
Then $\frac{RQ}{TQ} = \frac{3}{4}$

And $\frac{TR}{RQ} = \frac{TR}{TQ} \times \frac{TQ}{RQ}$

$= \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$

ACTIVITIES/ ASSESSMENT

1. The diagram below shows three parallel lines cut by two transversals EC and AC such that ED: DC = 4: 6.



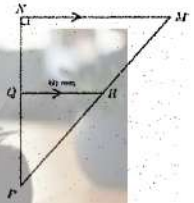
Determine:

(a) $\frac{BC}{AB}$ (b) $AB\!:\!AC$

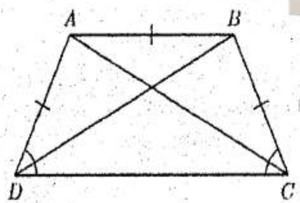
(c) The length of AC and ED if it is given that AB = 12mm

2. In right-angled ΔMNP, QR is drawn parallel to NM, with R the midpoint MP. NP = 16cm and RQ = 60 mm

Determine QP and RP.



3. Given trapezium ABCD with DA = AB = BC and $A\hat{D}C = B\hat{C}D$.



(a) Prove that BD bisects $\hat{D}$

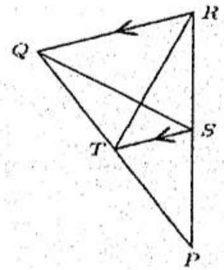
(b) Prove that the two diagonals are equal in length

(c) If DC: AB = 5: 4, show that

Area $ABCD = 2{,}25 \times$ Area ΔABC

4. In the diagram to the right, ΔPQR is given with QR k TS.

Show that area ΔPQS = area ΔPRT.



TOPIC: EUCLIDEAN GEOMETRY (Lesson 5)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Proportionality theorem				
RELATED CONCEPTS/ TERMS/VOCABULARY	Construct				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Ratio, proportional, corollary, parallel lines, midpoint

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Not doing construction correctly, not understanding why construction is done.

METHODOLOGY

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

Reason: **(line ‖ to one side of a Δ)**

Given: DE‖BC

Required to prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Proof:

In ΔADE, draw height h relative to base AD and height k relative to base AE. Join BE and DC to create ΔBDE and ΔCED.

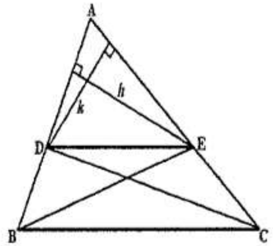
$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta BDE} = \frac{\frac{1}{2}AD \times h}{\frac{1}{2}DB \times h} = \frac{AD}{DB}$

$\frac{Area\ of\ \Delta AED}{Area\ of\ \Delta CED} = \frac{\frac{1}{2}AE \times k}{\frac{1}{2}EC \times k} = \frac{AE}{EC}$

$Area\ \Delta BDE = Area\ of\ \Delta CED$ (same base, same height and between parallel lines)

$\therefore \frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta BDE} = \frac{Area\ of\ \Delta AED}{Area\ of\ \Delta CED}$

$\therefore \frac{AD}{BD} = \frac{AE}{CE}$



NOTE: We use the same method to show that:

$$\frac{AD}{AB} = \frac{AE}{AC}\ and\ \frac{AB}{BD} = \frac{AC}{CE}$$

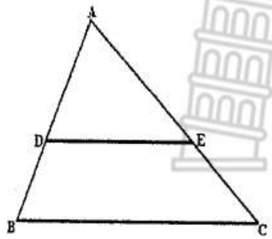
A corollary of the proportion theorem is the **mid-point theorem**: the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

Given ΔABC: If AB = BD and AC = CE, then BC ‖ DE and BC = $\frac{1}{2}$DE.

Examples:

1. In ΔABC, $DE \parallel BC$, $AB = 28$ mm and $AE : EC = 4 : 3$.

   Determine the length of BD.



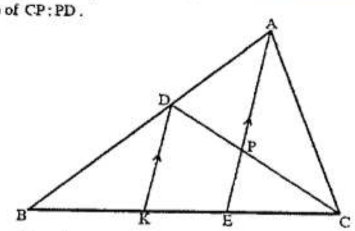
$\frac{AD}{DB} = \frac{AE}{EC} = \frac{4}{3}$ ... **(line ∥ to one side of a ΔABC)**

$\frac{BD}{AB} = \frac{CE}{AC} = \frac{3}{7}$

$\frac{BD}{28} = \frac{3}{7}$

$BD = \frac{3}{7} \times 28 = 12\ mm$

2. D and E are points on sides AB and BC respectively of ΔABC such that $AD : DB = 2 : 3$ and $BE = \frac{4}{3}EC$. If $DK \parallel AE$ and AE and CD intersect at P, find the ratio of CP : PD.



$\frac{CP}{PD} = \frac{CE}{EK}$ ... **(line ∥ to one side of a ΔCDK)**

$\frac{CP}{PD} = \frac{3p}{EK}$

$\frac{EK}{EB} = \frac{AD}{AB}$ ... **(line ∥ to one side of a ΔABE)**

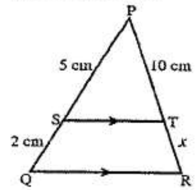
$\frac{EK}{4p} = \frac{2k}{5k}$

$EK = \frac{2}{5} \times 4p = \frac{8p}{5}$

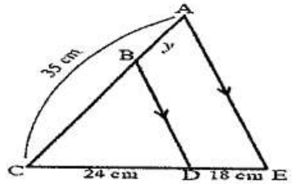
$\therefore \frac{CP}{PD} = 3p \div \frac{8p}{5} = 3p \times \frac{5}{8p} = \frac{15}{8}$

ACTIVITIES/ ASSESSMENT

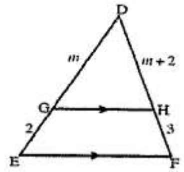
1. In ΔPQR, $ST \parallel QR$. Calculate the value of x.



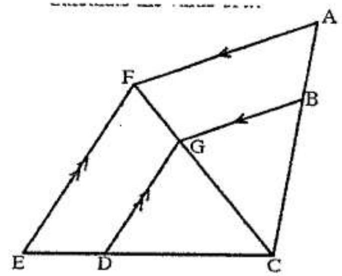
2. In ΔACE, $BD \parallel AE$. Calculate the value of y.



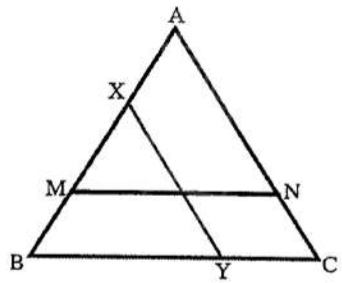
3. In ΔDEF, $GH \parallel EF$. Calculate the value of m.



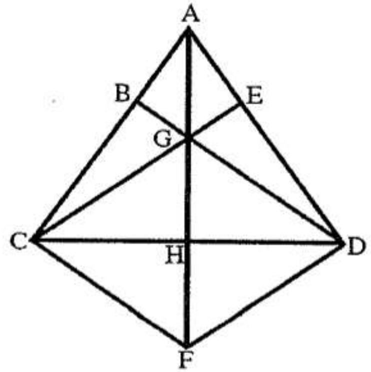
4. In ΔACF, $AF \parallel BG$ and in ΔCEF, $EF \parallel DG$. $ED = 22$cm, $DC = 33$cm, $BC = 15$cm and $AB = x$. Calculate the value of x.



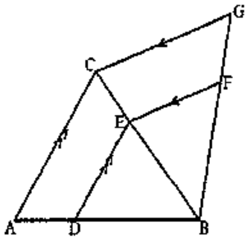
5. In ΔABC, $XY \parallel AC$ and $MN \parallel BC$. $AN : NC = 3 : 2$ and $BY = 2YC$. $AB = 15$ cm. Calculate: (a) AM (b) XB



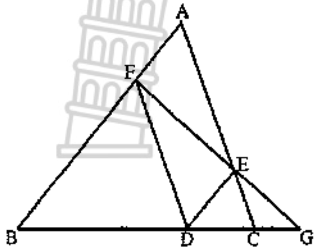
6. $DB \parallel FC$, $CE \parallel FD$ and $AG = \frac{1}{2}GF$. The diagonals of parallelogram GCFD intersect at H. Calculate: (a) GH : HF (b) AG : GH (c) AB : BC (d) AE : ED



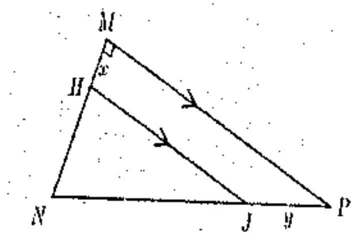
7. In ΔABC, AC∥DE. In ΔBCG, CG∥EF.
Prove that: AD: DB = GF: FB



8. In ΔABC, DE∥AB and DF∥AC. Prove that GB:GD = GD: GC



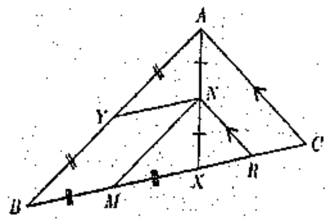
9. In ΔMNP, $\hat{M} = 90°$ and HJ ∥ MP.
HN: NH = 3: 1, HM = x and JP = y.



(a) Calculate JP: NP.

(b) Calculate $\frac{Area\ \Delta HNJ}{Area \Delta MNP}$

10. In ΔABC, X is a point on BC. N is the mid-point of AX, Y is the mid-point of AB and M is the mid-point of BX.



(a) Prove that Y BMN is a parallelogram.

(b) Prove that $MR = \frac{1}{2}BC$

TOPIC: EUCLIDEAN GEOMETRY (Lesson 6)		Weighting	40 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Proportionality Theorem**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Sides in proportion				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Proof of proportionality theorem, properties of polygons

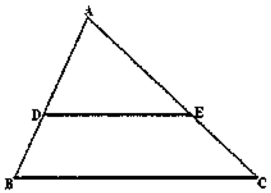
RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Assuming statements and reasons
- Incorrect or incomplete reasons for statements
- Incorrect naming of angles
- Can't differentiate between a parallelogram and a cyclic quadrilateral

METHODOLOGY

Given ΔABC.

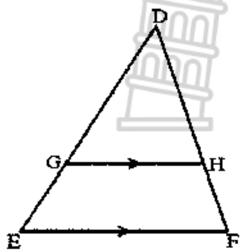


If DE ∥ BC, then $\frac{AD}{DB} = \frac{AE}{EC}$ or $\frac{AB}{AD} = \frac{AC}{AE}$ $\frac{AB}{DB} = \frac{AC}{EC}$

ACTIVITIES/ ASSESSMENT

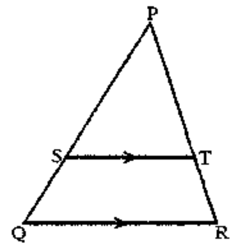
1. In ΔDEF, GH∥EF. DG: GE = 5: 3.
DE = 32mm and DF = 24mm

Determine the length of DG, GE, DH and HF.

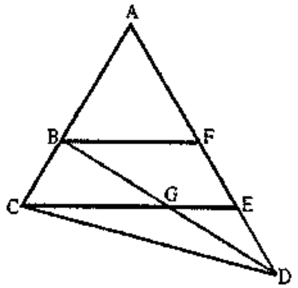


2. In ΔPQR, ST∥QR. PQ = 35 mm, PR = 25mm and QS = 14mm.

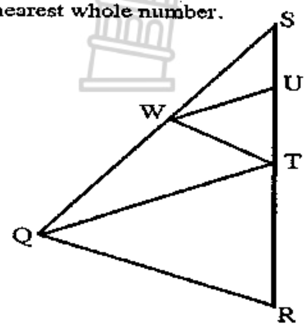
Determine the length of PT.



3. In ΔACE, BF||CE, BC = $\frac{3}{8}$AC
and AE: ED = 4: 3
Determine DG: GB

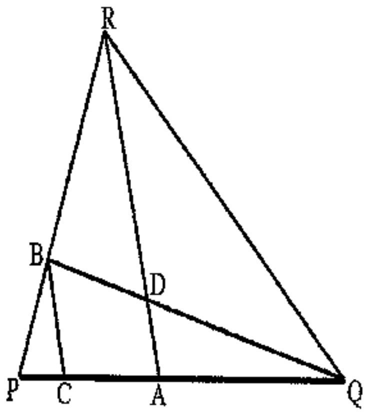


4. In ΔQRS, T and U are points on RS and W is a point on QS such that QT||WU and QR||WT. If QW = 12 mm, WS = 11 mm and TU = 6 mm, CALCULATE RS correct to the nearest whole number.



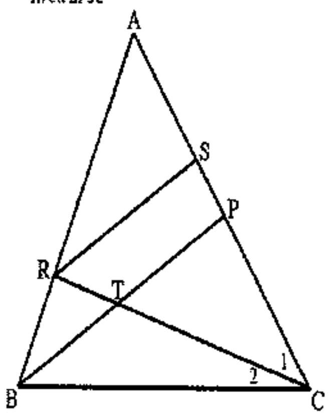
5. PA = $\frac{4}{9}$PQ and 2PB = BR.
BC||RA. Determine:

(a) BD: DQ
(b) $\frac{Area \Delta PRA}{Area \Delta QRA}$ (Hint: construct the common height)

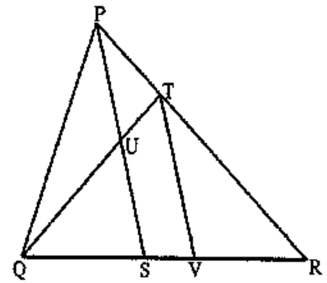


6. In ΔABC, P is the midpoint of AC. RS||BP and AR: AB = 3: 5. Determine:

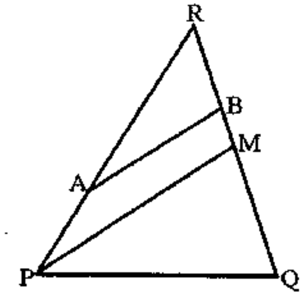
(a) AS: SP (b) AS: SC (c) RT: TC
(d) $\frac{Area \Delta TPC}{Area \Delta PSC}$ (Hint: Use area rule)



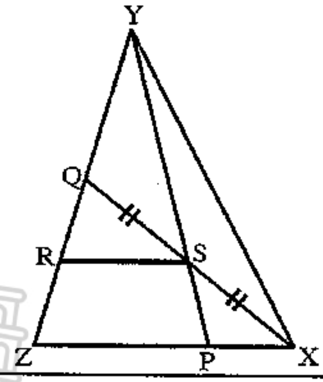
7. In ΔPQR, PS||VT and QS:SR = 2: 3.
T is a point on PR such that PT: TR = 2: 7.
Prove that: QU: UT = 3: 1



8. $\frac{RB}{RQ} = \frac{1}{3}$, PA: AR = 1: 2 and PM||AB.
(a) Write down values for RA: RP and RB: BQ
(b) Determine BM: BR
(c) Prove that RM = MQ



9. Q bisects YZ. SR||XZ.
(a) Show that QR = RZ (2) Prove that YS = $\frac{3}{4}$YP (3) If YS = 12 cm, find the length of SP.



TEST 2: EUCLIDEAN GEOMETRY

MATHSCIECAT TUTORING AND 2015 SEPT. P2 (WC)

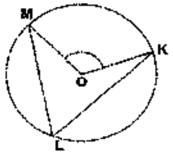
MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

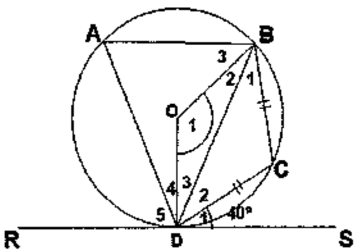
QUESTION 1 [15 Marks]

1. In the diagram, O is the centre of the circle. K, L and M are points on the circumference of the circle.
Prove that the obtuse angle at O, $K\hat{O}M = 2K\hat{L}M$. (6)



2. In the diagram, RDS is a tangent to circle O at D.
If BC = DC and $C\hat{D}S = 40°$, calculate with reasons
The size of:

1.1 $B\hat{D}C$ (3)
1.2 $\hat{C}$ (2)
1.3 $\hat{A}$ (2)
1.4 $\hat{O}_1$ (2)



QUESTION 2 [10 Marks]

In the diagram below, P is the midpoint of AC in ΔABC.

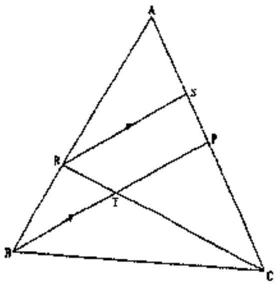
R is a point on AB such that RS || BP and $\frac{AR}{AB} = \frac{3}{5}$. RC cuts BP in T.

Determine, giving reasons, the following ratios;

2.1 $\frac{AS}{SC}$ (3)

2.2 $\frac{RT}{TC}$ (3)

2.3 $\frac{Area\ of\ \Delta TPC}{Area\ of\ \Delta RSC}$ (4)



TOPIC: EUCLIDEAN GEOMETRY (Lesson 7)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Similarity of Triangles				
RELATED CONCEPTS/ TERMS/VOCABULARY					
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					
Properties of triangles, sides in proportion					
RESOURCES					

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confuse similar and congruent
- Use of proportionality theorem instead of similar triangle to the ratio
- Incorrect labelling of similar triangles

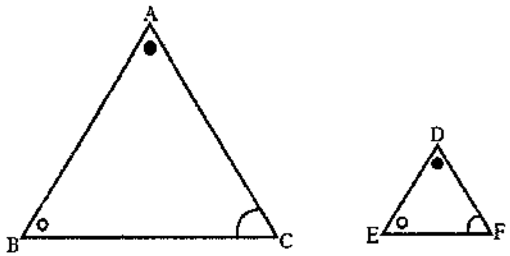
METHODOLOGY

Two polygons with the same number of sides are similar when:

- All pairs of **corresponding angles** are **equal**, and
- All pairs of **corresponding sides** are in the same **proportion.**

To **prove two triangles are similar**, we need only show that one of the conditions is true. If one of the **conditions is true** for two triangles, then the **other condition is also true.**

Equiangular triangles are similar.

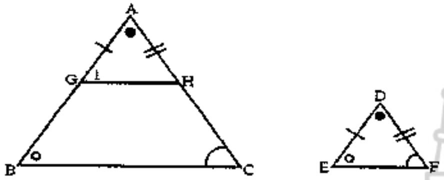


Given: ΔABC and ΔDEF with $\hat{A} = \hat{D}, \hat{B} = \hat{E}, \hat{C} = \hat{F}$

Required to prove: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Proof:

Construction: mark G on AB so that AG = DE, and mark H on AC so that AH = DF.



In $\Delta\, AGH$ and $\Delta\, DEF$

1. AG = DE ... Construction
2. $\hat{A} = \hat{D}$ ... Given
3. AH = DF ... Construction

$\therefore\ \Delta AGH \equiv \Delta DEF$ ... SAS

$\therefore\ A\hat{G}H = \hat{E}$

$\hat{B} = \hat{E}$ ... Given

$\therefore\ A\hat{G}H = \hat{B}$

$\therefore\ GH \parallel BC$ ... Corresponding angles equal

$\therefore\ \frac{AB}{AG} = \frac{AC}{AH}$ ... Prop. Theorem

$\therefore\ \frac{AB}{DE} = \frac{AC}{DF}$ ... AG = DE, AH = DF

Similarly, by constructing Q on CA so that CQ = FD, and mark P on BC so that CP = FE.

$\frac{CA}{FD} = \frac{CB}{FE}$

$\therefore\ \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Therefore, equiangular triangles are similar

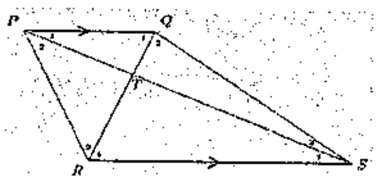
NOTE:

- The symbol for similar is ///.
- The symbol for congruency is ≡.
- Be careful to label similar triangles correctly.
- If we know that **two pairs of angles are equal**, then the **remaining angle** in each triangle must also be equal. Therefore, the two triangles are similar.
- In other words, to prove that two triangles are equiangular, we need only show that two pairs of angles are equal.

Examples:

1. PQSR is a trapezium, with PQ ∥ RS.

Prove that PT: RT = ST: QT.



In ΔPTQ and ΔSTR:

$\hat{Q}_1 = \hat{R}_1$ ... Alternate angles, PQ ∥ RS

$P\hat{T}Q = S\hat{T}R$ ... Vertically opposite angles

$\hat{P}_1 = \hat{S}_1$

$\therefore\ \Delta PTQ /// \Delta STR$ ... ∠, ∠, ∠

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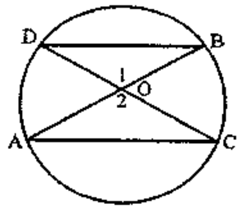
$\frac{PT}{ST} = \frac{TQ}{TR} = \frac{PQ}{SR}$ ... $\Delta PTQ /// \Delta STR$

$\therefore\ PT.TR = ST.TQ$

2. A, B, C and D are points on the circle.
DOC and AOB are chords.
DB and AC are joined. Prove that:

(a) ΔAOC|||ΔDOB

(b) $\frac{OB}{OD} = \frac{OC}{OA}$



(a) In ΔAOC and ΔDOB

$\hat{A} = \hat{D}$ ... ∠s on the same segment

$\hat{B} = \hat{C}$ ... ∠s on the same segment

$\hat{O}_1 = \hat{O}_2$ ... Sum of ∠s of Δ

∴ AOC|||ΔDOB ... ∠, ∠, ∠

(b) $\frac{AO}{DO} = \frac{OC}{OB} = \frac{AC}{DB}$ ... AOC|||ΔDOB

$\therefore\ \frac{OB}{DO} = \frac{OC}{AO}$

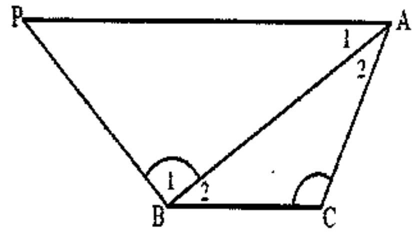
ACTIVITIES/ ASSESSMENT

1. In the diagram, PA||BC and $\hat{B} = \hat{C}$. Prove that:

(a) ΔPAB|||ΔABC

(b) PA: AB = PB: AC

(c) AB. AC = BP. BC

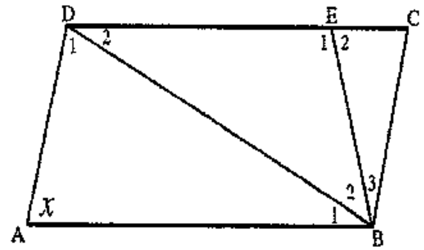


2. In the diagram, ABCD is a parallelogram.
BE = BC and BD = AB. $\hat{A} = x$. Prove that:

(a) ΔABD|||ΔCBE

(b) AB. BE = BD. BC

(c) $\frac{BD}{AD} = \frac{BE}{CE}$



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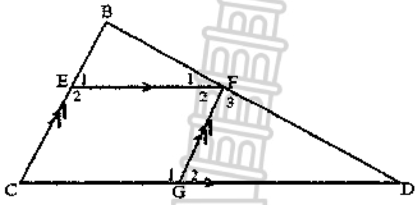
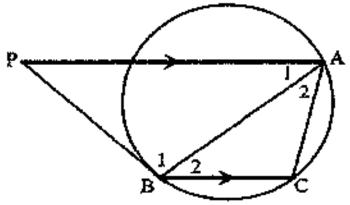
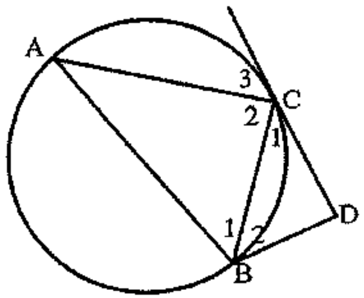
3. In the diagram, EF||CGD and BEC||FG.
Prove that:
(a) ΔBEF|||ΔFGD
(b) $\frac{DG}{FG} = \frac{EF}{BE}$



4. PB is a tangent to circle ABC. PA||BC.
Prove that:
(a) ΔPAB|||ΔABC
(b) PA: AB = AB: BC
(c) $\frac{AP}{BP} = \frac{AB}{AC}$



5. AB is a diameter of circle ABC.
DC is a tangent at C and BD ⊥ CD.
(a) Prove that ΔBDC|||ΔBCA
(b) If AB = 25cm and AC = 24cm
Calculate the length of BC and CD



TOPIC: EUCLIDEAN GEOMETRY (Lesson 8)	Weighting	40 ± 3	Grade	12
Term	1	Week no.		
Duration	1 hour	Date		
Sub-topics	Euclidean Geometry			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

RESOURCES

DBE PAST PAPERS

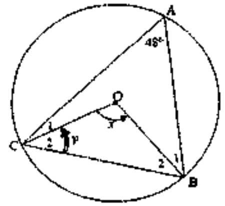
ERRORS/MISCONCEPTIONS/PROBLEM AREAS

METHODOLOGY

ALL THEOREMS

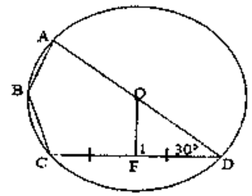
ACTIVITIES/ ASSESSMENT

1. In the diagram, O is the centre of the circle passing through A, B and C. $C\hat{A}B$ 48°, and . $C\hat{O}B = x$ and $\hat{C}_2 = y$.



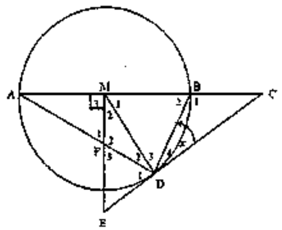
Determine with reasons the size of:
(a) x
(b) y

2. In the diagram, O is the centre of the circle passing through A, B, C and D. AOD is a straight line and F is the midpoint of chord CD. and OF are joined. $O\hat{D}F = 30°$, O and F are joined.



Determine with reasons, the size of
(a) $\hat{F}_1$
(b) $A\hat{B}C$

3. In the diagram, M is the centre of the circle and diameter AB is produced to C. ME is drawn perpendicular to AC such that CDE is a tangent to the circle at D. ME and chord AD intersect at F. MB = 2BC

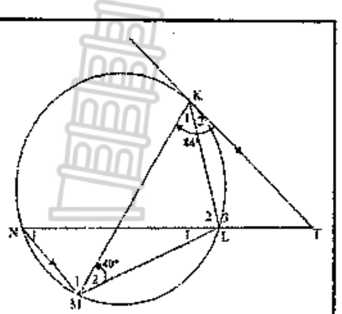


If $\hat{D}_4 = x$, write down with reasons another angle equal to x.

4. In the diagram, tangent KT to the circle at K is parallel to the chord NM. NT cuts the circle at L. Δ KML is drawn. $\hat{M}_2 = 40°$ and $M\hat{K}T = 84°$.

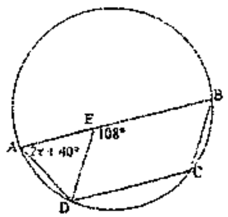
Determine, giving reasons, the size of:

(a) $\hat{K}_2$
(b) $\hat{N}_1$
(c) $\hat{T}$
(d) $\hat{L}_2$
(e) $\hat{L}_1$



5. In the diagram, AB and DC are chords of a circle. E is a point on AB such that BCDE is a parallelogram. $D\hat{E}B = 108°$ and $D\hat{A}E = 2x + 40°$.

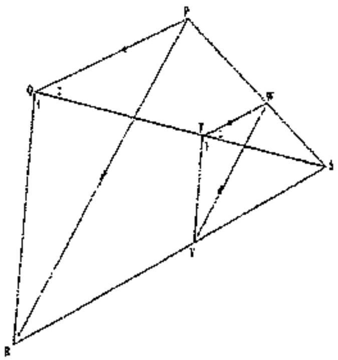
Calculate, giving reasons, the value of x.



6. In the diagram, PQRS is a quadrilateral with diagonals PR and QS drawn. W is a point on PS. WT is parallel to PQ with T on QS. WV is parallel to PR with V on RS. TV is drawn. PW:WS = 3:2.

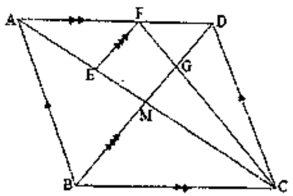
Write down the value of:

(a) $\frac{ST}{TQ}$
(b) $\frac{SV}{VR}$
(c) Complete the statement: $\Delta VWS /// \Delta$ ...
(d) Determine WV:PR



7. In the diagram, ABCD is a parallelogram. The diagonals of ABCD intersect in M. F is a point on AD such that AF: FD = 4: 3. E is a point on AM such that EF || BD. FC and MD intersect in G. Calculate, giving reasons, the ratio of:

(a) $\frac{EM}{AM}$
(b) $\frac{CM}{ME}$
(c) $\frac{Area\ \Delta FDC}{Area\ \Delta BDC}$



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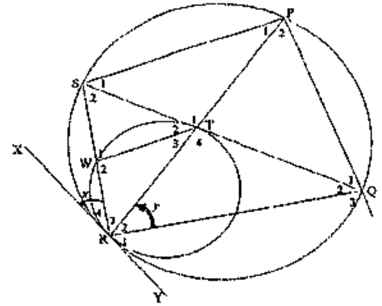
TOPIC: EUCLIDEAN GEOMETRY (Lesson 9)	Weighting	40 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Euclidean Geometry			
RELATED CONCEPTS/ TERMS/VOCABULARY				
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE				
RESOURCES				
DBE PAST PAPERS				
ERRORS/MISCONCEPTIONS/PROBLEM AREAS				
METHODOLOGY				
ALL THEOREMS				

ACTIVITIES/ ASSESSMENT

1.The two circles in the diagram have a common tangent XRY at R. W is any point on the small circle. The straight line RWS meets the large circle at S. The chord STQ is a tangent to the small circle, where T is the point of contact. Chord RTP is drawn. Let $\hat{R}_4 = x$ and $\hat{R}_2 = y$.

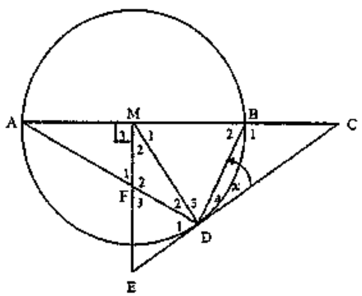
(a) Prove that $RT = \frac{WR.RP}{RS}$
(b) Identify, with reasons, another TWO angles equal to y
(c) Prove that $\hat{Q}_3 = \hat{W}_2$
(d) Prove that $\Delta RTS \equiv \Delta RQP$
(e) Hence, prove that $\frac{WR}{RQ} = \frac{RS^2}{RP^2}$



2. In the diagram, M is the centre of the circle and diameter AB is produced to C. ME is drawn perpendicular to AC such that CDE is a tangent to the circle at D. ME and chord AD intersect at F. MB = 2BC.

Prove that:

(a) CM is a tangent at M to the circle passing through M, E
(b) FMBD is a cyclic quadrilateral
(c) $DC^2 = 5BC^2$
(d) $\Delta DBC /// \Delta DFM$
(e) Hence, determine the value of $\frac{DM}{FM}$



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3. In the diagram, ΔABC is drawn. TA and TB are tangents to the circle. The straight line THK is parallel to AC with H on BA and K on BC. AK is drawn. Let $\hat{A}_3 = x$

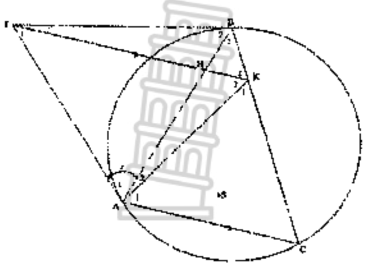
Prove that:

(a) $\hat{K}_3 = x$

(b) AKBT is a cyclic quadrilateral

(c) TK bisects $A\hat{K}B$

(d) TA is a tangent to the circle passing through the points A, K and H



4. ABC is a tangent to the circle BFE at B. From C, a straight line is drawn parallel to BF to meet FE produced to D. EC and BD are drawn. $\hat{E}_1 = \hat{B}_2 = x$ and $\hat{C}_2 = y$

(a) Give a reason, why EACH of the following is true:

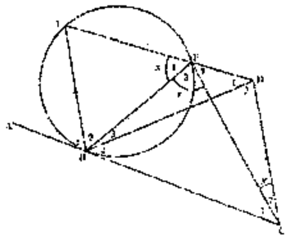
1) $\hat{B}_1 = x$

2) $B\hat{C}D = \hat{B}_1$

(b) Prove that BCDE is a cyclic quadrilateral

(c) Which TWO other angles are each equal to x?

(d) Prove that $\hat{B}_2 = \hat{C}_1$



5. In the diagram, HLKF is a cyclic quadrilateral. The chords HL and FK are produced to meet M. The line through F parallel to LK meets MH produced to G. MK = x, KF = $2x$, ML = y and LH = HG.

(a) Give a reason why $G\hat{F}M = L\hat{K}M$

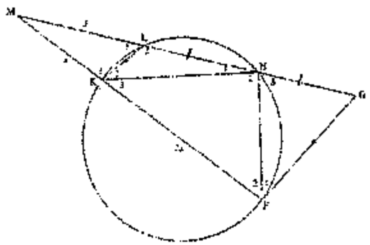
(b) Prove that:

1) GH = y

2) $\Delta MFH /// \Delta MGH$

3) $\frac{GF}{FH} = \frac{3x}{2y}$

(c) Show that $\frac{y}{x} = \sqrt{\frac{3}{2}}$



6. In the diagram, W is a point on the circle with centre O. V is a point on OW. Chord MN is drawn such that MV = VN. The tangent at W meets OM produced at T and ON produced at S.

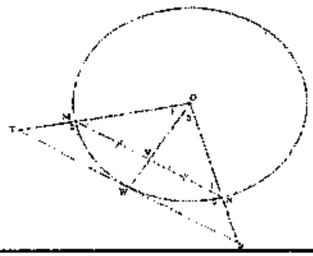
(a) Give a reason why OV ⊥ MN

(b) Prove that:

1) MN//TS

2) TMNS is cyclic quadrilateral

3. OS.MN = 2ON. WS



TEST 3: EUCLIDEAN GEOMETRY

FROM DBE PAST PAPERS

MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

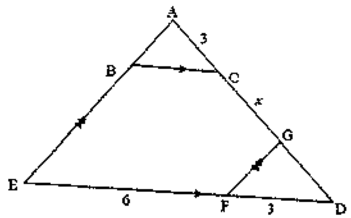
1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [17 Marks]

In the diagram, ADE is a triangle having BC//ED and AE//GF. It is also given that AB:BE = 1:3, AC = 3 units, EF = 6 units, FD = 3 units and CG = x units.

Calculate, giving reasons:

1.1	the length of CD	(3)
1.2	the value of x	(4)
1.3	the length of BC	(5)
1.4	The value of $\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta GFD}$	(5)

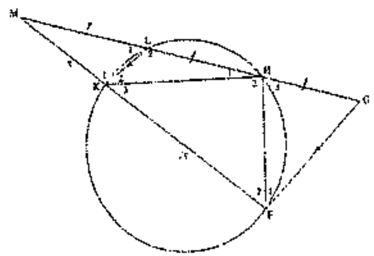


QUESTION 2 [08 Marks]

In the diagram, HLKF is a cyclic quadrilateral. The chords HL and FK are produced to meet M. The line through F parallel to LK meets MH produced to G. MK = x, KF = $2x$, ML = y and LH = HG.

2.1 Prove that GH = y (3)

2.2 Prove that $\Delta MFH /// \Delta MGH$ (5)



TOPIC: TRIGONOMETRY (Lesson 1)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Reduction Formulae				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Quadrants, trigonometric ratios, co-ratios

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

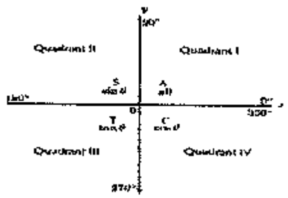
No understanding why trigonometric ratios are positive or negative in quadrants.

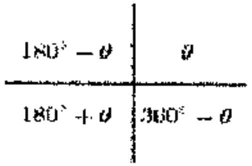
METHODOLOGY

Reduction formulae are used to **reduce** the trigonometric ratio of any angle to the trigonometric ratio of an **acute angle**.

An **acute angle** is the angle that is greater than 0° and less than 90° ($0° < \textit{acute angle} < 90°$) in measurement. Acute angle is usually represented by θ in a cartesian plane

- **Reduction formulae of the Function values $(180° \pm \theta)$ and $(360° - \theta)$**





Here is a summary of the reduction formulae:

Reduction formulae			
$180° - \theta$	$180° + \theta$	$360° - \theta$	$360° + \theta$
$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$	$\sin(360° + \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = \cos\theta$	$\cos(360° + \theta) = \cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = \tan\theta$	$\tan(360° - \theta) = -\tan\theta$	$\tan(360° + \theta) = \tan\theta$

- **Reduction of the Function values of $(-\theta)$**

An **anti-clockwise** rotation gives a **positive angle**. We can measure **negative angles** by rotating in a **clockwise** direction. For an acute angle θ, we know that $-\theta$ will lie in the fourth quadrant.

$\sin(-\theta) = -\sin\theta$ … 4th quadrant, sin negative
$\cos(-\theta) = \cos\theta$ … 4th quadrant, cos positive
$\tan(-\theta) = -\tan\theta$ … 4th quadrant, tan negative

- **Reduction of the Function values of $(360° + \theta)$**

We can also have an angle that is **greater than** 360°. The angle **completes a revolution** of 360° and then continues to give an angle of θ.

$$\sin(360° + \theta) = \sin\theta$$
$$\cos(360° + \theta) = \cos\theta$$
$$\tan(360° + \theta) = \tan\theta$$

NOTE:

When determining function values of $(180° \pm \theta)$, $(360° \pm \theta)$ and $(-\theta)$ the function (**name of the ratio**) **does not change.**

- **Reduction of the Function values $(90° \pm \theta)$**

Sine and **cosine** are known as **co-functions.** Two functions are called co-functions if f (A) = g (B) whenever A + B = 90° (that is, A and B are **complementary** angles).
Therefore, the function value of an angle is equal to the co-function of its complement.
Thus, for sine and cosine we have:

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$ … 2nd quadrant, sine positive
$\cos(90° + \theta) = -\sin\theta$ … 2nd quadrant, cosine negative

NOTE:

When determining function values of $(90° \pm \theta)$ the function (**name of the ratio**) **changes to its co-function.**

Examples:

1. $\frac{\sin(360°-\theta).\sin(90°-\theta).\cos(540°+\theta)}{\cos(90°+\theta).\cos(-\theta)}$

$= \frac{-\sin\theta.\cos\theta.\cos(360°+(180°+\theta))}{-\sin\theta.\cos\theta}$

$= \cos(180° + \theta)$

$= \cos\theta$

2. $\frac{\tan 135° - \cos 320°.\cos 140°}{\sin 220°.\cos(-50°)}$

$= \frac{\tan(180°-45°) - \cos(360°-40°).\cos(180°-40°)}{\sin(180°+40°).\cos 50°}$

$= \frac{-\tan 45° - \cos 40° \times -\cos 40°}{-\sin 40°.\cos 50°}$

$= \frac{-1+\cos^2 40°}{-\sin 40.\sin 40°}$ … $\cos 50° = \sin 40°$

$= \frac{-(1-\cos^2 40°)}{-\sin^2 40}$

$= 1$ … $(1 - \cos^2 40° = \sin^2 40°)$

ACTIVITIES/ ASSESSMENT

Simplify the following Expressions without the use of a calculator and to a single trigonometric function where necessary:

1. $\frac{\tan(180°-x).\sin(360°-x).\cos(90°-x)}{\sin(-x).\cos(90°+x).\tan(180°+x)}$

2. $\frac{2\sin(90°-\theta) + \cos(180°-\theta)}{\sin(90°-\theta) - \cos(x-180°)}$

3. $\frac{\tan(540°+\alpha).\sin(90°-\alpha)}{\sin(180°-\alpha)} - \frac{\cos(-\alpha)}{\cos(360°-\alpha)}$

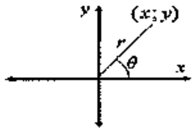
4. $\frac{2\tan 315° \sin 225° \sin 120°}{\cos 315° \cos 210°}$

5. $\frac{\cos 225°.\sin(-135°) - \sin 330°}{\tan 225°}$

6. $\frac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 220°}$

INVESTIGATION: COMPOUND ANGLES

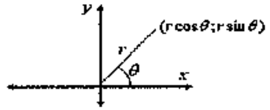
Polar coordinates

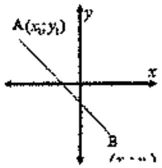


The trigonometric ratios are defined as: $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$

meaning that, with some simple algebra, that: $y = r\sin\theta$ and $x = r\cos\theta$

This results with polar coordinates, where a point is given in terms of the angle and the radius.



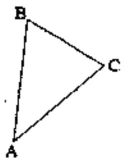


Distance formula:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

or (often more useful)

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$



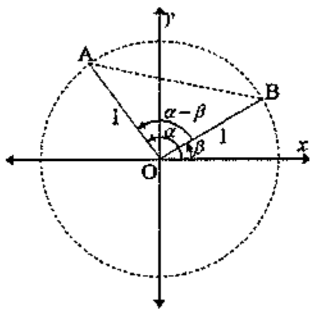
Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

ACTIVITY:

Refer to the figure alongside. Two angles, α ($A\hat{O}x$) and β ($B\hat{O}x$), are drawn with a radius of 1 unit on the Cartesian plane.

(a) Give the polar coordinates of points A and B.
(b) Use distance formula to calculate the length of AB^2 in terms of α and β.
(c) Use cosine rule in ΔAOB to find the length of AB^2 in terms of $\alpha - \beta$.
(d) Equate your answers in (b) and (c) to show that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.



TOPIC: TRIGONOMETRY (Lesson 2)		**Weighting**	50 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Compound angles**				
RELATED CONCEPTS/ TERMS/VOCABULARY					
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Failing to recognize compound angles
- Failing to simplify into a compound angle
- Failing to write an angle as an acute angle as a compound special angles

METHODOLOGY

Compound angles involve the trigonometric ratio of the sum of two angles or the difference between two angles.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Examples:

1. Expand the following:

(a) $\sin(A - 20°)$
$= \sin A\cos 20° - \cos A\sin 20°$

(b) $\cos(2\alpha + 45°)$
$= \cos 2\alpha\cos 45° - \sin 2\alpha\sin 45°$
$= \frac{\sqrt{2}}{2}\cos 2\alpha - \frac{\sqrt{2}}{2}\sin 2\alpha$

2. Express the following as a single trigonometric ratio:

(a) $\cos 70°\cos x + \sin 70°\sin x$
$= \cos(70° - x)$

(b) $\sin 3x\cos x - \cos 3x\sin x$
$= \sin(3x - x)$
$= \sin 2x$

(c) $\cos 65° \sin 85° + \sin 65° \cos 85°$
$= \sin(65° + 85°)$
$= \sin 150°$
$= \sin(180° - 30°)$
$= \sin 30$
$= \frac{1}{2}$

(d) $\cos 320° \cos 20° + \sin 140° \sin 200°$
$= \cos(360° - 40°) \cos 20° + \sin(180° - 40°) \sin(180° + 20°)$
$= \cos 40° \cos 20° - \sin 40° \sin 20°$
$= \cos(40° + 20°)$
$= \cos 60°$
$= \frac{1}{2}$

ACTIVITIES/ ASSESSMENT

1. Expand each of the following using the compound angle formulae:

(a) $\cos(x - 40°)$
(b) $\sin(10° + B)$
(c) $\cos(60° - A)$
(d) $\sin(x + 45°)$
(e) $\cos(x + x)$
(f) $\sin(\theta + \theta)$

2. Use the compound angle formulae to simplify each expression to one term only:

(a) $cos20°cos40° - sin20°sin40°$
(b) $\sin 10° \cos 80° + \cos 10° \sin 80°$
(c) $\cos(45° - x) \cos x - \sin(45° - x) \sin x$
(d) $cos^2 15 - sin^2 15$

3. Calculate each of the following without the use of a calculator:

(a) $\sin 50 \cos 10 + \cos 50 \sin 10$
(b) $\cos 70° \cos 40° + \sin 70° \sin 40°$
(c) $-\sin 40° \cos 10° + \cos 40° \sin 10°$
(d) $(\cos 20° \sin 65° - \sin 20° \cos 65°)^2$
(e) $-\cos 80° \cos 35° - \sin 80° \sin 35°$
(f) $\sin 55° \sin 5° - \cos 55° \cos 5°$

4. Show that $\cos(60° + \theta) - \cos(60° - \theta) = -\sqrt{3} \sin\theta$

5. Calculate $\sin 75°$ without using a calculator.

TOPIC: TRIGONOMETRY (Lesson 3)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Double Angles				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Compound angles, square identity $sin^2\theta + cos^2\theta = 1$

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Failing to recognize a double angle if not written as a double angle
- Failing to simplify to a double angle/work backwards

METHODOLOGY

Double angels are expanded using compound angle identity.

$\sin 2\theta = \sin(\theta + \theta)$
$\sin 2\theta = \sin\theta \cos\theta + \cos\theta \sin\theta$
$\therefore \mathbf{\sin 2\theta = 2 \sin\theta \cos\theta}$ … add like terms

$\cos 2\theta = \cos(\theta + \theta)$
$\cos 2\theta = \cos\theta \cos\theta - \sin\theta \sin\theta$
$\therefore \mathbf{\cos 2\theta = cos^2\theta - sin^2\theta}$

From the identity $sin^2\theta + cos^2\theta = 1$, $\begin{cases} sin^2\theta = 1 - cos^2\theta \\ cos^2\theta = 1 - sin^2\theta \end{cases}$

$\mathbf{\cos 2\theta = cos^2\theta - sin^2\theta}$
$\cos 2\theta = (1 - sin^2\theta) - sin^2\theta$
$\mathbf{\cos 2\theta = 1 - 2sin^2\theta}$

AND

$\cos 2\theta = cos^2\theta - (1 - cos^2\theta)$
$\mathbf{\cos 2\theta = 2cos^2\theta - 1}$

Therefore,

$\mathbf{\sin 2\theta} = 2 \sin\theta \cos\theta$

$$\mathbf{\cos 2\theta} = \begin{cases} cos^2\theta - sin^2\theta \\ 1 - 2sin^2\theta \\ 2cos^2\theta - 1 \end{cases}$$

Examples:

1. Expand the following:

(a) $\sin 50°$
$= \sin(25° + 25°)$
$= \sin 2(25°)$
$= 2 \sin 25° \cos 25°$

(b) $\cos 4\theta$
$= \cos(2\theta + 2\theta)$
$= \cos 2(2\theta)$
$= cos^2 2\theta - sin^2 2\theta$

2. Simplify the following without the use of a calculator:

(a) $cos^2 15° - sin^2 15°$
$= \cos(15° + 15°)$
$= \cos 30°$
$= \frac{\sqrt{3}}{2}$

(b) $2 \sin 22,5° \cos 22,5°$
$= \sin(22,5° + 22,5°)$
$= \sin 45°$
$= \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

(c) $(\sin 75° - \cos 75°)^2$
$= sin^2 75° - 2.\sin 75° \cos 75° + cos^2 75°$
$= 1 - \sin 2(75°)$
$= 1 - \sin 150°$
$= 1 - \sin 30°$ … $\sin 150° = \sin(180° - 30°)$
$= 1 - \frac{1}{2} = \frac{1}{2}$

3. Simplify $\frac{\sin 2x - \sin x}{2\cos x - 1}$
$= \frac{2\sin x \cos x - \sin x}{2\cos x - 1}$ … expand $\sin 2x$
$= \frac{\sin x(2\cos x - 1)}{2\cos x - 1}$ … take out a common factor
$= \sin x$

ACTIVITIES/ ASSESSMENT

1. Expand the following using double angle formulae:

(a) $\cos 4x$ (b) $\sin 70°$
(c) $4 \sin 22°$ (d) $2\cos 44°$ (e) $-\cos 86°$

2. Determine the value of the following without using a calculator:

(a) $cos^2 22,5° + sin^2 22,5°$ (b) $2 \sin 75° \sin 75°$ (c) $(sin\ 15° + cos\ 15°)^2$

3. Simplify the following expressions:

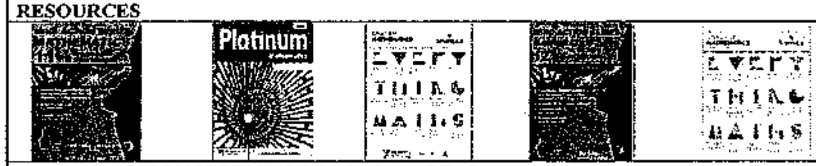
(a) $\frac{\cos 2x}{\cos x + \sin x}$ (b) $\frac{\sin 2\theta}{1 + \cos\theta}$ (c) $\frac{1 - \cos 2A}{\sin 2A}$

TOPIC: TRIGONOMETRY (Lesson 4)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Trigonometric Ratios and Pythagoras				
RELATED CONCEPTS/ TERMS/VOCABULARY	Quadrants, Cartesian plane				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Right-angled triangle, hypotenuse, opposite side, adjacent side,

RESOURCES

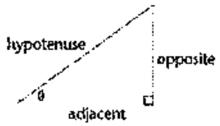


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to choose the correct quadrant, don't understand the meaning of the given condition.

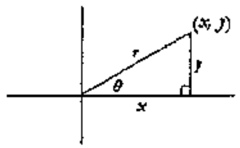
METHODOLOGY

In a right-angled triangle:



$\sin\theta = \frac{opposite\ side}{hypotenuse}$ $\cos\theta = \frac{adjacent\ side}{hypotenuse}$ $\tan\theta = \frac{opposite\ side}{adjacent\ side}$

For angles are on a cartesian plane:



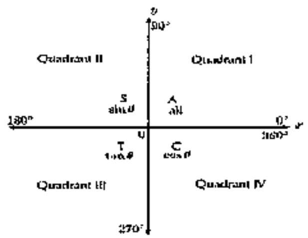
$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

Quadrant 1: All ratios are positive

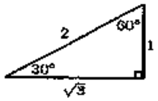
Quadrant 2: $\sin\theta$ positive
($\tan\theta$ *and* $\cos\theta$ *are negative*)

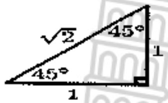
Quadrant 3: $\tan\theta$ positive
($\cos\theta$ *and* $\sin\theta$ *are negative*)

Quadrant 4: $\cos\theta$ positive
($\sin\theta$ *and* $\tan\theta$ *are negative*)



Special Angles:





$\text{Sin } 30° = \frac{1}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\sin 45° = \frac{1}{\sqrt{2}}$

$\sin 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$ $\quad$ $\tan 45° = \frac{1}{1} = 1$

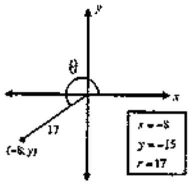
Examples:

1. If $17\cos\theta = -8$ and $\theta \in (180°;360°)$. Without the use of a calculator and **with the aid of a diagram,** calculate the value of

(a) $8\tan\theta - 17\sin\theta$ $\quad$ (b) $\cos 2\theta$ $\quad$ (c) $\sin(180° - 2\theta)$ $\quad$ (d) $\tan 2\theta$

$\cos\theta = \frac{-8}{17} = \frac{x}{r}$ … x is negative in quadrant 2 and 3

$r^2 = x^2 + y^2$ … Pythagoras
$(17)^2 = (-8)^2 + y^2$
$225 = y^2$
$y = \pm 15$
$y = -15$ … 3rd quadrant



(a) $8\tan\theta - 17\sin\theta$

$= 8 \times \frac{-15}{-8} - 17 \times \frac{-15}{17} = 30$

(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{-8}{17}\right)^2 - \left(\frac{-15}{17}\right)^2 = \frac{-161}{289}$

(c) $\sin(180° - 2\theta = \sin 2\theta$

$\sin\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{-15}{17} \times \frac{-8}{17} = \frac{240}{289}$

(d) $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$

$= \frac{240}{289} \div \frac{-161}{289}$

$= \frac{240}{289} \times \frac{289}{-161} = -\frac{240}{161}$

2. If $\cos 25° = m$, determine the following in terms of m without the use of a calculator

(a) $\sin(-25°)$ $\quad$ (b) $\cos 50°$ $\quad$ (c) $\cos 155°$

$\cos 25° = \frac{m}{1} = \frac{x}{r}$ $\quad$ Therefore, we need to determine y by Pythagoras Theorem

$x^2 + y^2 = r^2$ … Pythagoras Theorem
$m^2 + y^2 = 1^2$
$\therefore y = \sqrt{1 - m^2}$

(a) $\sin(-25°) = -\sin 25°$
$= -\frac{\sqrt{1-m^2}}{1}$

(b) $\cos 50° = \cos 2(25°) = \cos^2 25° - \sin^2 25°$
$= \left(\frac{m}{1}\right)^2 - \left(\frac{\sqrt{1-m^2}}{1}\right)^2 = 2m^2 - 1$

(d) $\cos 155° = \cos(180° - 25°) = -\cos 25° = -m$

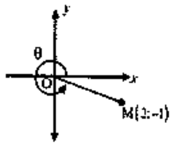
2 ACTIVITIES/ ASSESSMENT

1. Simplify the following without the use of a calculator:

(a) $2\sin 45° \times 2\cos 45°$
(b) $\cos^2 30° - \sin^2 60°$
(c) $\sin 60° \times \sqrt{2\tan 45° + 1} - \sin 30°$

2. Point M (2; -1) is a point on the cartesian plane. Such that $X\hat{O}M = \theta$.

Calculate the following without the use of a calculator.



(a) $\cos\theta$
(b) $1 - \sin^2\theta$

3. If $5\tan\theta = 12$ and $\theta \in (90°; 270°)$, determine the value of $\frac{10}{\cos\theta} - \frac{5}{\cos\theta}$

4. Given: $5\sin\theta + 4 = 0$ and $\cos\theta > 0$. Determine:

(a) $\frac{\sin\theta}{\cos\theta}$
(b) $10\sin\theta - 25\cos^2\theta$

5. If $\sin\theta = -\frac{3}{5}$ for $\theta \in (0°; 270°)$, and $\cos\beta = \frac{5}{13}$ for $\sin\beta < 0$, determine the value of $\cos\theta + \sin\beta$

6. If $\cos 38° = p$, with the aid of the diagram write the following in terms of p.

(a) $\tan(-38)$ $\quad$ (b) $\cos 52°$ $\quad$ (c) $\sin 38°$

(d) $\cos 76°$ $\quad$ (e) $\sin 104°$ $\quad$ (f) $\cos 68°$

(g) $\cos 142°$ $\quad$ (h) $\tan 236°$ $\quad$ (i) $2\sin 19°\cos 19°$

(j) $2\cos^2 26° - 1$ $\quad$ (k) $1 - 2\sin^2 19°$ $\quad$ (l) $(\sin^2 26° - \cos^2 26°)^2$

7. If $\tan 22 = k$, write the following in terms of k by first drawing a diagram.

(a) $\cos 22°$ $\quad$ (b) $\tan 68°$ $\quad$ (c) $\sin 22°$

(d) $\tan 202°$ $\quad$ (e) $\sin 44°$ $\quad$ (f) $\cos 82°$

TOPIC: TRIGONOMETRY (Lesson 5)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Trigonometric Identities including Identities Involving Compound and Double Angles				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Compound angles, Double angles

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Incorrect use of identities

METHODOLOGY

An **identity** is a mathematical statement that equates one quantity with another.

Trigonometric identity – an equality which is true for all values of an unknown variable, for which both sides of the identity are defined (so no zero denominators).

Trigonometric identities allow us to simplify a given expression so that it contains sine and cosine ratios only. This enables us to solve equations and also to **prove other identities.**

Prove the following Identities:

- Apply algebraic **factorization**, where necessary
- Identify **LCD**, where necessary
- Use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$, where necessary
- Use **compound** and **double** angles, where necessary

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ AND $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$ ... make $\sin^2\theta$ the subject of the formula

$\cos^2\theta = 1 - \sin^2\theta$ ... make $\cos^2\theta$ the subject of the formula

COMPOUND ANGLES:

$$\begin{cases}\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{cases}$$

$$\begin{cases}\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{cases}$$

DOUBLE ANGLES: $\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad \cos 2\alpha = \begin{cases}\cos^2\alpha - \sin^2\alpha \\ 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha\end{cases}$

Whenever expanding $\cos 2\theta$, one must decide whether to use $\cos^2\theta - \sin^2\theta$ OR $2\cos^2\theta - 1$ OR $1 - 2\sin^2\theta$

Examples:

Prove the following identities:

1. $\frac{1}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta} = \tan\theta$

L.H.S: $\frac{1}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta}$ ... choose one side and simplify it to get the other side

$= \frac{1(1+\sin\theta) - \cos\theta(\cos\theta)}{\cos\theta(1+\sin\theta)}$ ... LCD is $[\cos\theta\,(1+\sin\theta)]$ product of denominators

$= \frac{1+\sin\theta-\cos^2\theta}{\cos\theta(1+\sin\theta)}$

$= \frac{1+\sin\theta-(1-\sin^2\theta)}{\cos\theta(1+\sin\theta)}$

$= \frac{\sin\theta+\sin^2\theta}{\cos\theta(1+\sin\theta)}$

$= \frac{\sin\theta(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$ ... take out common factor on the numerator

$= \tan\theta = \text{RHS}$

2. $\frac{1-\cos 2x}{\sin 2x} = \tan x$

LHS: $\frac{1-\cos 2x}{\sin 2x}$

$= \frac{1-(1-2\sin^2 x)}{2\sin x\cos x}$ ... apply double angle formula, numerator must have **sin** because of RHS

$= \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$

3. $\frac{1+\sin 2\theta}{\cos 2\theta} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

LHS: $\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$ ... apply identity and double angle formulae

$= \frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}{\cos^2\theta-\sin^2}$

$= \frac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$ ... factorise perfect square trinomial and difference of two squares

$= \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$

ACTIVITIES/ ASSESSMENT

Prove the following Identities

1. $\sin\theta.\cos\theta.\tan\theta = 1 - cos^2\theta$
2. $tan^2x + 1 = \frac{1}{cos^2x}$
3. $\frac{\cos x}{1-\sin x} = \frac{1+\sin x}{\cos x}$
4. $\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \tan x$
5. $\frac{1-\sin 2\theta}{\sin\theta-\cos\theta} = \sin\theta - \cos\theta$
6. $\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$
7. $\tan x + \frac{\cos x}{\sin x} = \frac{2}{\sin 2x}$
8. $\frac{1-\tan\theta}{1+\tan\theta} = \frac{\cos 2\theta}{1-\sin 2\theta}$

TEST 1: TRIGONOMETRY

FROM: MIND ACTION SERIES [NEW EDITION]

GRADE 12 LEARNER SUPPORT STEP AHEAD – 2021

DBE FEBRUARY/MARCH 2015

MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [22 Marks]

1.1 If $\cos 16° = p$, write the following in terms of p:

1.1.1 $\cos 32°$ (4)

1.1.2 $\sin(-74°)$ (2)

1.1.3 $\sin 8° \cos 8°$ (3)

1.2 Simplify without using a calculator

1.2.1 $\sin(180° - x).\sin(-x) - \cos(180° + x).\sin(90° + x)$ (5)

1.2.2 $\frac{\tan(-60°)\cos 156°\cos 294°}{\sin 492}$ (8)

QUESTION 2 [3 Marks]

Prove that $\frac{\sin 2x}{1-sin^2x} = 2\tan x$ (3)

TOPIC: TRIGONOMETRY (Lesson 6)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Trigonometric Equations and General Solution				
RELATED CONCEPTS/ TERMS/VOCABULARY	General solution				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Trigonometric ratios, period of the graph, equation,

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Application of basic algebra is a problem

METHODOLOGY

Trigonometric equation is true only for certain values of the unknown variable.

The **periodicity** of the trigonometric functions means that there are an **infinite** number of positive and negative angles that satisfy an equation. If solution is not **restricted**, then the **general solution** needs to be determined to the equation. It is known that the **sine** and **cosine** functions have a **period of 360°** and the **tangent** function has a **period of 180°.**

The **period** is the number of degrees needed for a trigonometric function to complete one cycle.

Steps for determining general solution:

- Simplify the equation as far as possible
- Determine the reference angle
- Identify the possible quadrants (where the function is positive or negative, based on the sign of the given function)

- $If\ \sin\theta = x$, then
  $\theta = sin^{-1}x + k.360°\ for\ k \in Z$
  Or $\theta = 180° - sin^{-1} + k.360°\ for\ k \in Z$

- $If\ \cos\theta = x$, then
  $\theta = cos^{-1}x + k.360°\ for\ k \in Z$
  Or $\theta = 360° - cos^{-1}x + k.360°\ for\ k \in Z$

$If\ \tan\theta = x$, then
$\theta = tan^{-1} + k.180°$

Examples:

1. $3\cos\theta = 2\sin\theta$

Note that cos and sin have the same angle, therefore, divide by cos θ *on both sides to get* $\tan\theta$

$3 = 2\frac{sin\theta}{cos\theta}$

$\tan\theta = \frac{3}{2}$

$\theta = tan^{-1}\left(\frac{3}{2}\right)$ … shift tan $\left(\frac{3}{2}\right)$ on the calculator

$\theta = 56{,}31°$ … Reference angle

$\theta = 56{,}31° + k.180°, k \in Z$

2. $\cos 3x = \sin x$

Note that cos and sin have different angles, therefore, write sin in terms of cos

$\cos 3x = \cos(90° - x)$

$3x = (90° - x)$ … Reference angle

I: $3x = (90° - x) + k.360°, k \in Z$

$3x = 90° - x + k.360°$

$4x = 90° + k.360°$

$x = 22{,}5° + k.90°$

IV: $3x = 360° - (90° - x) + k.360, k \in Z$

$3x = 360° - 90° + x + k.360°$

$2x = 270 + k.360°$

$x = 135° + k.180°$

3. $2sin^2\theta + \sin\theta = 0$

Note that the quadratic is in a standard form

$sin\theta(2\sin\theta + 1) = 0$ … common factor $\sin\theta$.

$\sin\theta = 0$ *or* $2\sin\theta + 1 = 0$

$\theta = 0°$ … Ref. angle

I: $\theta = 0° + k.360°, k \in Z$

II: $\theta = 180° + k.360°, k \in Z$

$\sin\theta = -\frac{1}{2}$

$\theta = 30°$ … Ref. angle (shift sin $\left(\frac{1}{2}\right)$ on calculator)

III: $\theta = 180° + 30° + k.360°, k \in Z$

$= 210° + k.360°$

IV: $\theta = 360° - 30° + k.360°, k \in Z$

$= 330° + k.360°$

4. $2cos^2x + \cos x = 3$

Remember that quadratic equation must be in a **standard form.**

$2cos^2x + \cos x - 3 = 0$

$(2\cos x + 3)(\cos x - 1) = 0$ ... factorise

$2\cos x + 3 = 0$ or $\cos x - 1 = 0$

$\cos x = -\frac{3}{2}$ $\quad\quad$ $\cos x = 1$

No solution cos has a minimum value of -1 $\quad\quad$ $x = 0° + k.360°, k \in Z$

ACTIVITIES/ ASSESSMENT

1. $\sin\theta + \cos\theta = 0$

2. $\sin(3x + 30°) = -\cos 2x$

3. $2sin^2x + \sin x = 3$

4. $1 + \sin\theta = cos^2\theta$

5. $2cos^2x = \cos x$

6. Consider the identity $\frac{1+2sinx.cosx}{sinx+cosx} = 1$

6.1 For which values of x will the identity be undefined?

6.2 State the general solution of $\frac{1+2sinx.cosx}{sinx+cosx} = 1$

TOPIC: TRIGONOMETRY (Lesson 7)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Trigonometric Equation Involving Compound and Double angles				
RELATED CONCEPTS/ TERMS/VOCABULARY	General solution				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Trigonometric ratios, period of the graph, equation, Compound and Double angles

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Can not recognize factorization in trigonometric equations

METHODOLOGY

Trigonometric identities allow us to simplify a given expression so that it contains sine and cosine ratios only. This enables us to **solve equations** and also to prove other identities.

COMPOUND ANGLES:

$$\begin{cases}\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{cases}$$

$$\begin{cases}\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{cases}$$

DOUBLE ANGLES: $\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad\quad$ $\cos 2\alpha = \begin{cases}cos^2\alpha - sin^2\alpha \\ 2cos^2\alpha - 1 \\ 1 - 2sin^2\alpha\end{cases}$

Examples:

1. $\sin 2\alpha + 2\sin\alpha = 0$

$2\sin\alpha\cos\alpha + 2\sin\alpha = 0$

$2\sin\alpha(\cos\alpha + 1) = 0$ ... common factor

$2\sin\alpha = 0$ $\quad$ *or* $\quad$ $\cos\alpha + 1 = 0$

$\alpha = 0°$ ... reference angle $\quad\quad$ $\cos\alpha = -1$

$\alpha = 0° + k.360°$ *or* $\alpha = 180° + k.360°$ $\quad\quad$ $\alpha = 0°$ ... reference angle

$\alpha = 180° - 0° + k.360°$

or $\alpha = 180° + 0° + k.360°, k \in Z$

$\therefore$ *the solution is* $\alpha = 0° + k.360°$ *or* $\alpha = 180° + k.360°, k \in Z$

2. $2\sin 2\theta = 3\cos\theta$

$2(2\sin\theta\cos\theta) - 3\cos\theta = 0$

$4\sin\theta\cos\theta - 3\cos\theta = 0$

$\cos\theta(4\sin\theta - 3) = 0$

$\cos\theta = 0$ $\quad$ or $\quad$ $4\sin\theta - 3 = 0$

$\theta = 90°$ ... Ref. Angle

$\theta = 90° + k.360$

$or\ \theta = 270° + k.360°, k \in Z$

$\sin\theta = \frac{3}{4}$

$\theta = 48{,}59°$ ... Ref. Angle

$\theta = 48{,}59° + k.360°, k \in Z$

Or $\theta = 180° - 48{,}59° + k.360°, k \in Z$

$\theta = 131{,}41° + k.360°$

3. $1 + \sin 2x - 4\sin^2 x = 0$

$1 + 2\sin x\cos x - 4\sin^2 x = 0$

$\sin^2 x + \cos^2 x + 2\sin x\cos x - 4\sin^2 x = 0$

$\cos^2 x + 2\sin x\cos x - 3\sin^2 x = 0$

$(\cos x - \sin x)(\cos x + 3\sin x) = 0$

$\cos x - \sin x = 0$ or $\cos x + 3\sin x = 0$

$\cos x = \sin x$

$1 = \tan x$ ... divide by $\cos x$ on both sides

$x = 45°$

$x = 45° + k.360°, k \in Z$

Or $x = 360° - 45° + k.360°, k \in Z$

$\cos x = -3\sin x$

$1 = -3\tan x$

$\tan x = -\frac{1}{3}$

$x = 18{,}43°$

$x = 180° - 18{,}43° + k.360°, k \in Z$

Or $x = 360° - 18{,}43° + k.360°, k \in Z$

ACTIVITIES/ ASSESSMENT

Determine the general solution (correct to two decimal places):

1. $\sin\theta = \sin 2\theta$

2. $\sin 60°\cos x + \cos 60°\sin x = 1$

3. $3\sin 2x - 2\sin x = 0$

4. $\cos 2\theta = \sin\theta + 1$

5. $\sin^2\beta - 2\sin 2\beta + \cos^2\beta = 0$

6. (a) Show that $\frac{\sin 2x}{\cos 2x - 1} = -\frac{\cos x}{\sin x}$

(b) Hence, determine the general solution if $\frac{\sin 2x}{\cos 2x - 1} = 1$

7. Given: $\cos 2\theta\sin\theta - \cos\theta\sin 2\theta = -0{,}5$. Solve for θ $if\ \theta \in [-180°; 270°]$.

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TOPIC: TRIGONOMETRY (Lesson 8)		Weighting	50 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	The values for which Identities are undefined				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Asymptotes, undefined

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Can't explain when or why is the expression undefined. Meaning of asymptote

METHODOLOGY

An identity is an equation which is always true for any value substituted into the variable, except for those values the identity is not defined for.

- The tangent function is undefined at its asymptotes (at $90° + k.180°\ where\ k \in Z$)
- An identity is undefined for values for which the **denominator** is **zero**.
- Therefore, to determine values of the variable for which an identity is undefined, **solve for the denominator.**

Examples:

1. For which values of θ is the identity $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$ undefined?

$\sin 2\theta = 0$ ... undefined if the denominator is "0" for $\tan\theta$ if $\theta = 90° + k.180°\ where\ k \in Z$

$2\theta = 0° + k.360°$ or $2\theta = 180° + k.360°, k \in Z$

$\theta = 0° + k.180°$ $\theta = 90° + k.180°$

2. Determine the values of x for which the identity $\frac{\cos x - \cos 2x - 1}{\sin x - \sin 2x} = \frac{1}{\tan x}$ undefined?

$\sin x - \sin 2x = 0$ ... denominator equal to 0

$\sin x - 2\sin x\cos x = 0$ ... double angle formula

$\sin x(1 - 2\cos x) = 0$ ... common factor

$\sin x = 0$ or $1 - 2\cos x = 0$

$x = 0° + k.360°, k \in Z$

Or $x = 180° + k.360°, k \in Z$

$\cos x = \frac{1}{2}$

I: $x = 60° + 360°, k \in Z$

IV: $x = 360° - 60° + k.360°, k \in Z$

$= 300° + k.360°$

$\tan x = 0$

$x = 0° + k.180°, k \in Z$

$x = 90° + k.180°$ ... asymptote

ACTIVITIES/ ASSESSMENT

Determine the values of the variable for which each of the following identities are undefined.

1. $\frac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

2. $\tan\theta + \frac{\cos\theta}{\sin\theta} = \frac{2}{\sin 2\theta}$

3. $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

4. For which values of θ in the interval $[0°; 360°]$ is $\frac{2\sin^2\theta}{1+\cos 2\theta} = \tan^2\theta$ undefined?

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TOPIC: TRIGONOMETRY (Lesson 9)	Weighting	50 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Trigonometric Functions: Sketching			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Amplitude, period, domain, range, vertical and horizontal shift

RESOURCES



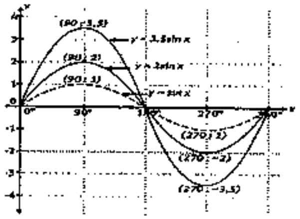
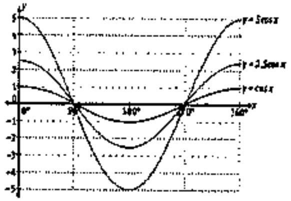
ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Find it difficult to shift the graph horizontally. Effect of q on minimum and maximum values

METHODOLOGY

In Grade 10 you did amplitude (a), period and vertical shifts (q) for the three trigonometric functions. Effects of a and q on the graph of $y = a\sin x + q, y = a\cos x + q$ *and* $y = a\tan x + q$ *for* $x \in [0°; 360°]$.

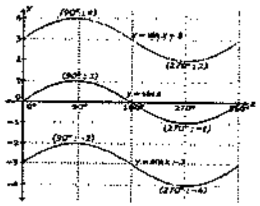
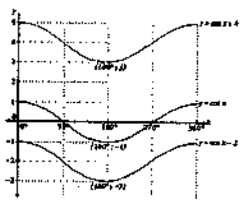
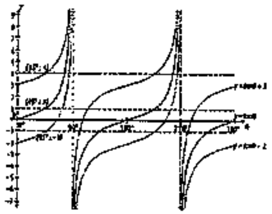
1. THE EFFECT OF "a"

a changes the minimum and the maximum value.

Write down the range of each function represented by the graph.

2. THE EFFECT OF "q"

$q > 0$ shifts the graph up and $q < 0$ shifts the graph down

Refer to the graphs above and write down the amplitude and the range of each function.

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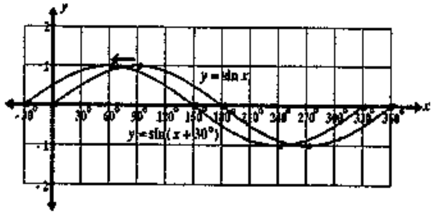
3. THE EFFECT OF "p" on the graph defined by $y = a\sin(x + p) + q$, $y = a\cos(x + p) + q$ *and* $y = a\tan(x + p) + q$ *for* $x \in [0°; 360°]$.

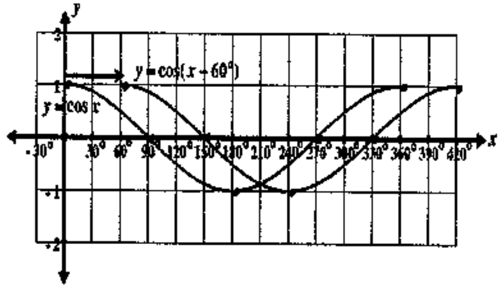
1. The graph of $y = \sin x$ and
   The graph of $y = \sin(x + 30°)$

   Write down the domain and the period Of each graph.



2. The graph of $y = \cos x$ and
   The graph of $y = \cos(x - 60°)$

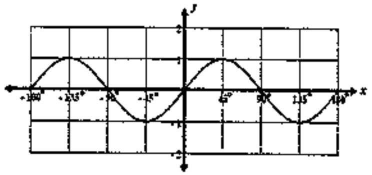
   Write down the domain and the period of each graph.



$p > 0$ shifts the graph to the left and $p < 0$ shifts the graph to the right.

4. THE EFFECT OF "k" on the graph defined by $y = \sin 2x$, $y = \cos 2x$ *and* $y = \tan 2x$

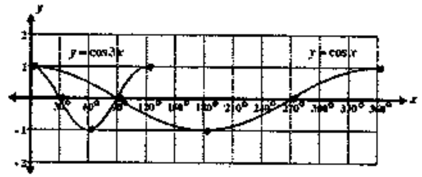
Sketch the graph of $y = \sin 2x$ *for* $x \in [-180°; 180°]$
Use a calculator: MODE→ *Table* → $\sin 2x$ →=→start $(-180°)$ →= *end* $(180°)$ →= *step* $(45° = \frac{90°}{2})$
Therefore, all your x- values (domain) must be 45° apart.



The period $= \frac{360°}{2} = 180°$

Sketch the graph of $y = \cos 3x$ *for* $x \in [0°; 360°]$
Use a calculator to find the corresponding y-values.
First x- value after 0° *is* $\frac{90°}{3} = 30°$. Therefore, all your x- values (domain) must be 30° apart.

The period $= \frac{360°}{3} = 120°$



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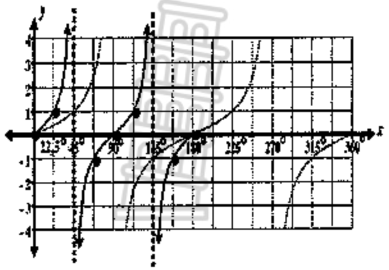
Sketch the graph of $y = \tan 2x$ for $x \in [0°; 180°]$

The period $= \frac{180°}{2} = 90°$



ACTIVITIES/ ASSESSMENT

In each case, sketch the graphs on the same set of axes:

1. $f(x) = -2sinx$ and $g(x) = \sin x - 2$ for $x \in [-90°; 180°]$

   (a) Write down the amplitude of g.
   (b) Write down the period of f
   (c) Determine the domain and the range of f and g
   (d) Determine for which values of x is
      1) $f(x) = g(x)$
      2) $f(x) > g(x)$
      3) $f(x).g(x) \leq 0$

2. $f(x) = 2tanx$ and $g(x) = 1 - tanx$ for $x \in [-90°; 180°]$

   (a) Write down the period of f and g
   (b) Determine for which values of x is
      1) $f(x) < g(x)$
      2) $f(x) = 0$
      3) $f(x) \geq 0$

3. $f(x) = -2\sin(x - 30°)$ and $g(x) = \frac{1}{2}\cos(x + 30°)$ for $x \in [-90°; 180°]$

   (a) Write down the range of g
   (b) Write down the period of f
   (c) Determine the x values for which
      1) $f(x) \leq 0$
      2) $g(x) > 0$
      3) $f(x).g(x) \geq 0$

TEST 2: TRIGONOMETRY

FROM PAST PAPERS

MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [21 Marks]

1.1 Determine the general solution of:

1.1.1 $2sin^2x - 5\sin x + 2 = 0$ (6) F/M 2017

1.1.2 $4\sin x + 2\cos 2x = 2$ (6) F/M 2016

1.2.1 For which values of x will $\frac{2\tan x - \sin 2x}{2sin^2x}$ be undefined in the interval $0° \leq x \leq 180°$ (3) F/M 2015

1.2.2 Prove the identity $\frac{2\tan x - \sin 2x}{2sin^2x} = \tan x$ (6) F/M 2015

QUESTION 2 [4 Marks]

Consider: $g(x) = -4\cos(x + 30°)$

2.1 Write down the maximum value of $g(x)$ (1)

2.2 Sketch the graph of $g(x)$, indicating the intercepts with the axes and the turning points (3)

TOPIC: TRIGONOMETRY (Lesson 10)	Weighting	50 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Trigonometric Functions: Graph Interpretation			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

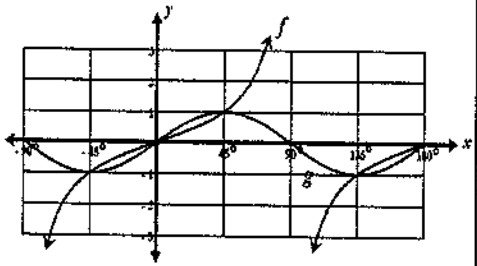
Forgetting the effect of the coefficient of the angle. Writing a negative amplitude

METHODOLOGY

Examples:

The diagram below represents the graphs of: $f(x) = \tan x$ and $g(x) = \sin 2x$ for $x \in [-90°; 180°]$

1. Write down the period of f
2. Write down the range of g
3. Determine the general solution of the equation $\tan x = \sin 2x$
4. Solve this equation for $x \in [-90°; 180°]$
5. Determine the value(s) of x for which:
   (a) $f(x) < 0$ ...
   (b) $g(x) \leq f(x)$
   (c) $f(x).g(x) > 0$



1. 180°

2. $-90° \leq y \leq 180°$

3. $\frac{sinx}{cosx} = 2sinxcosx$

$\sin x = 2\sin x\, cos^2x$
$sinx - 2sinxcos^2x = 0$
$sinx(1 - 2cos^2x) = 0$
$\sin x = 0$ or $\cos 2x = 0$
$x = 0° + k.360°, k \in Z$
Or $x = 180° + k.360°$ ... $(180° - 0°)$

$2x = 90° + k.360°, k \in Z$
$x = 45° + k.180°$
Or IV: $2x = 270° + k.360°$ ... $(360° - 90°)$
$x = 135° + k.180°$

4. $k \in Z$: for k = 1 360°, 540°, 225°, 315°
k = 0 0°, 180°, 45°, 135°
k = -1 −360°, -180°, -135°, −45°
$\therefore x = -45°, 0°, 45, 135°, 180°$

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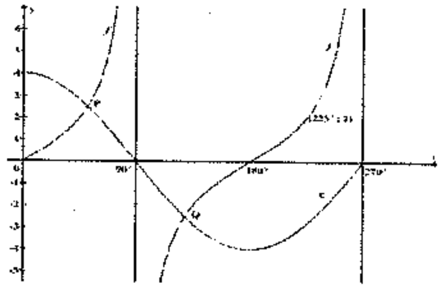
(a) $-90° < x < 0°$ or $90° < x < 180°$

(b) $-45° < x < 0°$ or $135° < x < 180°$

(c) $-90° < x < 0°$ or $0° < x < 90°$ or $90° < x < 180$

2. The graphs of functions $f(x) = atanx$ and $g(x) = bcosx$ for $0° \leq x \leq 270°$ are shown in the diagram below. The point (225°; 2) lies on f. The graphs intersect at points P and Q.
   (a) Determine the values of a and b.
   (b) Determine the range of $g(x) + 2$
   (c) Determine the period of $f(\frac{1}{2}x)$



(a) $a = 2$ and $b = 4$
(b) $y \in [-2; 6] / -2 \leq y \leq 6$
(c) 360°

ACTIVITIES/ ASSESSMENT

1. Given: $f(x) = \cos(x - 30°)$ and $g(x) = sin3x$

   (a) Sketch the graphs of $f(x) = \cos(x - 30°)$ and $g(x) = sin3x$ on the same set of axes for the interval $-60° \leq x \leq 120°$

   (b) Hence, determine graphically the values of x for which $\cos(x - 30°) = \sin 3x$

2. Given: $f(x) = -tanx$ and $g(x) = sin2x$

   (a) Sketch the graphs of $f(x) = -tanx$ and $g(x) = sin2x$ on the same set of axes for the interval $[-90°; 90°]$

   (b) Hence, determine the values of x for which $-\tan x = \sin 2x$

   (c) Write down the period of $g(\frac{x}{2})$

   (d) Write down the asymptote of $f(x - 30°)$

3. The diagram below represents the graphs of:
$f(x) = \tan x - 1$ and $g(x) = \sin 3x$ for $x \in [-90°; 180°]$

(a) Determine graphically the value(s) of x for which:

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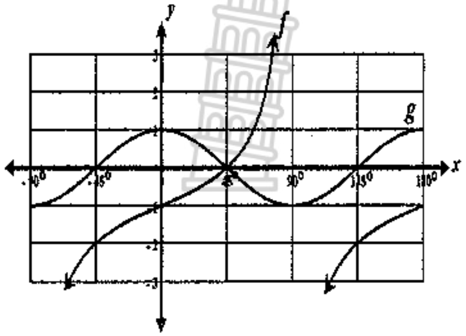
1) $\tan x = \cos 2x + 1$
2) $f(x) < 0$
3) $f(x) \geq g(x)$
4) $f(x) \geq -1$
5) $f(x).g(x) < 0$
6) $g(x) - f(x) = 2$

(b) For which values of x is:

1) Graph of f increasing?
2) Graph of g decreasing?

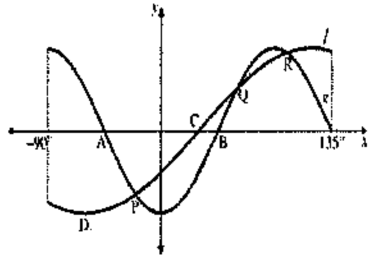


4. The sketch below shows the graphs of $f(x) = \sin(x - 30°)$ and $g(x) = -\cos 2x$.

(a) Write down the

1) Period of f
2) Amplitude of g

(b) Write down the coordinates of A, B and C
(c) Determine x coordinates of P, Q and R
(d) For which values of x, in the interval $[-90°; 135°]$, is;
1) $f(x) < g(x)$
2) $f(x).g(x) \geq 0$

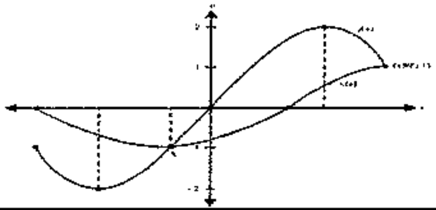


(e) Use the graphs and calculations above to determine the values of x, in the interval $[-90°; 135°]$, for which
1) $\sqrt{3}\sin x - \cos x = -2$ 2) $\cos x - \sqrt{3}\sin x = 2\cos 2x$

(f) Explain how the graph of f must be transformed to produce the graph of $h(x) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

(g) Explain how the graph of g must be transformed to produce the graph of $p(x) = 2sin^2x$

3. The diagram shows $f(x) = a.\sin x$ and $g(x) = \sin(x - b)$.
(a) Determine the values of a and b
(b) Write down the coordinates of A.
(c) Determine the coordinates of x for which $f(x) \geq g(x)$
(d) What is the period of $g(x)$?
(e) What is the range of $f(x)$?



TOPIC: TRIGONOMETRY (Lesson 11)	Weighting	50 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	2-Dimensional and 3-Dimensional Problems			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Sine, Cosine and Area Rule, quotient identity and square identity

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Confusing the sine rule and the area rule. Don't know when to use which rule.

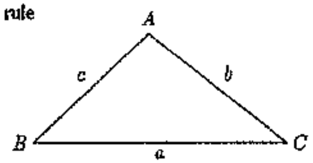
METHODOLOGY

$\sin\theta = \frac{opposite\ side}{hypotenuse}$ $\cos\theta = \frac{adjacent\ side}{hypotenuse}$ $\tan\theta = \frac{opposite\ side}{adjacent\ side}$

NOTE: The definitions of opposite, adjacent and hypotenuse are only applicable when working with **right-angled triangles**! Always check to make sure your triangle has a right-angle.

The sine, cosine and area rules can also be used to solve problems in three-dimensional space.

Area, sine and cosine rule



AREA RULE	SINE RULE	COSINE RULE
Area of $\Delta ABC = \frac{1}{2}ab\sin c$	$\frac{\sin a}{a} = \frac{\sin b}{b} = \frac{\sin c}{c}$	$c^2 = a^2 + b^2 - 2ab\cos C$
Area of $\Delta ABC = \frac{1}{2}bc\sin a$	$\frac{a}{\sin a} = \frac{b}{\sin b} = \frac{c}{\sin c}$	$b^2 = a^2 + c^2 - 2ac\cos B$
Area of $\Delta ABC = \frac{1}{2}ac\sin b$		$a^2 = b^2 + c^2 - 2bc\cos A$

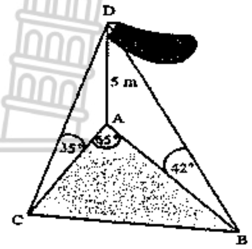
Use sine rule when:
- two sides and an angle are given (not the included angle)
- two angles and a side are given

Use cosine rule when:
- two sides and the included angle are given
- three sides are given

Use area rule when:
- no perpendicular height is given

Examples:

1. In the diagram, AD represents a flag pole of length 5 metres which is perpendicular to the horizontal plane. An observer C notes that the angle of elevation of D is 35°, while another observer at B, in the same horizontal plane as C, finds that the angle of elevation of D is 42°. $B\hat{A}C = 65°$
   (a) What are the angles $D\hat{A}C$ and $D\hat{A}B$ equal to?
   (b) How far is each observer from the foot of the flag pole? Round off to two decimal places.
   (c) Calculate the distance between the two observers at C and B (two decimal places).
   (d) Calculate the area of ΔABC to the nearest whole number.



(a) $D\hat{A}C = 90°$ and $D\hat{A}B = 90°$

(b) The length of AC and AB needs to be calculated.

In ΔDAC

$\frac{AC}{\sin C\hat{D}A} = \frac{AD}{\sin A\hat{C}D}$

$\frac{AC}{\sin 55°} = \frac{5}{\sin 35°}$

$AC = \frac{5\times\sin 55°}{\sin 35°} = 7{,}14m$

AND

ΔDAB

$\frac{AB}{\sin A\hat{D}C} = \frac{AD}{\sin A\hat{B}D}$

$\frac{AB}{\sin 48°} = \frac{5}{\sin 42°}$

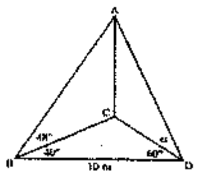
$AB = \frac{5\times\sin 48°}{\sin 42°} = 5.55m$

Then, fill in your answers on the sketch.

(c) $CB^2 = AC^2 + AB^2 - 2\times AB\times AC\cos C\hat{A}B$
$= (7{,}14)^2 + (5{,}55)^2 - 2\times 7{,}14\times 5{,}55\cos 65$
$= 48{,}28791229$
$BC = 6{,}95m$

(d) Area of $\Delta ABC = \frac{1}{2}AC\times AB\sin 65°$
$= \frac{1}{2}\times 7{,}14\times 5{,}55\times\sin 65°$
$= 18m^2$

2. In the sketch alongside, AC is a vertical line. ΔBCD lies in a horizontal plane. $BD = 10m$, $C\hat{B}D = 40°$ and $C\hat{D}B = 60°$. The angle of elevation of A from B is 48° and the angle of elevation of A from D is α
(a) Calculate the length of: 1) AB 2) AC 3) AD
(b) Calculate the size of α
(c) Write down the angle of depression of B from A.
(d) Calculate the size of $A\hat{B}D$
(e) Calculate the area of of ΔABD



(a) In ΔBCD: $B\hat{C}D = 80°$

$\frac{BC}{\sin 60°} = \frac{10}{\sin 80°}$

$BC = \frac{10\times\sin 60°}{\sin 80°} = 8{,}79m$

In ΔABC: $\cos 48° = \frac{8{,}79}{AB}$

$AB = \frac{8{,}79}{\cos 48°} = 13{,}14m$

2) $\tan 48° = \frac{AC}{8{,}79}$

AC= $8{,}79\times\tan 48° = 9{,}76m$

3) In ΔBCD: $\frac{CD}{\sin 40°} = \frac{10}{\sin 80°}$

$BC = \frac{10\times\sin 40°}{\sin 80°} = 6{,}52m$

In ΔACD: $AD^2 = AC^2 + CD^2$ ... Pythagoras
$= 9{,}76^2 + 6{,}52^2$
$= 137{,}768$
$AD = 11{,}74m$

(b) $\tan\alpha = \frac{9{,}76}{6{,}52}$
$\alpha = 56{,}26°$

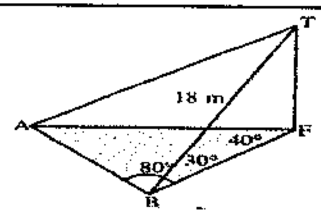
(c) Angle of depression from A from B
= Angle of elevation of A from B = 48°

(d) In ΔABD: $AD^2 = AB^2 + BD^2 - 2AB.BD\cos A\hat{B}D$
$11{,}74^2 = 13{,}14^2 + 10^2 - 2(13.14)(10)\cos A\hat{B}D$
$\cos A\hat{B}D = \frac{13.14^2+10^2-11{,}74^2}{2(13{,}14)(10)} = 0{,}5130593607$
$A\hat{B}D = 59{,}13°$

(e)Area ΔABD= $\frac{1}{2}.AB.BD\sin A\hat{B}D$
$= \frac{1}{2}.13{,}14.10.\sin 59{,}13°$
$= 56{,}39\ m^2$

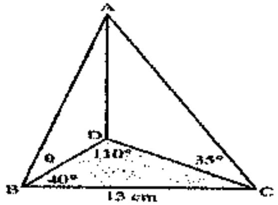
ACTIVITIES/ ASSESSMENT

1. A and B lie in the same horizontal plane as F, the foot of a building TF. TB = 18 m, $A\hat{F}B = 40°$ and $A\hat{B}F = 80°$
AB is perpendicular to BC.
(a) Calculate the height of TF (two decimal places).
(b) What is the distance between A and B?
(c) Calculate the angle of elevation of T from A.



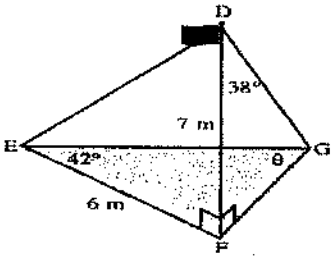
2. In the diagram below, AB is a straight line 1 500 m long. AD is a vertical tower with C, B and D points in the same horizontal plane. $BC = 13\ m$, $D\hat{B}C = 40°$ and $B\hat{D}C = 110°$. The angles of elevation of A from C is 35° and the angle of elevation of A from B is θ.

(a) Determine the length of
1) AC 2) AD 3) AB
(b) Find the value of θ (two decimal places)



3. In the sketch alongside, DF is a vertical flagpole and E, F and G are three points on ground level. DF = 7m, EF = 6m, $F\hat{D}G = 38°$ and $F\hat{E}G = 42°$. $E\hat{G}F = \theta$, where is an acute angle. Calculate:

(a) the magnitude
(b) the length of EG
(c) the magnitude of $E\hat{D}G$
(d) the area of ΔEDG
(e) the angle of elevation of D from E.



TOPIC: TRIGONOMETRY (Lesson 12)	Weighting	50 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	2-Dimensional and 3-Dimensional Problems			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Sine, Cosine and Area Rule, quotient identity and square identity

RESOURCES

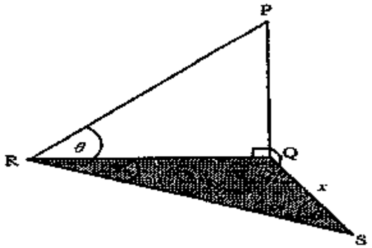
ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Confusing the sine rule and the area rule. Don't know when to use which rule.

METHODOLOGY

Examples:

1. In the diagram, PQ is perpendicular to the horizontal plane RQS. QS = x metres; $Q\hat{R}S = \alpha$, $Q\hat{S}R = \beta$ and the angle of elevation of P from R is θ.
   (a) Determine the length of RQ in terms of α, β and x
   (b) Hence, show that: $PQ = \frac{x\sin\beta.\tan\theta}{\sin\alpha}$



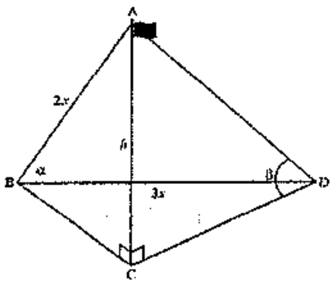
(a) In ΔRQS: $\frac{RQ}{\sin\beta} = \frac{QS}{\sin\alpha}$

$RQ = \frac{x \times \sin\beta}{\sin\alpha}$

(b) $\tan\theta = \frac{PQ}{RQ}$ then $PQ = RQ \times \tan\theta$

$= \frac{x\sin\beta}{\sin\alpha} \times \tan\theta$

2. The sketch shows a vertical flagpole AC with height. B, C and D are three points on ground level. AB = $2x$, BD = $3x$ and $A\hat{B}D = \alpha$. The angle of elevation of the top of the flagpole (A) from point D is β.
   (a) Show that $h = x\sin\beta\sqrt{13 - 12\cos\alpha}$
   (b) Calculate the angle of elevation of the flagpole (A) from point B if $\alpha = 48°$, $\beta = 56°$ and $x = 10\ m$.



(a) In ΔABD: $AD^2 = 2x^2 + 3x^2 - 2 \times 2x \times 3x\cos\alpha$

$= 4x^2 + 9x^2 - 12x^2\cos\alpha$

$AD^2 = 13x^2 - 12x^2\cos\alpha$

$AD = \sqrt{x^2(13 - 12\cos\alpha)}$

$\sin\beta = \frac{h}{AD}$ therefore, $h = \sin\beta \times AD$

$= \sin\beta \times x\sqrt{13 - 12\cos\alpha}$

(b) The angle of elevation of A from B is $A\hat{B}C$.

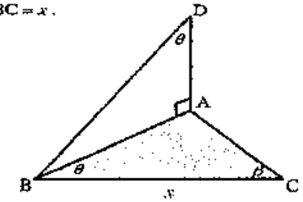
$\sin A\hat{B}C = \frac{h}{2x} = \frac{x\sin\beta\sqrt{13-12\cos\alpha}}{2x}$

$\sin A\hat{B}C = \frac{10\sin 56°\sqrt{13-12\cos 48°}}{2(10)} = 0{,}9241475528$

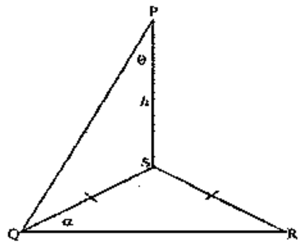
$\therefore A\hat{B}C = 67{,}54°$

ACTIVITIES/ ASSESSMENT

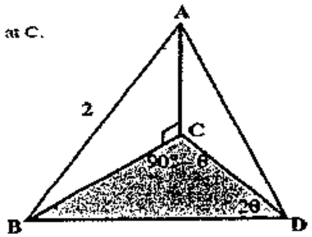
1. In the figure alongside A, B and C are three points in the same horizontal plane. AD represents a lamp pole that is perpendicular to the horizontal plane. $B\hat{D}A = A\hat{B}C = \theta$, $B\hat{C}A = \beta$ and BC = x.
   (a) Write $B\hat{A}C$ in terms of θ and β
   (b) Show that AB= $\frac{x\sin\beta}{\sin(\theta+\beta)}$
   (c) If AB = AC, show that
   1) $AB = \frac{x}{2\cos x}$ 2) $AD = \frac{x}{2\sin x}$



2. In the diagram shown, PS is a vertical line. ΔSQR lies in a horizontal plane. $Q\hat{P}S = \theta$, $S\hat{Q}R = \alpha$ and PS = h. SQ = SR
   (a) $QR = 2h\tan\theta\cos\alpha$
   (b) If $\theta = 45°$, $\alpha = 30°$ and $h = \sqrt{3}$ units, determine the length of QR without the use of a calculator.



3. AC represents a vertical tower which is perpendicular to the horizontal plane BCD at C. AB is 2 units. $B\hat{C}D = 90° - \theta$, $B\hat{D}C = 2\theta$ and $B\hat{A}C = \theta$.
   (a) Determine BC in terms of θ
   (b) Show that BD = 1 unit.
   (c) If AD = 3 units, calculate the size of $A\hat{B}D$ without using a calculator.



TEST 3: TRIGONOMETRY

FROM PAST PAPERS

MARKS: 25 DURATION: 30 MIN.

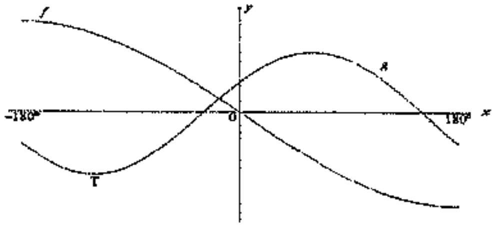
INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [7 Marks]

Drawn are the graphs of $f(x) = -3\sin\frac{1}{2}x$ and $g(x) = 2\cos(x - 60°)$ for $x \in [-180°; 180°]$.

T $(p; q)$ is a turning point of g with $p < 0$.



1.1 Write down the period of f and g (2)

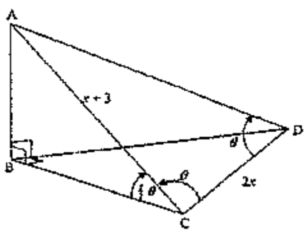
1.2 Write down the range of g. (2)

1.3 Use the graphs to determine the value(s) in the interval $x \in [-180°; 180°]$ for which:

$f(x) > 0$ (3)

QUESTION 2 [18 Marks] M/J 2015 and M/J 2017

2.1 A corner of a rectangular block of wood is cut off and shown in the diagram. The inclined plane, that is, ΔACD, is an isicseles triangle having $A\hat{D}C = A\hat{C}D = \theta$. Also $A\hat{C}B = \frac{1}{2}$, $AC = x + 3$ and $CD = 2x$.

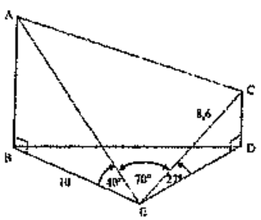


2.1.1 Determine an expression for $C\hat{A}D$ in terms of θ (1)

2.1.2 Prove that $\cos\theta = \frac{x}{x+3}$ (4)

2.1.3 If it is given that $x = 2$, calculate AB. (5)

2.2 In the diagram, B, E and D are points in the same horizontal plane. AB and CD are vertical poles. Steel cables AE and CE anchor the poles at E. Another steel cable connects A and C. CE = 8,6m, BE = 10m, $A\hat{E}B = 40°$, $A\hat{E}C = 70°$ and $C\hat{E}D = 27°$. Calculate the:



2.2.1 height of pole CD (2)

2.2.2 length of cable AE (2)

2.2.3 length of cable AC (4)

TOPIC: ANALYTICAL GEOMETRY (Lesson 1)	Weighting	40 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Revision of Grade 10 and 11 Work			
RELATED CONCEPTS/ TERMS/VOCABULARY				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Distance/length of a line, Midpoint a line segment, Gradient of a straight line, Inclination, Equation of a straight line.

RESOURCES

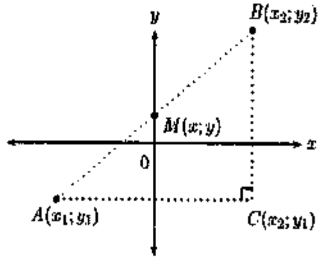
ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Writing gradient as change in x divided by change in y. Subtracting x value and y value in the midpoint

METHODOLOGY

1. DISTANCE/LENGTH OF A LINE

Formula: $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



2. MIDPOINT OF A LINE

Formula: $M\left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right)$

3. GRADIENT OF A LINE

Formula: $m = \frac{y_2-y_1}{x_2-x_1}$

(a) Parallel lines: Same/equal gradients

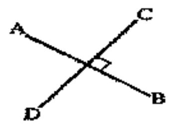
$m_{AB} = m_{CD}$

(c) Collinear points:

$m_{AB} = m_{BC} = m_{AC}$

(b) Perpendicular lines: Product of the gradients is – 1

$m_{AB} \times m_{CD} = -1$

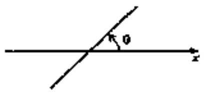


4. EQUATION OF A LINE

Formula: $y = mx + c$ (m is the gradient and c is the y-intercept)

$y - y_1 = m(x - x_1)$ [passing through point $(x_1; y_1)$]

5. **INCLINATION θ OF A LINE** is the angle between the line and the positive x-axis, measured in the anticlockwise direction. $\tan\theta = m$ and $\theta \in [0°; 180°]$

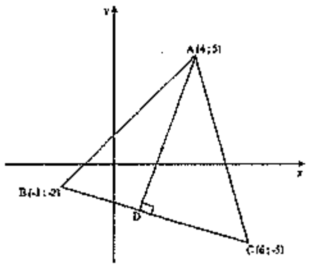


$m > 0 : \theta = \tan^{-1} m$

$m < 0 : \theta = \tan^{-1} m + 180°$

Example:

1. In the diagram alongside, A (4;5), B (-3; -2) and C (6; -5) are the vertices of Δ ABC. AD is drawn perpendicular to BC.



(a) Calculate the gradient of BC
(b) Determine the equation of BC
(c) Determine the equation of AD
(d) Determine the coordinates of D
(e) Calculate the size of $B\hat{A}D$
(f) Calculate the coordinated of point E if D is the midpoint of AE.

(a) $m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$ … gradient formula

$= \frac{-5-(-2)}{6-(-3)} = \frac{-3}{9} = -\frac{1}{3}$

(b) $y - y_1 = m(x - x_1)$ … equation formula

$y - (-2) = -\frac{1}{3}(x - (-3))$

$y + 2 = -\frac{1}{3}(x + 3)$

$y = -\frac{1}{3}x - 1 - 2$

$y = -\frac{1}{3}x - 3$

(c) $AD \perp BC$

$\therefore m_{AD} = 3$

$y - y_1 = m(x - x_1)$

$y - 5 = 3(x - 4)$

$y = 3x - 12 + 5$

$y = 3x - 7$

(d) D is on line AD and on line BC

$3x - 7 = -\frac{1}{3}x - 3$

$9x - 21 = -x - 9$

$10x = 12$

$x = \frac{6}{5}$

$y = 3\left(\frac{6}{5}\right) - 7 = -\frac{17}{5}$ … substitute x-value

$\therefore D\left(\frac{6}{5}; -\frac{17}{5}\right)$

(e) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{5+2}{4+3} = \frac{7}{7} = 1$

$\tan\theta = 1$

$\theta = 45°$

$m_{AD} = 3$

$\tan\theta = 3$

$\theta = 71{,}57°$

(f) $\frac{4+x}{2} = \frac{6}{5}$ AND $\frac{5+y}{2} = -\frac{17}{5}$

$20 + 5x = 12$ $\quad$ $25 + 5y = -34$

$5x = -8$ $\quad$ $5y = -59$

$x = -\frac{8}{5}$ $\quad$ $y = -\frac{59}{5}$

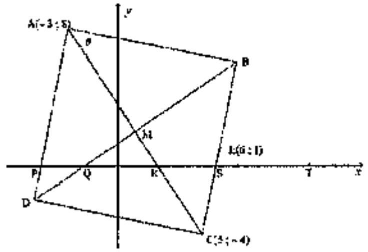
$B\hat{A}D = 71{,}57 - 45$
$= 26{,}57°$

ACTIVITIES/ ASSESSMENT

1. Consider the following points on a Cartesian plane: A (1; 2), B (3;1), C (-3; k) and D (2; - 3). Determine the value(s) of k if:
   (a) (-1; 3) is the midpoint of AC
   (b) AB is parallel to CD
   (c) AB is perpendicular to CD
   (d) A, B and C are collinear
   (e) $CD = 5\sqrt{2}$

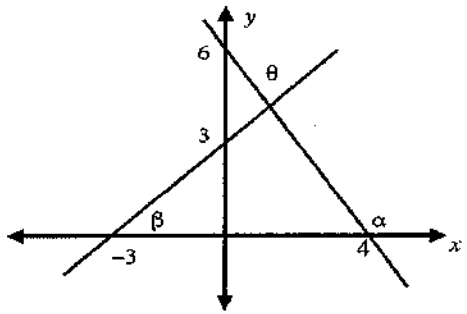
2. A $(1; 4)$, B $(-2; 2)$, C$(4;1)$ and M $(x; y)$ are the points on the Cartesian plane. Use analytical methods to determine the coordinates of M so that ABCM, in this order, is a parallelogram.

3. ABCD is a rhombus with A (– 3; 8) and C (5; – 4). The diagonals of ABCD bisect each other at M. The point E (6; 1) lies on BC.
   (a) Calculate the coordinates of M.
   (b) Calculate the gradient of BE.
   (c) Determine the equation of the line AD in the form $y = mx + c$
   (d) Determine the size of θ



4. Determine

   (a) The size of α
   (b) The size of β
   (c) The size of θ



TOPIC: ANALYTICAL GEOMETRY (Lesson 2)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Circles				
RELATED CONCEPTS/ TERMS/VOCABULARY	Centre at the origin				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Radius, centre of the circle, Distance formula

RESOURCES



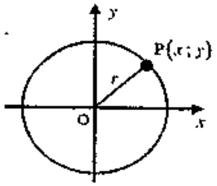
ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to understand that for the equation, you need to calculate the radius.

METHODOLOGY

1. Equation of the circle with Centre at the origin

Consider a point P (x; y) on the circumference of a circle of radius r with centre at O (0; 0).



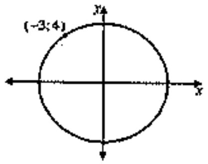
$OP^2 = (x-0)^2 + (y-0)^2$ ... Distance formula
$r^2 = x^2 + y^2$ ... $OP^2 = r^2$
∴ Equation of the circle with centre at (0; 0)

Examples:

1. Determine the equation of the circle with centre the origin and radius 5.

$x^2 + y^2 = r^2$ ... Equation of the circle with centre at (0; 0)
$x^2 + y^2 = 5^2$
$x^2 + y^2 = 25$

2. Determine the equation of the circle with centre (0; 0) and passing through the point (-3; 4).



$x^2 + y^2 = r^2$
$(-3)^2 + 4^2 = r^2$
$9 + 16 = r^2$
$25 = r^2$
$\therefore x^2 + r^2 = 25$

3. $(b; -\sqrt{8})$ is a point on the circle with equation $x^2 + y^2 = 17$. Determine the possible values of b.

$b^2 + (-\sqrt{8})^2 = 17$
$b^2 = 9$
$b = \pm 3$

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M.B. MPISI, Z.I. SHANGASE, P.T.C. ZUNGU & S.M. MQADI

ACTIVITIES/ ASSESSMENT

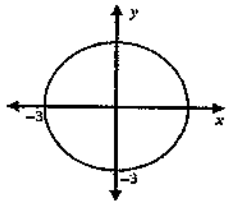
1. Determine the equation of the circle with centre at the origin, and:

(a) a radius of 10 units
(b) passing through the centre (5; -1)
(c) a diameter of 30 units
(d)

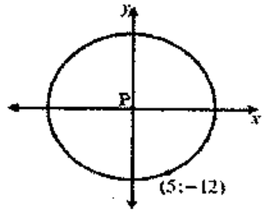
2. Write down the radius and the coordinates of the centre of the circle with equation $x^2 + y^2 = 4$

3. Determine the equation of the circle in:

(a)



(b)



4. Point $(a; \sqrt{5})$ lies on the circle $x^2 + y^2 = 21$. Determine the value(s) of a.

5. P (-2; 3) lies on a circle with centre at (0; 0).

(a) Determine the equation of the circle.
(b) Sketch the circle and label point P.
(c) If PQ is a diameter of the circle, determine the coordinates of Q.
(d) Calculate the length of PQ.
(e) Determine the equation of the line PQ.
(f) Determine the equation of the line perpendicular to PQ and passing through the point P.

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TOPIC: ANALYTICAL GEOMETRY (Lesson 3)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Circle with Centre at any point (*a; b*)				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Distance formula, radius, midpoint, diameter, completing the square

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Forgetting that the equation of the circle needs value of the centre and the radius

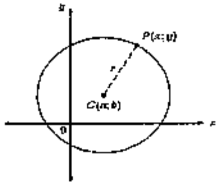
METHODOLOGY

2. **Equation of the circle with centre (*a; b*)**

Consider a point P (x; y) on the circumference of a circle of radius r with centre at C (*a*; *b*).

If P (x; y) is a point on a circle with centre C (a; b) And radius r, then the **equation of the circle** is:

$(x-a)^2+(y-b)^2=r^2$ … distance formula



Examples:

1. F (6; -4) is a point on the circle with centre (3; -4).

(a) Determine the equation of the circle.

$(x-a)^2+(y-b)^2=r^2$

$(x-3)^2+(y--4)^2=r^2$ … substitute centre (*a*; *b*) by (3; -4)

$(6-3)^2+(-4+4)^2=r^2$ … Substitute point F (6; -4) to *x* and *y*

$9+0=r^2$

Therefore, $(x-3)^2+(y+4)^2=9$

(b) does the point G $\left(\frac{3}{2};-2\right)$ lie on the circle? Motivate your answer.

If G$\left(\frac{3}{2};-2\right)$ lies on the circle, then it will satisfy the equation of the circle:

$(x-3)^2+(y+4)^2=9$

LHS = $\left(\frac{3}{2}-3\right)^2+(-2+4)^2$

$=\frac{9}{4}+4=\frac{25}{4}$

RHS = 9 $\quad\therefore\quad$ LHS ≠ RHS

Therefore, point G $\left(\frac{3}{2};-2\right)$ does not lie on the circle.

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2. Determine the co-ordinates of the centre and the length of the radius in:

(a) $(x-4)^2+(y+3)^2=18$

Centre (4; -3)

Radius = $\sqrt{18}=3\sqrt{2}$

(b) $x^2+y^2-6x+5y-4=0$ … complete the square

$x^2-6x+y^2+5y=4$

$x^2-6x+\left(\frac{-6}{2}\right)^2+y^2+5x+\left(\frac{5}{2}\right)^2=4+\left(\frac{-6}{2}\right)^2+\left(\frac{5}{2}\right)^2$

$(x)^2-6x+(-3)^2+(y)^2+5x+\left(\frac{5}{2}\right)^2=4+9+\frac{25}{4}$

$(x-3)^2+\left(y+\frac{5}{2}\right)^2=\frac{77}{4}$

Centre $\left(3;-\frac{5}{2}\right)$ and radius = $\sqrt{\frac{77}{4}}=\frac{\sqrt{77}}{2}$

3. Determine the equation of the circle with (-3; -7) and (3; 1) the endpoints of one of the diameters.

The midpoint of the diameter is the centre of the circle.

Centre $\left(\frac{-3+3}{2};\frac{-7+1}{2}\right)$ … midpoint formula

$(0;-3)$

$\therefore\ (x-0)^2+(y--3)^2=r^2$

$(-3-0)^2+(-7+3)^2=r^2$

$9+16=r^2$

$r^2=25$

Therefore, equation is $x^2+(y+3)^2=25$

ACTIVITIES/ ASSESSMENT

1. Determine the coordinates of the centre and the length of the radius in:

(a) $(x-4)^2+(y+3)^2=25$

(b) $x^2+(y-2)^2-10=0$

(c) $(x)^2+(y)^2-4x+2y-5=0$

(d) $(x+2)^2+(y)^2=49$

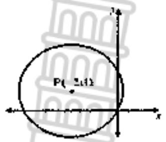
2. Determine the equation of the circle

(a) with centre (-2; -1) and radius of 4 units.

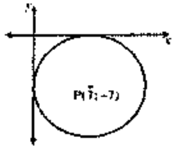
(b) with centre (-2; 3) and passing through the point (4; 5).

(c) with (2; -3) and (8; -1) the endpoints of one of its diameters.

(d)



(e)



3. The equation of a circle with radius $3\sqrt{2}$ units $x^2-6x+y^2+2y-m=0$.

(a) Determine the coordinates of the centre of the circle.

(b) Calculate the value of *m*.

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TOPIC: ANALYTICAL GEOMETRY (Lesson 4)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Equation of Tangent to a circle				
RELATED CONCEPTS/ TERMS/VOCABULARY	Tangent is a straight line				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Equation of a straight line, tangent ⊥ radius

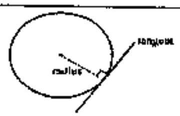
RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Forgetting that gradients of perpendicular lines have a product of -1.

METHODOLOGY

A **tangent to a circle** is a straight line that touches the circle at only one point.
A tangent is always **perpendicular to the radius** at the point of contact.
Therefore, $m_{tan} \times m_{rad} = -1$



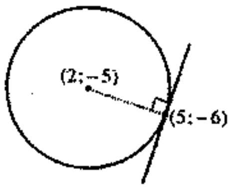
Examples:

1. Determine the equation of the tangent to the circle $(x-2)^2 + (y+5)^2 = 10$ at the point (5; -6).

Centre: (2; -5) passing through point (5; -6)

$m_{rad} = \frac{-6-(-5)}{5-2} = -\frac{1}{3}$

$m_{rad} = 3$ ... tangent ⊥ radius



Equation of tangent: $y = mx + c$

$-6 = 3(5) + c$ ... passing through (5; -6)

$c = 21$

Therefore, $y = 3x + 21$

2. The straight line y = x + 2 cuts the circle $x^2 + y^2 = 20$ at P and Q.

a) Calculate the coordinates of P and Q.

Solve equations simultaneously: $x^2 + (x+2)^2 = 20$ ... substitute $y = x + 2$

$x^2 + x^2 + 4x + 4 - 20 = 0$ ... simplify the bracket

$2x^2 + 4x - 16 = 0$ ... standard form

$x^2 + 2x - 8 = 0$ ... common factor

$(x+4)(x-2) = 0$

$x = -4 \ or \ x = 2$

Substitute x values to any of the two equations to get y values: $y = x + 2$ or $y = x + 2$

$-4 + 2 = -2$ $\quad$ $= 2 + 2 = 4$

Therefore, P (-4; -2) and Q (2;4)

b) Determine the length of PQ.

$PQ = \sqrt{(-4-2)^2 + (-2-4)^2}$

$= \sqrt{36+36}$

$= \sqrt{72} = 6\sqrt{2}$

c) Determine the coordinates of M, the mid-point of chord PQ.

$M_{PQ}\left(\frac{-4+2}{2}; \frac{-2+4}{2}\right)$, $\therefore M_{PQ}(-1; 1)$

d) If O is the centre of the circle, show that PQ ⊥ OM.

$O(0; 0)$ ... centre at the origin

$m_{OM} = \frac{1-0}{-1-0} = -1$ and $m_{PQ} = \frac{-2-4}{-4-2} = \frac{-6}{-6} = 1$

$m_{OM} \times m_{PQ} = -1 \times 1 = -1$

Therefore, PQ ⊥ OM

e) Determine the equations of the tangents to the circle at P and Q.

$m_{OP} = \frac{-2-0}{-4-0} = \frac{1}{2}$, then $m_{\tan at P} = -2$ ... tangent ⊥ radius

Equation of the tangent at P: $y - y_1 = m(x - x_1)$ ... equation of straight line

$y - (-2) = -2(x - (-4))$

$y + 2 = -2x - 8$

$y = -2x - 10$

$m_{OQ} = \frac{4-0}{2-0} = 2$, then $m_{\tan at Q} = -\frac{1}{2}$ ... tangent ⊥ radius

Equation of tangent at Q: $y = mx + c$ ... equation of a straight line

$4 = -\frac{1}{2}(2) + c$ ... substitute gradient of tangent at Q and Q (2;4)

$c = 4 + 1 = 5$

$y = -\frac{1}{2}x + 5$

f) Determine the coordinates of S, the point where the two tangents intersect.

$-2x - 10 = -\frac{1}{2}x + 5$ ... tangents are equal at the point of intersection

$-4x - 20 = -x + 10$ ... multiply by LCD all terms

$-3x = 30$

$x = -10$

Substitute $x = -10$ to ant of the two equations to get y: $y = -2x - 10$

$y = -2(-10) - 10 = 10$

Therefore, S (-10; 10)

g) Show that PS = QS.

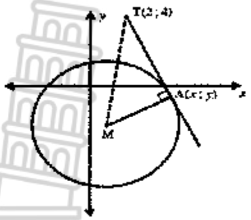
$PS = \sqrt{(-10-(-4))^2 + (10-(-2))^2}$ P (-4; -2) and $QS = \sqrt{(-10-2)^2 + (10-4)^2}$ Q(2;4)

$= \sqrt{36+144}$ $\quad$ $= \sqrt{144+36}$

$= 6\sqrt{5}$ $\quad$ $= 6\sqrt{5}$

Therefore, PS = QS

3. Consider a circle with equation $x^2-2x+y^2+6y=4$ A tangent is drawn from (2; 4) to point A $(x; y)$ on the circle. Determine the length of the tangent TA.



First, determine the centre of the circle and the length of the radius.

$x^2-2x+\left(\frac{-2}{2}\right)^2+y^2+6y+\left(\frac{6}{2}\right)^2=4+\left(\frac{-2}{2}\right)^2+\left(\frac{6}{2}\right)^2$

$(x-1)^2+(y+3)^2=14$

Therefore, centre is M (1; -3) and the radius AM is $\sqrt{14}$

$TM^2=(2-1)^2+(4-(-3))^2$

$TM^2=1+49=50$ and $AM^2=14$

$TA^2=TM^2-AM^2$

$=50-14=36$ units

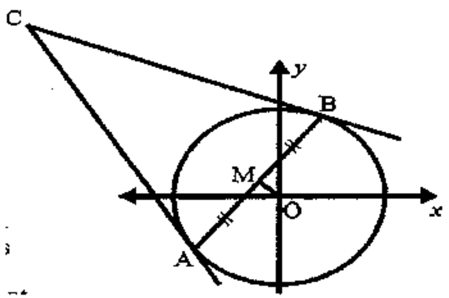
ACTIVITIES/ ASSESSMENT

1. Determine the equation of the tangent to the circle:
   (a) $(x+4)^2+(y+2)^2=50$ at the point (-9; 3).
   (b) $x^2-2x+y^2+4y-5=0$ at the point (-2; -1)

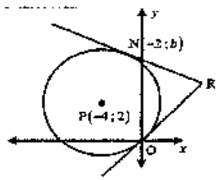
2. (a) A circle with centre (8; -7) and the point (5, -5) on the circle are given. Determine the gradient of the radius.
   (b) Determine the gradient of the tangent to the circle at the point (5; -5)

3. Given the equation of the circle $(x+4)^2+(y+8)^2=136$
   (a) Determine the gradient of the radius at the point (2; 2) on the circle
   (b) Determine the gradient of the tangent to the circle at the point (2; 2)

4. The straight line $y=x=2$ cuts the circle $x^2+y^2=20$ at A and B.
   (a) Determine the coordinates of A and B
   (b) Determine the length of the chord AB
   (c) Determine the coordinates of M, the midpoint of the chord AB
   (d) Show that OM ⊥ AB if O is the origin
   (e) Determine the equations of the tangents to the circle at A and B
   (f) Determine the coordinates of C, the point of intersection of the tangents



5. A circle with centre P (-4; 2) has the point O (0;0) and N (-2; b) on the circumference. The tangents at O and N meet at R.
   Determine;
   (a) The equation of the circle
   (b) The value of b
   (c) The equation of OR
   (d) The coordinates of R



TOPIC: ANALYTICAL GEOMETRY (Lesson 5)		Weighting	40 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	The Relationship between Two Circles				
RELATED CONCEPTS/ TERMS/VOCABULARY	Relationship between two circles				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Radius, distance between two points

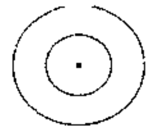
RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

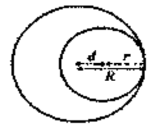
Failing to calculate the distance between the radiuses of two circles and describe the relationship.

METHODOLOGY

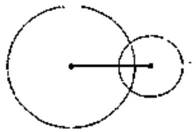
TWO CIRCLES



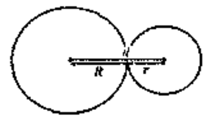
Two circles with the same centre



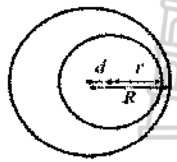
Circles touch internally. $d=R-r$



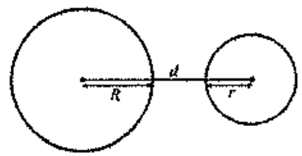
Circles intersect at two points: $R-r<d<R+r$



Circles touch externally. $d=R+r$



The circles do not intersect, the smaller circle Lies inside the larger circle. $d<R-r$



Circles do not intersect and the smaller circle Lies outside the larger circle. $d>R+r$

Examples:

1. Describe the relationship between the two circles in:

(a) $(x+1)^2+(y+3)^2=9$ and $(x-2)^2+(y-1)^2=4$

R = 3 r = 2
R + r = 5 R − r = 1

Centres: (-1; -3) and (2; 1): $d=\sqrt{(-1-2)^2+(-3-1)^2}=5$

Therefore, d = R + r, the circles touch externally

(b) $(x+1)^2+(y-3)^2=9$ and $(x+3)^2+(y-2)^2=36$

r = 3 R = 6
R + r = 6 R − r = 3

Centres: (-1; 3) and (-3; 2): $d=\sqrt{(-1+3)^2+(3-2)^2}=\sqrt{5}=2{,}24$

Therefore, d < R − r, the circles do not intersect and the smaller circle lies **inside** the larger circle.

(c) $x^2-10x+y^2-14y+49=0$ and $x^2+y^2=9$

$x^2-10x+\left(\frac{-10}{2}\right)^2+y^2-14y+\left(\frac{-14}{2}\right)^2=-49+\left(\frac{-10}{2}\right)^2+\left(\frac{-14}{2}\right)^2$

$(x-5)^2+(y-7)^2=25$

R = 5 r = 3
R + r = 8 R − r = 2

Centres: (5; 7) and (0; 0): $d=\sqrt{(5-0)^2+(7-0)^2}=\sqrt{74}=8{,}60$

Therefore, d > R + r, the circles do not intersect and the smaller circle lies **outside** the larger circle

2. Show that the circles $(x-1)^2+(y-2)^2=25$ and $(x+1)^2+(y-1)^2=16$ intersect in two points without calculating the points of intersection.

R = 5 r = 4
R + r = 9 R − r = 1

Centres: (1; 2) and (-1; 1): $d=\sqrt{(1+1)^2+(2-1)^2}=\sqrt{5}=2{,}24$

Therefore, R − r < d < R + r, the circles **intersect in two points**

3. For which values of k will the circle $(x-1)^2+(y+3)^2=k$ touch the circle $(x+2)^2+(y-1)^2=49$ internally if $k<59$?

R = 7 r = $\sqrt{k}$ Centres: (1; -3) and (-2; 1)

$d=\sqrt{(1+2)^2+(-3-1)^2}=5$

For the circles to touch internally d = R − r

Therefore, $5=7-\sqrt{k}$

$\sqrt{k}=2$

$k=4$

ACTIVITIES/ ASSESSMENT

1. Describe the relationship between the two circles in:

(a) $(x+3)^2+(y-1)^2=1$ and $(x-1)^2+(y-4)^2=16$
(b) $(x+2)^2+y^2=4$ and $(x-3)^2+(y+3)^2=9$
(c) $x^2+(y-2)^2=64$ and $(x-3)^2+(y+2)^2=9$
(d) $x^2+4x+y^2+6y=12$ and $x^2+2x+y^2+4y=-1$
(e) $x^2+y^2-6x+4y-12=0$ and $x^2+y^2+4x+2y-95=0$

2. Determine the value of a if the circle $x^2-4x+y^2-2y=a$ touches circle $(x-14)^2+(y-10)^2=100$ externally.

3. For which value of p will the circle $(x+5)^2+(y+2p)^2=225$ touch the circle $x^2+(y-p)^2=4$ internally.

TEST: ANALYTICAL GEOMETRY

FROM: PAST PAPERS

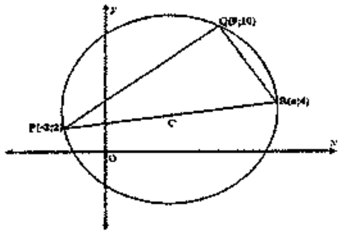
MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [15 Marks]

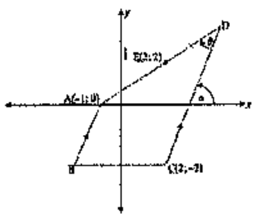
1.1 The points P (−3; 2), Q (9; 10) and R (a; 4) lie on the circumference of the circle, as shown in the figure below. PR is a diameter of the circle with centre C.



1.1.1 Show that $a = 13$ (3)

1.1.2 Determine the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$ (4)

1.2 In the diagram below, A (−1; 0), B, C (2; −2) and D are the vertices of a trapezium having AB ∥ DC. The length of DC is three times the length of AB (i.e., DC = 3AB). $A\hat{D}C = \theta$. E (2; 2) is the midpoint of AD. The angle of inclination of DC is α.



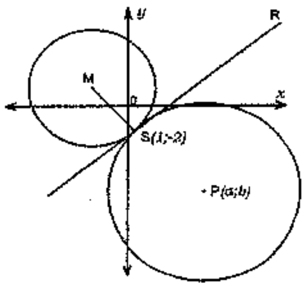
1.2.1 Determine the coordinates of D (2)

1.2.2 Determine the equation of AB in the form $y = mx + c$ (3)

1.2.3 Calculate the size of θ, correct to ONE decimal place. (3)

QUESTION 2 [10 Marks]

In the figure below, a circle with centre M is drawn. The equation of the circle $(x+2)^2 + (y-1)^2 = r^2$. S (1; −2) is a point on the circle. SR is a tangent to the circle.



2.1 Write down the coordinates of M and the radius of the circle centre M. (2)

2.2 Determine the equation of the tangent RS in the form $y = mx + c$. (3)

2.3 The circles having centres P and M touch externally at point S. SR is a tangent to both these circles. If MS: MP=1:3, determine the coordinates (a; b) of point P. (5)

TOPIC: FUNCTIONS, GRAPHS, INVERSES (Lesson 1)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Functions and Inverses				
RELATED CONCEPTS/ TERMS/VOCABULARY	Set-builder notation, Interval Notation, Functional Notation, Vertical Line Test				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Domain, Range, intercepts with the axes, gradient.

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Writing the inverse in the form y = …. Understanding of set-builder or interval notation

METHODOLOGY

What is a function?

A **function** is a relation in which no x- value has more than one y- value … i.e. no element of the domain is repeated.

This means that functions are relations with **one-to-one** or **many-to-one** correspondence.

Vertical Line Test is used on the graph to see whether it is a function or not.

- If any vertical line cuts the graph only once, then the relation is a function (one-to-one or many-to-one).
- If the vertical line cuts the graph more than once, then the relation is not a function.

The **domain** of a function is the set of **all independent** x**-values** from which the function produces a single y-value for each x-value.

The **range** is the set of **all dependent** y**-values** which can be obtained using an independent x-value

Set-Builder Notation

$x \in R, x > 0$: The set of all x-values such that x is an element of the set of real numbers and is greater than 0.

$3 < y \leq 5$: The set of all y-values such that y, is greater than 3 and is less than or equal to 5.

Interval Notation

It is important to note that this notation can only be used to represent an interval of real numbers.

(3; 11): Round brackets indicate that the number is not included. This interval includes all real numbers greater than but not equal to 3 and less than but not equal to 11.

$(-\infty; -2)$: Round brackets are always used for positive and negative infinity. This interval includes all real numbers less than, but not equal to -2.

[1; 9): A square bracket indicates that the number is included. This interval includes all real numbers greater than or equal to 1 and less than but not equal to 9.

Functional Notation

This is a very useful way to express a function. Another way of writing $y = 2x + 1$ is $f(x) = 2x + 1$. We say "f of x is equal to $2x + 1$". Any letter can be used, for example. $g(x)$, $h(x)$, $p(x)$, etc.

ACTIVITIES/ ASSESSMENT

1. Write the following in se-builder notation:

(a) $(-\infty; 7]$ (b) $[-13; 4)$

(c) $(35;\infty)$ (d) $[\frac{3}{4}; 21)$ (e) $[-\frac{1}{2}; \frac{1}{2}]$

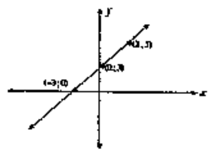
2. Write the following in interval notation:

(a) $p \leq 6$ (b) $-5 < k < 5$

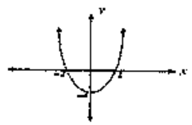
(c) $x > \frac{1}{5}$ (d) $21 \leq x \leq 41$

3. Determine whether each of the following relations is a function or not. Also write down the domain and range for each relation.

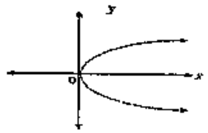
(a)



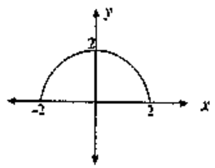
(b)



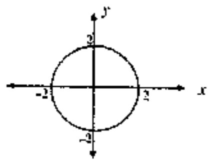
(c)



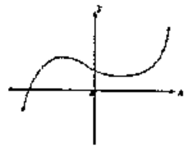
(d)



(e)



(f)



4. If $f(x) = -4x + 5$, determine:

(a) $f(2)$ (b) $f(-3)$ (c) $f\left(\frac{3}{4}\right)$

(d) $f(x)$ if $x = -1$ (e) x if $f(x) = 9$ (f) $f(x + h)$

TOPIC: FUNCTIONS, INVERSES, GRAPHS (Lesson 2)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Functions and Inverses**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Inverse of a function, one-to-one function, line of symmetry				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Functional Notation, Domain, Range, line of symmetry, intercepts with the axes, gradient.

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confusing f^{-1} with f'.
- Failing to sketch the graph if there are fractions involved.

METHODOLOGY

The Inverse of a Function (f^{-1})

An inverse function is a function which **does the "reverse"/ opposite** of a given function.

To determine the inverse of a function:

- **Swop x and y values in the given equation**
- Then, make **y the subject of the formula/solve for y**

$f(x) = \cdots$ indicates a function and $f^{-1}(x) = \cdots$ indicates the inverse

Important: for f^{-1} the superscript -1 is not an exponent. It is the notation for indicating the inverse of a function. Do not confuse this with exponents, such as $\left(\frac{1}{2}\right)^{-1}$ or $x + x^{-1}$

Inverse of a Linear Function: $y = ax + q$ … One-to-one function

Consider the function $y = 2x - 4$

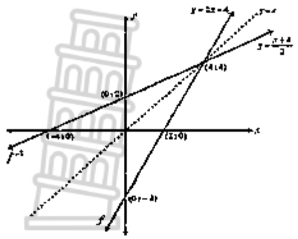
(a) Determine the inverse of $y = 2x - 4$

Interchange x and y in the equation: $x = 2y - 4$

Make y the subject of the formula: $x + 4 = 2y$

$\frac{x+4}{2} = y$

(b) Sketch both graphs on the same set of axes.
The two graphs are **reflection** of each other about the line $y = x$, meaning they are **symmetrical** in the line $y = x$



$y = 2x - 4$
x- intercept ($y = 0$) and y-intercept ($x = 0$)
$2x - 4 = 0$ $\quad$ $y = 2(0) - 4$
$x = 2$ ... (2; 0) $\quad$ $y = -4$ ... (0; -4)

$y = \frac{x+4}{2}$
x-intercept (y = 0) and y-intercept (x = 0)
$x = -4$ ... (-4; 0) $\quad$ $y = 2$ ... (0; 2)

ACTIVITIES/ ASSESSMENT

1. Consider the function: $f(x) = 5x + 4$

   (a) Determine the equation of f^{-1}.
   (b) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry
   (c) Determine the coordinates of the point of intersection of both graphs.

2. Given the function $f(x) = \frac{1}{2}x + 3$

   (a) Determine the equation of f^{-1}.
   (b) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry
   (c) Determine the coordinates of the point of intersection of both graphs.

3. (a) Sketch the graph of the function f (x) = 3x - 1 and its inverse on the same system of axes. Indicate the intercepts and the axis of symmetry of the two graphs.

   (b) T $(\frac{4}{3}; 3)$ is a point on f and R is a point on f^{-1}. Determine the coordinates of R if R and T are symmetrical.

4. Given: $f^{-1}(x) = -2x + 4$

   (a) Determine $f(x)$.
   (b) Determine the intercepts of $f(x)$ and $f^{-1}(x)$
   (c) Determine the coordinates of T, the point of intersection of $f(x)$ and $f^{-1}(x)$
   (d) Sketch the graphs of f and f^{-1} on the same system of axes. Indicate the intercepts and point T on the graph.
   (e) Is f^{-1} an increasing or a decreasing function?

TOPIC: FUNCTIONS, INVERSES, GRAPHS (Lesson 3)		**Weighting**	35 ± 3	**Grade**	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Inverse of a Quadratic Function: $y = ax^2$**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Restrict Domain,				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Inverse of a function, Functional notation, maxima, minima, intercepts with the axes, average gradient

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confusing $y = x^2$, $x \geq 0$ with $y = 2^x$
- Failing to recognize that $y = \sqrt{x}$ is a quadratic function

METHODOLOGY

A quadratic function is a **many-to-one** function and its inverse is not a function.

Examples:

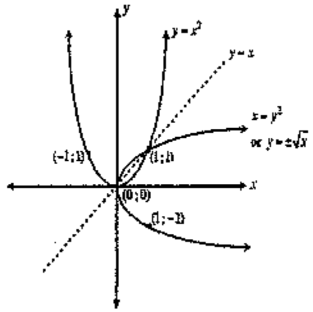
Given the function: $y = x^2$

(a) Determine the equation of the inverse of this function.

$x = y^2$ ... swop x and y
$y = \pm\sqrt{x}$, $x \geq 0$

(b) Sketch the graphs of $y = x^2$ and its inverse on the same set of axes. Draw the line of symmetry

Function	Inverse
(-1; 0)	(0; -1)
(0; 0)	(0; 0)
(1; 1)	(1; 1)



(c) Write down the domain and the range of $y = x^2$.

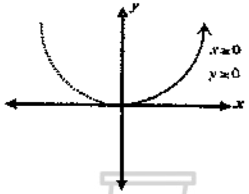
Domain: $x \epsilon (-\infty; \infty)$ / $x \epsilon R$ $\qquad$ Range: $y \geq 0$

(d) Is the inverse of a function a function? Motivate

The inverse of $y = x^2$ is not a function because a vertical line will cut the graph in two points as it moves from left to right.

The vertical line test shows that the inverse of a parabola is not a function. However, we can **limit/ restrict the domain** of the function $y = ax^2$ (parabola) so that the inverse of the parabola is a function.

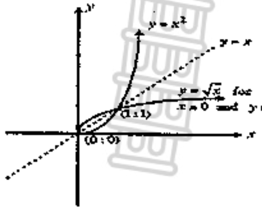
1. Restrict the domain of $y = x^2$ where $x \geq 0$.



Whenever the domain of the original function is restricted, it is important to ensure that the **range** of the restricted function **remains the same** as the original function.

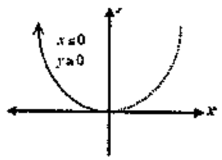
By keeping the range, the same for the restricted function, **the inverse of this restricted function will also be a function.**

Notice that both graphs are **one-to-one** functions.
The equation of the inverse function is then defined as $y = \sqrt{x}$ where $x \geq 0$ and $y \geq 0$



2. Restrict domain of $y = x^2$ where $x \leq 0$.

The inverse of $y = x^2$ where $x \leq 0$ will also be a function and is defined as $y = -\sqrt{x}$ where $x \geq 0$ and $y \leq 0$



Examples:

1. Given the function: $h(x) = 3x^2$ where $x \geq 0$.

(a) Write down the range of h.

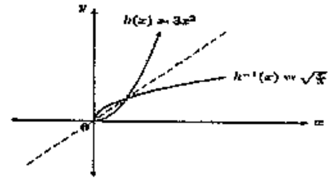
$y \geq 0$ ... x^2 can never be negative

(b) Determine the equation of h^{-1} and write in the form $y = ...$

$y = 3x^2, x \geq 0$ and $y \geq 0$ Inverse $x = 3y^2$ ... swop x and y

$y = \sqrt{\frac{x}{3}}$ ... $y \geq 0$

(c) Sketch the graphs of (a) and (b) on the same set of axes and show the line of symmetry.



(d) Another h(x) is the reflection of given $h(x) = 3x^2$ in the y – axis

1) Determine the equation of the new h(x).

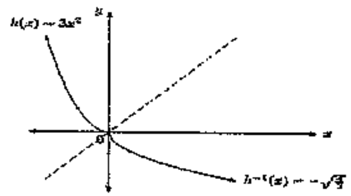
$y = 3(-x)^2$ ... reflection on y-axis, x changes the sign $[-x \geq 0$ and $y \geq 0]$
$h(x) = 3x^2$ where $-x \leq 0$

2) Determine the equation of the inverse of the function in 1)

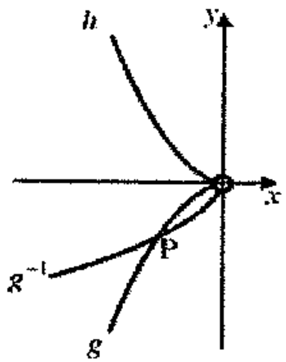
$h(x) = 3x^2$ where $-x \leq 0$ and $y \geq 0$ Inverse $x = 3y^2$ ... swop x and y

$y = -\sqrt{\frac{x}{3}}$ ... $y \leq 0$

3) Sketch the graph in 1) and 2) on the same set of axes and show the line of symmetry.



2. The sketch alongside shows the graphs of g, g^{-1} and h.
$g(x) = -\frac{1}{2}x^2; x < 0$. h is the reflection of g in the x-axis.



(a) Write down the range of g

$y < 0$ ... remember $x = 0$ is excluded, $y = 0$ is also excluded

(b) Determine the equation of h

$-y = -\frac{1}{2}x^2$ ... y changes the sign $x < 0$ $-y < 0$
$y = \frac{1}{2}x^2$ ... $y > 0$

(c) Determine the equation of g^{-1}

$x = -\frac{1}{2}y^2$ $y < 0$ $x < 0$
$y^2 = -2x$ $\therefore y = \pm\sqrt{-2x}$
Since $y < 0, y = -\sqrt{-2x}$

(d) Calculate the coordinates of P

Solve $g(x)$ and $g^{-1}(x)$ simultaneously

$-\frac{1}{2}x^2 = -\sqrt{-2x}$

$x^2 = 2\sqrt{-2x}$

$(x^2)^2 = \left(2\sqrt{-2x}\right)^2$ ... squaring on both sides

$x^4 = -8x$

$x^4 + 8x = 0$

$x(x^3 + 8) = 0$ ... common factor

$x = 0$ or $x^3 = -8$

$x = -2$

$y = -\frac{1}{2}(-2)^2 = -2$ ... substitute $x = -2$ to any of the two equations

P (-2; -2)

(e) For which values of x is $g(x) \geq g^{-1}(x)$

$-2 \leq x < 0$

ACTIVITIES/ ASSESSMENT

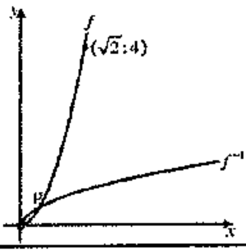
1. Consider the function $f(x) = 2x^2$

(a) Write down the inverse of $f(x)$ in the form $y = \ldots$
(b) Sketch the graph of the function and its inverse on the same set of axes.
(c) Is the inverse a function? Justify your answer.
(d) Restrict the domain of the given function in two different ways to form one-to-one functions
(e) Sketch the graphs of each new function and inverse on the same set of axes.
(f) Rewrite the equation of each inverse function in the form $f^{-1}(x)$
(g) Write down the domain and the range of each graph drawn

2. Given $f(x) = -\frac{1}{2}x^2, x \geq 0$

(a) Determine $f^{-1}(x)$
(b) Determine the point of intersection of $f(x)$ and $f^{-1}(x)$
(c) Sketch $f(x)$ and $f^{-1}(x)$ on the same set of axes
(d) Use the sketch to determine if $f(x)$ and $f^{-1}(x)$ are increasing or decreasing functions
(e) Determine the gradient of $f^{-1}(x)$ between the two points of intersection.

3. The sketch shows the graph of $f(x) = ax^2$, with a restriction on its domain, and the graph of f^{-1}. f passes through the point $(\sqrt{2}; 4)$. P is the point of intersection of f and f^{-1}.



(a) Write down the domain of f
(b) Determine the value of a
(c) Calculate the coordinates of P
(d) For which values of x is $f(x) > f^{-1}(x)$?

TOPIC: FUNCTIONS, INVERSES, GRAPHS (Lesson 4)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Inverse of the Exponential Function: $y = b^x$ for $b > 0, b \neq 1$**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Definition of Logarithm, vertical asymptote				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Asymptotes, average gradient, increasing/decreasing function

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

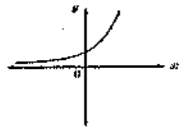
- Find it difficult to write log function as an exponential function
- Failing to differentiate between an increasing and a decreasing exponential/log function

METHODOLOGY

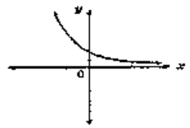
The graph of the exponential function: $f(x) = b^x$

NOTE: The value of b affects the direction of the graph:

- If $b > 1$, $f(x)$ is an increasing function



- If $0 < b < 1$, $f(x)$ is a decreasing function.



Consider the function $y = b^x$

The inverse is $x = b^y$ ... interchange the values of x and y. The **logarithmic** function allows us to rewrite the inverse with y as the subject of the formula:

x is the number
b is the base e.g., $8 = 2^3$
y is the exponent

$exponent = \log_{base} number$

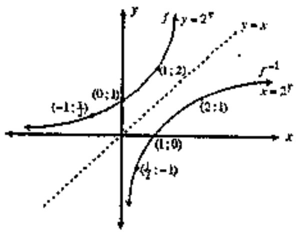
$\therefore y = \log_b x$

Examples:

1. Consider the function: $f(x) = 2^x$
(a) Determine the equation of the inverse in the form $y = \ldots$
(b) Sketch the function and its inverse on the same set of axes, clearly show the line of symmetry.
(c) Write down the domain and the range of $f(x)$ and of its inverse
(d) State whether the function of $f(x)$ and its inverse are increasing or decreasing.

(a) $x = 2^y$ ... interchange x and y $[f(x) = y]$
$y = \log_2 x$

(b)



(c) Domain of f: $x \in (-\infty;\infty)$
Domain f^{-1}: $x \in (0;\infty)$

Range of f: $y \in (0;\infty)$
Range of f^{-1}: $y \in (-\infty;\infty)$

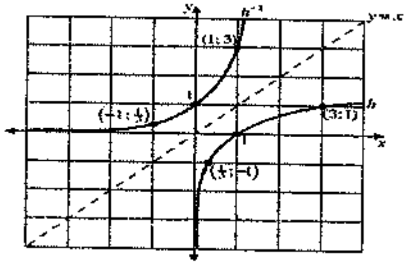
(d) Both functions are increasing

2. Given: $h(x) = \log_3 x$

(a) Write down the equation of the inverse
(b) Sketch the function and its inverse on the same set of axes. Clearly show the line of symmetry

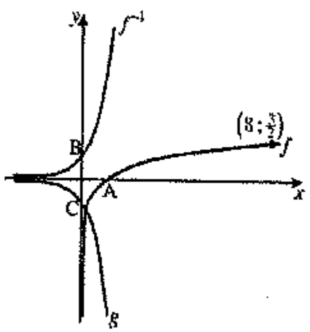
(a) $x = \log_3 y$ ... interchange x and y
$y = 3^x$ ... (x is the exponent, 3 is the base and y is the number) e.g., $8 = 2^3$

(b)



3. The sketch shows the graph of $f(x) = \log_b x$, f^{-1} and g, the reflection of f^{-1} in the x-axis. $\left(8;\frac{3}{2}\right)$ is a point on f.

(a) Determine the value of b.
(b) Determine the equation of f^{-1}
(c) Determine the equation of g
(d) Write down the domain of f
(e) Write down the range of
1) f^{-1} 2) g
(f) Write down the coordinates of A, B and C
(g) For which values of x is
1) $f(x) < 0$ 2) $g(x) \geq -1$ 3) $f^{-1}(x) < 8$
(h) If the graph of $f(x+t)$ passes through the point $\left(11;\frac{3}{2}\right)$. Write down the value of t.



(a) $\frac{3}{2} = \log_b 8$ ... substitute $\left(8;\frac{3}{2}\right)$ into $y = \log_b x$
$8 = b^{\frac{3}{2}} \rightarrow b = (2^3)^{\frac{2}{3}}$ $\therefore b = 4$

(b) Inverse of $y = \log_4 x$ is $x = \log_4 y$
$y = 4^x$ $\therefore f^{-1}(x) = 4^x$

(c) For g, $-y = 4^x$ ... reflection on x-axis, y changes the sign
$y = -4^x$

(d) $x > 0$

(e) 1) $y > 0$ 2) $y < 0$

(e) A (1;0), B (0;1) and C (0;-1)

(g) 1) $0 < x < 1$ 2) $x \leq 0$ 3) $x < \frac{3}{2}$

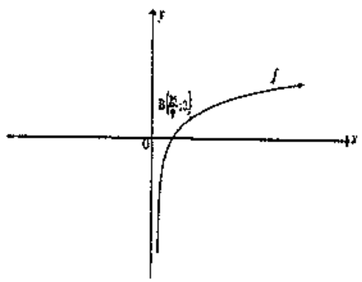
(f) The graph of f must be shifted 3 units to the right. Therefore, $t = 3$

ACTIVITIES/ ASSESSMENT

1. Given: $f(x) = 3^x, g(x) = \left(\frac{1}{5}\right)^x$ and $h(x) = \log_2 x$
(a) Determine the equation of the inverse of each function in the form y = ...
(b) Sketch the function and its inverse on the same set of axes. Show the line of symmetry.
(c) Determine:
1)The domain of h 2) the range of g^{-1} 3) Whether f is increasing or decreasing

2. The graph of $f(x) = a^x$ passes through the point $\left(2;\frac{9}{4}\right)$.
(a) Calculate the value of a
(b) Write down the equation of the inverse in the form y = ...
(c) Determine the equation of g, the reflection of f about the y-axis.
(d) Determine the equation of h, the reflection of f about the x-axis.
(e) Sketch the graph of f and its inverse on the same set of axes.
(f) Write down:
1) The domain of g 2) the range of h

3. In the diagram, the graph of $f(x) = \log_a x$ is drawn.
$B\left(\frac{25}{9};2\right)$ is a point on f.
(a) Determine the value of a
(b) Determine the value(s) of x for which $f(x) \leq 0$
(c) Write down the equation of f^{-1}, the inverse of f in the form y = ...
(d) $B^{//}$ is the reflection of B on the graph $g(x) = \left(\frac{3}{5}\right)^x$.
(e) Determine for which value(s) of x will $f^{-1} > \frac{25}{9}$



TOPIC: FUNCTIONS, INVERSES, GRAPHS (Lesson 5)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Functions, Graphs and Inverses				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Properties and characteristics of linear, quadratic, hyperbolic or exponential functions/graphs
- Asymptotes and turning points

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Writing the equation of asymptotes as p = … and/or q = …
- No real understanding of line of symmetry
- Horizontal shift in graphs is also a challenge

METHODOLOGY

Linear Function: $y = ax + q$, $y = mx + c$, where c *or* q is the y-intercepts (vertical shift)
a *or* m is the gradient of a line

Quadratic Function (Parabola):

$y = a(x+p)^2 + q$, where $(p; q)$ is the turning point.
p is the axis of symmetry, also shifts the graph horizontally
q is the maximum/minimum value, also shifts the graph vertically

$y = ax^2 + bx + c$, where c is the y-intercept
Equation of axis of symmetry: $x = -\frac{b}{2a}$
$\left(-\frac{b}{2a}; f(-\frac{b}{2a})\right)$ is the turning point

$y = a(x - x_1)(x - x_2)$, where x_1 *and* x_2 are x-intercepts

Hyperbolic Function: $y = \frac{a}{x+p} + q$, where p is the vertical asymptote.
q is the horizontal asymptote
Equation of horizontal asymptote: $x = \cdots$
Equation of vertical asymptote: $y = \cdots$
Equation of line of symmetry: $y = \pm(x+p) + q$
Or substitute $(p; q)$ into $y = \pm x + c$

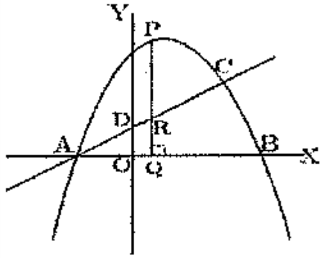
Exponential Function: $y = a.b^{x+p} + q$
q is a horizontal asymptote
q is the horizontal asymptote, also shifts the graph vertically

Examples:

1. Given: $f(x) = \frac{-8}{x+2} - 3$

(a) Write down the equations of the asymptotes for f.
(b) Calculate the x- and y-intercepts of f.
(c) Sketch the graph of f.
(d) If $y = -x + k$ the equation of the symmetry of f, determine the value of k.
(e) Write down the domain and the range for f.
(f) Determine the equation of h where h is the reflection of f in the x-axis.

2. The diagram shows $f(x) = -x^2 + 10x + 24$ and $g(x) = 2x + 4$.



(a) Calculate the length of AB
(b) Calculate the coordinates of C
(c) Express the equation of f in the form $y = a(x+p) + q$
(d) Determine the length of PR if OQ = 3.
(e) For which values of x will $\frac{f(x)}{g(x)} > 1$?
(f) Write down the equation of $h(x)$ if $f(x)$ is moved 2 units to the right and 3 units down.
(g) Determine the equation of $q(x)$, the reflection of $h(x)$ in the x-axis

3. The graph of an increasing exponential $f(x) = a.b^x + q$ function with equation has the following properties:

- Range: $y > -3$
- The points $(0; -2)$ and $(1; -1)$ lie on the graph of f.

(a) Determine the equation that defines $f(x)$

$q = -3$ …horizontal asymptote
$f(x) = a.b^x - 3$

$-2 = a.b^0 - 3$ … substitute point (0; -2)
$a = 1$
$f(x) = b^x - 3$

$-1 = b^1 - 3$ … substitute (1; -1)
$b = 2$
Therefore, $f(x) = 2^x - 3$

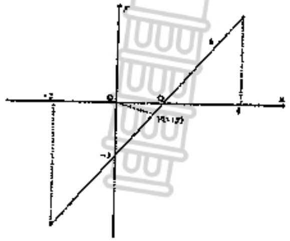
(b) Describe the transformation from $f(x)$ to $h(x) = 2.2^x + 1$

ACTIVITIES/ ASSESSMENT

1. Consider the function: $f(x) = \frac{2}{x+1} - 3$

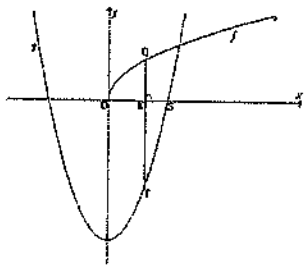
(a) Calculate the coordinates of the y-intercept of f.
(b) Calculate the coordinates of the x-intercept of f.
(c) Sketch the graph of f, showing clearly the asymptotes and the intercepts with the axes.
(d) One of the axes of symmetry of f is a decreasing function. Write down the equation of this axis of symmetry.

1. Given: $h(x) = 2x - 3 \; for -2 \leq x \leq 4$.
The x-intercept of h is Q.



(a) Determine the coordinates of Q
(b) Write down the domain of h^{-1}
(c) Sketch the graph of h^{-1}, clearly indicating the y-intercept and the end points
(d) For which value(s) of x will $h(x) = h^{-1}(x)$?

2. Sketched are the graphs of $g(x) = 4x^2 - 6$ and $f(x) = 2\sqrt{x}$.
S is an x-intercept of g and K is a point between O and S. The Straight line QKT with Q on the graph of f and T on the Graph of g, is parallel to the y-axis.



(a) Determine the x-coordinate of S.
(b) Write down the coordinates of the turning point of g.
(c) Write down the length of QKT in terms of x, where x is the x-coordinate of K.
(d) Calculate the maximum length of QT
(e) Write down the inverse of f in the form $y = \ldots$

3. Given: $f(x) = \left(\frac{1}{4}\right)^x$

(a) Determine the value of $f(-2)$
(b) Writ down the equation of f^{-1} in the form $y = \ldots$
(c) Andrew has no idea how to draw the graph of f^{-1}. Explain to Andrew how he may use the graph of f to draw the graph of f^{-1}.
(d) Hence or otherwise, sketch the graph of f^{-1}. Clearly indicate ALL intercepts with the axes.
(e) Writ down the domain of f^{-1}.
(f) For which values x will $f^{-1} \geq -2$?
(g) Given: $q = \log_{\frac{1}{4}} \frac{1}{2}$
  1) Determine the value of q
  2) Hence or otherwise, determine the coordinates of the point of intersection of f and f^{-1}

TEST : FUNCTIONS AND INVERSES

FROM: Eastern Cape September 2014

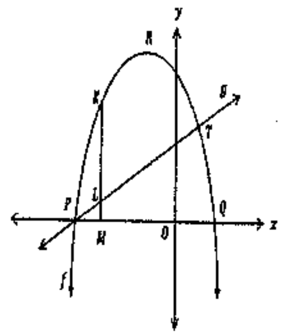
MARKS: 26 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [14 Marks]

The sketch shows the graphs of the functions $f(x) = -x^2 - x + 12$ and $g(x) = x + 4$. P and Q are the -intercepts of f, while R is the turning point of f. The functions intersect at T and P. K is a point on f, L is a point on g and M lies on the x-axis such that KLM is a straight line parallel to the y-axis.



1.1 Determine the coordinates of P and Q (3)

1.2 Determine the coordinates of R (2)

1.3 Determine the coordinates of M if KL $= 6\frac{3}{4}$ units (3)

1.4 Determine for which value(s) of x will $x.f(x) > 0$ (2)

1.5 For which real value(s) of k will $-x^2 - x + 12 = k$ have two negative, unequal roots? (2)

1.6 Give the equation of h in the form $h(x) = a(x+p)^2 + q$ if h is the reflection of f in the straight line $x = 1$. (2)

QUESTION 2 [12 Marks]

2.1 Given: $f(x) = \left(\frac{1}{5}\right)^x$

2.1.1 Write down the range of g if $g(x) = f(x) - 2$ (1)

2.1.2 Write down the equation of the f^{-1} in the form y = ... (2)

2.1.3 Draw a neat sketch of f^{-1}. Clearly show all intercepts with the axes. (2)

2.1.4 F o r which value(s) of x is $f^{-1}(x) \geq -1$? (2)

2.2 Consider: $f(x) = \frac{2}{x+p} + q$ with A (-5; -2) a point of intersection of vertical and horizontal asymptotes.

Sketch the graph of $f(x)$ clearly showing the asymptotes and the intercepts. (5)

TOPIC: CALCULUS (Lesson 1)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	The limit concept				
RELATED CONCEPTS/ TERMS/VOCABULARY	Limit				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

- Factorisation
- Simplification of Algebraic Expressions

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Incorrect multiplication of a binomial square e.g., $(x-4)^2 = x^2 + 16$
- Having zero on the denominator
- Forgetting to put a negative sign when changing e.g., $(3-x) = (x-3)$

METHODOLOGY

Calculus is one of the central branches of mathematics and was **developed from algebra and geometry**. It is built on the **concept of limits.**

There are some functions where the **value of the function** gets close to or **approaches a certain value** as the number of terms increases.

The limit of the function is therefore the value of y to which the graph approaches as the values of x approach a certain value from both the left and right.

Examples:

Determine the following limits:

1. $\lim\limits_{x\to 1}(x+4) = 5$

2. $\lim\limits_{x\to 2}\frac{x^2-4}{x+1} = \frac{0}{3} = 0$

3. $\lim\limits_{x\to 3}\frac{x^3-27}{3-x} = \frac{(x-3)(x^2+3x+9)}{-(x-3)} = -18$

Simplify the expression as the denominator will be zero if $x=3$ and division by zero is undefined.

4. $\lim\limits_{h\to 0}\frac{(2h+3)^2-9}{h}$

$= \lim\limits_{h\to 0}\frac{4h^2+12h+9-9}{h}$

$= \lim\limits_{h\to h}\frac{h(4h+12)}{h}$

$= \lim\limits_{h\to 0} 4h + 12 = 12$

5. $\lim\limits_{x\to 2} 10$

$f(x) = 10$ is a horizontal line. For any point on this line, the y-values will always be 10.

$\therefore \lim\limits_{x\to 2} 10 = 10$

ACTIVITIES/ ASSESSMENT

Evaluate:

1. $\lim\limits_{x\to 1}(2x^2+4)$
2. $\lim\limits_{x\to 0}(3-2x)^2$
3. $\lim\limits_{x\to 4}(x^3-1)$
4. $\lim\limits_{x\to 3}\frac{x^2-9}{x+3}$
5. $\lim\limits_{x\to -3}\frac{x+3}{x^2+3x}$
6. $\lim\limits_{x\to 2}\frac{3x^2-4x}{3-x}$
7. $\lim\limits_{\theta\to 60^\circ} cos2\theta$
8. $\lim\limits_{h\to 0}(2ah+h^2+a)$
9. $\lim\limits_{h\to 0}\frac{3h+h^2}{h}$
10. $\lim\limits_{x\to 0} 4$
11. $\lim\limits_{x\to 0}\frac{1}{x}$
12. $\lim\limits_{x\to 0}\frac{2x}{x^2+x}$
13. $\lim\limits_{x\to \pi}\frac{x^2-\pi^2}{x-\pi}$
14. $\lim\limits_{x\to \frac{1}{2}}\frac{4x^2-1}{1-2x}$

16. $\lim\limits_{x\to -\frac{3}{2}}\frac{4x^2-9}{2x+3}$

TOPIC: CALCULUS (Lesson 2)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Average gradient and the gradient of a tangent to a given function				
RELATED CONCEPTS/ TERMS/VOCABULARY	Gradient at a point, derivative, differentiation				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Gradient formula, tangent, straight line

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Notational error, simplification error (removing brackets)

METHODOLOGY

The **average gradient between any two points** on a curve is given by the **gradient of the straight line** that passes through both points.

To determine a **general formula for finding the average gradient** of a function f between two points x and $x+h$.

$$m = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

The **gradient at a single point** on a curve is the **gradient of the tangent** to the curve at the given point.

We use our **knowledge of limits to let h tend towards 0** to determine the **gradient of the tangent** to the curve at a point.

The **gradient of the tangent to a curve** with equation y = f(x) at x can be determine using the formula:

$$\text{Gradient at a point} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This formula can be used to determine an expression that describes the **gradient of the graph** (or the gradient of the tangent to the graph) at any point on the graph. This expression (or **gradient function**) is called the **derivative.**

The **derivative of a function** $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The process of determining the derivative of a given function is called **differentiation** and this method is called **differentiation from first principles** or using the definition.

The symbol $f'(x)$ represents the **gradient of the tangent** to a curve at x as well as the **gradient of the curve** itself at x. The symbol $f(x)$ represents the **y-value** of the function at x.

Examples:

1. Calculate the derivative of $f(x) = 3x$ from first principles.

$$f(x+h) = 3(x+h) = 3x + 3h$$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ... formula for finding the derivative using first principles

$= \lim_{h\to 0}\frac{3x+3h-3x}{h}$ ... Substitute into the formula and simplify

$= \lim_{h\to 0}\frac{3h}{h}$

$\therefore f'(x) = 3$

2. Determine the gradient of $f(x) = 2x - 3$ from first principles.

$$f(x+h) = 2(x+h) - 3 = 2x + 2h - 3$$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{2x+2h-3-(2x-3)}{h}$ ... put a bracket when subtracting more than one term

$= \lim_{h\to 0}\frac{2x+2h-3-2x+3}{h}$ ... remove brackets

$= \lim_{h\to 0}\frac{2h}{h}$ ... h divides h

$\therefore f'(x) = 2$

3. (a)Determine, from the first principles, the derivative of $f(x) = 1 - 3x^2$.

$$f(x+h) = 1 - 3(x+h)^2 = 1 - 3(x^2 + 2xh + h^2) = 1 - 3x^2 - 6xh - 3h^2$$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{1-3x^2-6xh-3h^2-(1-3x^2)}{h}$

$= \lim_{h\to 0}\frac{1-3x^2-6xh-3h^2-1+3x^2}{h}$

$= \lim_{h\to 0}\frac{-6xh-3h^2}{h}$

$= \lim_{h\to 0}\frac{h(-6x-3h)}{h}$ ... take out common factor h

$= \lim_{h\to 0} -6x + 3h$

$f'(x) = -6x$ ... gradient of the graph at any point

(b) Hence, calculate the $f'(-4)$ [derivative of f at $x = -4$].

$$f'(x) = -6x$$

$$f'(-4) = -6(-4) = 24$$

(c) What is the gradient of the tangent at $x = 5$?

$$f'(5) = -6(5) = -30$$

ACTIVITIES/ ASSESSMENT

1. Determine the derivatives from first principles if:

(a) $f(x) = x$ (b) $f(x) = -3x$ (c) $f(x) = 2x^2 + 1$

(e) $f(x) = -5x^2$ (e) $f(x) = 3 - x^2$ (f) $f(x) = 2x^2 + 3x + 1$

2. Consider: $f(x) = x^2 - x$

(a) Find the average rate of change of f over the interval $[1; 2]$.

(b) Find the instantaneous rate of change of f at $x = 1$ by using first principles.

(c) Determine $\lim\limits_{h\to 0}\frac{f(3+h)-f(3)}{h}$ and interpret your answer.

3. Consider the function $f(x) = -2ax^2 + 2x$

(a) Determine $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

(b) Determine $f(3)$ and $f'(3)$

(c) the gradient of the tangent to f at $x = -4$

(d) the rate of change of f at $x = 2$

TOPIC: CALCULUS (Lesson 3)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Derivative from First Principles				
RELATED CONCEPTS/ TERMS/VOCABULARY	Gradient of a horizontal line				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

$f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ... formula for finding the derivative using first principles

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Notational error, simplification error (removing brackets)

METHODOLOGY

Examples:

Differentiate the following from first principles:

1. $f(x) = 3$

The graph of $y = 3$ is a horizontal line with a gradient of zero.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = 3$$

$$f'(x) = \lim_{h\to 0}\frac{3-3}{h}$$

$$= 0$$

2. $f(x) = -x^3$

$$f(x+h) = -(x+h)^3$$
$$= -(x^3 + 3x^2h + 3xh^2 + h^3$$
$$= -x^3 - 3x^2h - 3xh^2 - h^3$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{-x^3-3x^2h-3xh^2-h^3-(-x^3)}{h}$$

$$= \lim_{h\to 0}\frac{-3x^2h-3xh^2+h^3}{h}$$

$$= \lim_{h\to 0}\frac{h(-3x^2-3xh+h^2)}{h}$$

$$f'(x) = -3x^2$$

3. $f(x) = 2x^3$

$$f(x+h) = 2(x+h)^3$$
$$= 2(x+h)(x+h)(x+h)$$
$$= 2(x+h)(x^2+2xh+h^2)$$
$$= 2(x^3+2x^2h+xh^2+hx^2+2xh^2+h^3)$$

4. $f(x) = -\frac{2}{x}$

$$f(x+h) = -\frac{2}{x+h}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$= 2(x^3 + 3x^2h + 3xh^2 + h^3)$

$= 2x^3 + 6x^2h + 6xh^2 + 2h^3$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+3h^3-2x^3}{h}$

$= \lim_{h\to 0}\frac{6x^2h+6xh^2+h^3}{h}$

$= \lim_{h\to 0}\frac{h(6x^2+6xh+h^2)}{h}$

$= \lim_{h\to 0} 6x^2 + 6xh + h^2$

$f'(x) = 6x^2$

$= \lim_{h\to 0}\frac{-\frac{2}{x+h}-\left(-\frac{2}{x}\right)}{h}$

$= \lim_{h\to 0} -\frac{2}{x+h} + \frac{2}{x} \div h$

$= \lim_{h\to 0}\frac{-2x+2x+2h}{x(x+h)} \times \frac{1}{h}$ ... L.C.D

$= \lim_{h\to 0}\frac{2h}{x(x+h)} \times \frac{1}{h}$

$= \lim_{h\to 0}\frac{2}{x(x+h)}$

$f'(x) = \frac{2}{x^2}$

Differentiate using first principles:

- If the question specifically states to use **first principles.**
- If we are required to differentiate **using the definition of a derivative.**

ACTIVITIES/ ASSESSMENT

Determine the derivatives using first principles

1. $f(x) = 5$
2. $f(x) = -7$
3. $f(x) = x^3$
4. $f(x) = -2x^3$
5. $f(x) = \frac{1}{x}$
6. $f(x) = -\frac{3}{x}$

TOPIC: CALCULUS (Lesson 4)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Rules of Differentiation:** $\frac{d}{dx}(ax^n) = anx^{n-1}$, **for any real number** n				
RELATED CONCEPTS/ TERMS/VOCABULARY	Rules of Differentiation Derivative of a constant				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Exponents, rational exponents, Factorisation, Algebraic Fractions

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

1. Notational error. 2. Performing different processes in the same step (rewriting one term and derivative of the other term). 3. Incorrectly differentiating the denominator.

METHODOLOGY

Determining the derivative of a function from first principles requires a long calculation. However, rules of **differentiation** can be used in finding the derivative which make finding the derivative of a function much simpler.

General rule for differentiation: $\frac{d}{dx}[x^n] = nx^{n-1}$

The derivative of a constant is equal to zero. $\frac{d}{dx}[k] = 0$

The following definitions will be helpful to you in this section of calculus.

DEFINITION	EXAMPLE
• $\frac{x^n}{a} = \frac{1}{a}x^n$ [a is a constant]	$\frac{x^2}{2} = \frac{1}{2}x^2$
• $\frac{a}{x^n} = ax^{-n}$ [a is a constant]	$\frac{2}{x^2} = 2x^{-2}$
• $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$ [a and b are constants]	$\frac{2}{3x^5} = \frac{2}{3}x^{-5}$
• $\frac{a}{(bx)^n} = \frac{a}{b^nx^n} = \frac{a}{b^n}.x^{-n}$ [a and b are constants]	$\frac{2}{(3x)^4} = \frac{2}{3^4x^4} = \frac{2}{81}x^{-4}$
• $\frac{a}{\sqrt[n]{x^m}} = \frac{a}{x^{\frac{m}{n}}} = a.x^{-\frac{m}{n}}$ [a is a constant]	$\frac{5}{\sqrt[3]{x^2}} = \frac{5}{x^{\frac{2}{3}}} = 5.x^{-\frac{2}{3}}$
• $\sqrt[n]{ax^m} = \sqrt[n]{a}.x^{\frac{m}{n}}$ [a is a constant]	$\sqrt[4]{3x^5} = \sqrt[4]{3}.x^{\frac{5}{4}}$

If the question does not specify how, we must determine the derivative, then we use the rules for differentiation.

Examples:

1. Determine the following

(a) $f(x) = 3x^5$

$f'(x) = 5.3x^{5-1}$

$f'(x) = 15x^4$

(b) $g(x) = \frac{1}{4}x^3$

$g'(x) = \frac{3.1x^{3-1}}{4}$

$g'(x) = \frac{3}{4}x^2$

(c) $k(x) = \frac{x^2}{5}$

$k'(x) = \frac{2x^{2-1}}{5}$

$k'(x) = \frac{2}{5}x$

(d) $f(x) = \frac{5}{x^2}$

$f(x) = 5x^{-2}$

$f'(x) = -2.5x^{-2-1}$

$f'(x) = -10x^{-3}$

(e) $g(x) = \sqrt{x^3} - \frac{3}{\sqrt{x}}$

$g(x) = x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}$ … remove root signs

$g(x) = x^{\frac{3}{2}} - 3x^{-\frac{1}{2}}$

$g'(x) = \frac{3}{2}x^{\frac{3}{2}-1} - (-\frac{1}{2}x^{-\frac{1}{2}-1})$

$g'(x) = \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

(f) $h(x) = \frac{x^2 + 2\sqrt[3]{x} - 3}{x}$

$h(x) = \frac{x^2}{x} + \frac{2x^{\frac{1}{3}}}{x} + \frac{3}{x}$ … common denominator

$h(x) = x + 2x^{\frac{1}{3}-1} + 3x^{-1}$

$h(x) = x + 2x^{-\frac{2}{3}} + 3x^{-1}$

$h'(x) = 1 - \frac{2}{3}.2x^{-\frac{2}{3}-1} - 1.3x^{-1-1}$

$h'(x) = 1 - \frac{4}{3}x^{-\frac{5}{3}} - 3x^{-2}$

ACTIVITIES/ ASSESSMENT

1. Determine the derivative of the following:

(a) $f(x) = 8x^4$ (b) $g(x) = \frac{1}{3}x^6$ (c) $k(x) = \frac{x^3}{4}$

(d) $h(x) = \frac{4}{x^3}$ (e) $f(x) = 6$ (f) $g(x) = \frac{1}{\sqrt[3]{x}}$

(g) $h(x) = 12x^3 + 7x$ (h) $k(x) = \frac{3}{2}x^4 - 1$ (i) $f(x) = \frac{5}{4}x^4 - \frac{5}{4}$

(j) $g(x) = -7x$ (k) $h(x) = \frac{(x+2)^2}{\sqrt{x}}$ (l) $k(x) = 1 - x + 3x^4$

2. Determine $f'(x)$ if

(a) $f(x) = x^2 - 2\sqrt{x}$ (b) $f(x) = \frac{x^3 - x^2 + 1}{x}$

(c) $x(x-2) + 5x$ (d) $f(x) = \sqrt{x^3} + \frac{1}{3x^3}$

Given: $g(x) = \frac{x^2 - 5x + 6}{x - 2}$, determine $g'(x)$

COMPILED BY PINETOWN DISTRICT MATHEMATICS ADVISORS:
M.B. MPISI, Z.I. SHANGASE, P.T.C. ZUNGU & S.M. MQADI

TOPIC: CALCULUS (Lesson 5)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Determining the derivative using the Rules				
RELATED CONCEPTS/ TERMS/VOCABULARY	Differential Operators				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

$\frac{d}{dx}[x^n] = nx^{n-1}$ and $\frac{d}{dx}[k] = 0$

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

1. Performing different processes in the same step (rewriting one term and derivative of the other term).
2. Incorrectly differentiating the denominator. 3. Incorrect exponent when removing a root sign.

METHODOLOGY

There are a few **different notations** used to refer to derivatives. The following symbols all mean the same thing (gradient, derivative, slope, rate of change):

$f'(x)$ $\frac{dy}{dx}$ $D_{(x)}$ $\frac{d}{dx}[f(x)]$

The symbols D and $\frac{d}{dx}$ are called **differential operators** because they indicate the operation of differentiation.

$\frac{dy}{dx}$ means differentiate y with respect to x. It is not a fraction and does not mean $dy \div dx$.

Examples:

Determine:

1. $\frac{dy}{dx}$ if $y = \frac{x^2-x-6}{2x^2}$

$y = \frac{x^2}{2x^2} - \frac{x}{2x^2} - \frac{6}{2x^2}$

$y = \frac{1}{2} - \frac{x^{-1}}{2} - 3x^{-2}$

$\frac{dy}{dx} = \frac{x^{-2}}{2} + 6x^{-3}$

2. $D_x[\frac{1}{2\sqrt[4]{x^3}}]$

$D_x[\frac{1}{2x^{\frac{3}{4}}}]$

$D_x[\frac{1}{2}x^{-\frac{3}{4}}]$

$= -\frac{3}{8}x^{-\frac{7}{4}}$

3. $\frac{dy}{dx}$ if $x = 2y + 3$

$2y = x - 3$

$y = \frac{x}{2} - \frac{3}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

4. $D_x[\frac{3x^3-7x^2-6x}{3-x}]$

$D_x[\frac{x(3x^2-7x-6)}{3-x}]$… common factor

$D_x[\frac{x(x-3)(3x+2)}{-(x-3)}]$ … $(a-x) = -(x-a)$

$D_x[-3x^2 - 2x]$

$= -6x - 2$

ACTIVITIES/ ASSESSMENT

Determine:

1. $f'(x)$ if $f(x) = x^3 - 15x^2 + 2x + 3$
2. $D_x[\frac{3x^2}{2} + \frac{3x}{2}]$
3. $D_x[(2x-3)(x+4)]$
4. $D_x[\frac{x^2-9}{3-x}]$
5. $\frac{dy}{dx}$ if $y = \frac{2x^2+3x-1}{x+1}$
6. $f'(x)$ if $f(x) = \frac{2x^3+2x^2-24x}{x-3}$
7. $\frac{dy}{dx}$ if $y = \frac{2x^2-\sqrt{x}+5}{\sqrt{x}}$
8. $\frac{dy}{dx}$ if $xy + y = x^2 - 1$
9. If $y = 2x(3-x)$ and $z = \frac{y}{2}$, determine:

(a) $\frac{dy}{dx}$ (b) $\frac{dz}{dy}$ (c) $\frac{dz}{dx}$

10. If $\sqrt{x} = y^{\frac{1}{3}}$ and $y = z^{-3}$, determine:

(a) $\frac{dy}{dx}$ (b) $\frac{dx}{dz}$ (c) $\frac{dy}{dz}$

TOPIC: CALCULUS (Lesson 6)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Equations of tangents to graphs of functions.				
RELATED CONCEPTS/ TERMS/VOCABULARY	Gradient of the curve				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Gradient of the tangent, differentiation, equation of a straight line, subject of the formula

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to equate the gradient of a function to a given gradient or to the gradient of a line parallel to the tangent.

METHODOLOGY

At a given point on a curve, the **gradient of the curve is equal to the gradient of the tangent** to the curve. $m_{tangent} = f'(x)$

To determine the equation of a tangent to a curve:

- Find the derivative using the rules of differentiation.
- Substitute the x-coordinate of the given point into the derivative to calculate the gradient of the tangent.
- Substitute the gradient of the tangent and the coordinates of the given point into an appropriate form of the straight line equation.
- Make y the subject of the formula.

Examples:

1. Determine the equation of the tangent to the curve $y = 3x^2$ at the point (1; 3).

$\frac{dy}{dx} = 6x$ ... derivative

$= 6(1) = 6$ ... substitute $x = 1$ into gradient

$y = mx + c$

$3 = 6(1) + c$ ... substitute gradient and a point into appropriate straight line form

$c = -3$

$\therefore y = 6x - 3$ ... equation of tangent

2. Determine the equation of the tangent to $f(x) = x^2 - 2x + 1$ if the gradient of the tangent is negative and the y-coordinate of the point of tangency is 4.

$m = ?$ $\quad x = ?$ $\quad y = 4$

$x^2 - 2x + 1 = 4$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

To get gradient of the tangent, determine $f'(x)$ and then substitute the two x-values into this expression to determine the gradient of the required tangent.

$f'(x) = 2x - 2$ ... derivative

$f'(-1) = 2(-1) - 2 = -4$ $\qquad f'(3) = 2(3) - 2 = 4$

$y = mx = c$

$4 = -4(-1) + c$ ... gradient is given as negative

$c = 0$ $\qquad \therefore y = -4x$ ... equation of tangent

3. Determine the value(s) of p if the line $y = 3x + p$ is a tangent to the graph of $f(x) = 2x^2 - 3x - 1$.

$m = 3$ $\qquad f'(x) = 4x - 3$

$4x - 3 = 3$

$x = \frac{6}{4} = \frac{3}{2}$ $\qquad f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 1 = \frac{7}{2}$

Equation of tangent: $y = mc + c$

$\frac{7}{2} = 3\left(\frac{3}{2}\right) + c$

$c = -1$

$y = 3x - 1$ $\qquad \therefore p = -1$

ACTIVITIES/ ASSESSMENT

1. (a) Determine the equation of the tangent to $f(x) = x^2 - 6x + 5$ at $x = 2$
   (b) Determine the equation of the tangent to the curve $y = 2x^3 - 21x^2 + 59x - 20$ at $x = 5$.

2. (a) Determine the equation of the tangent to the curve $y = 3x^2 - 2x + 2$ at $x = -4$.
   (b) Hence determine where this tangent cut the x-axis.

3. Determine the point where the gradient of the tangent to the curve:
   (a) $f(x) = 1 - 3x^2$ is equal to 5
   (b) $g(x) = \frac{1}{3}x^2 + 2x + 1$ is equal to 0.

4. Determine the equation of the tangent to the curve $y = x^3 - x^2 - 35x - 50$ at the point where $x = -$[illegible]. Find the coordinates of the point where the tangent meets the curve again.

5. (a) Determine the equation of the tangent to the curve $g(x) = x^2 + 4x - 5$ if the gradient of the tangent is negative and the y-coordinate of the point of tangency is -8.

   (b) Determine the equation of the tangents to the curve $f(x) = x^2 - 3x - 4$ at the points where $f(x = 0$.

6. (a) Determine the point on the graph of $y = 4\sqrt{x}$ for which the slope is 1.
   (b) Determine the points on the graph of $xy = 4$ for which the gradient is -4 .

7. Determine the point(s) on the curve $f(x) = (2x - 1)^2$ where the tangent is:
   (a) parallel to the line y = 4x - 2.
   (b) perpendicular to the line 2y + x - 4 = 0.

8. Determine the equation of the tangent to the curve $f(x) = -x^2 + 3x$ which is parallel to the line $y = x + 2$.

9. (a) Find the value of p if $y = p - 9x$ is a tangent to $y = -x^2 + 3x - 2$.
   (b) The graph of $f(x) = x^2 + 6$ has a tangent at $x = a$. The tangent passes through the point (2 ;1). Determine the equation of the tangent.

10. The tangent to $f(x) = 4 - x^2$ at $(a; f(a))$ passes through (4; - 3). Determine the equations of the tangents.

TEST 1: CALCULUS

FROM: PAST PAPERS

MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [14 Marks]

1.1 Determine the derivative of $f(x) = 1 - 2x^2$ using first principles. (4)

1.2 Determine the following:

1.2.1 $\frac{dy}{dx}$ if $y = \frac{2x^2-1}{\sqrt{x}}$ (3)

1.2.2 $D_x[(3x-2)^2]$ (3)

1.2.3 $f^{-1}(x)$ if $f(x) = \frac{5x^3-6\sqrt{x^3}+5}{x}$ (4)

QUESTION 2 [11 Marks]

2.1 If it is given that $f(x) = x^2 - \frac{4}{x^2}$

2.1.1 Determine the gradient of the tangent at the point $x = 2$ (3)

2.1.2 Determine the equation of the tangent in question 2.1.1 (3)

2.2 A tangent to the graph of $f(x) = -\frac{x^2}{2} + x$ has a gradient of -5 and x-intercept (a; 0).

Determine the value of a. (5)

TOPIC: CALCULUS (Lesson 7)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Second derivative of a function: $f''(x) = \frac{d}{dx}f'(x)$				
RELATED CONCEPTS/ TERMS/VOCABULARY	Second derivative, point of inflection, concavity				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Differentiation, Derivative

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Solving for x without equating the derivative to zero.

METHODOLOGY

The second derivative of a function is the **derivative of the first derivative.**

$f''(x) = \frac{d}{dx}f'(x)$.

The sign of the second derivative tells us if the **gradient of the original function** is increasing, decreasing or remaining constant.

The point where the **second derivative is equal to zero** is called the **point of inflection.**

$f''(x) = 0$

A **point of inflection** on the graph of a cubic function is the point at which the **concavity** of the function changes.

Concavity indicates whether the gradient of a curve is increasing, decreasing or stationary.

Cubic functions can change from being **concave down** to **concave up** or from concave up to concave down.

Examples:

1. Calculate the second derivative:

(a) $f(x) = 2x^3 - 4x^2 + 9$

$f'(x) = 6x^2 - 8x$

$f''(x) = 12x - 8$

(b) $y = \frac{3}{x} = 3x^{-1}$

$\frac{dy}{dx} = -3x^{-2}$

$\frac{d^2y}{dx^2} = 6x^{-3}$

2. Determine the x-value of point of inflection of:

(a) $f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

$f''(x) = 0$ … point of inflection

$6x - 12 = 0$

$x = 2$ … x-value of the point of inflection

(b) $y = -x^3 - 4x^2 + 3x + 8$

$\frac{dy}{dx} = -3x^2 - 8x + 3$

$\frac{d^2y}{dx^2} = -6x - 8$

$\frac{d^2y}{dx^2} = 0$…point in inflection

$-6x - 8 = 0$

$x = -\frac{4}{3}$

ACTIVITIES/ ASSESSMENT

1. Calculate the second derivative for each of the following:

(a) $g(x) = 5x^2$

(b) $y = 8x^3 - 7$

(c) $f(x) = x(x-6) + 10$

(d) $g(x) = x^5 - x^3 + x - 1$

(e) $y = -\frac{10}{x^2}$

(f) $f(x) = \sqrt{x} + 5x^2$

2. Determine the x-value of point of inflection of:

(a) $g(x) = 2x^3 - 5x^2 - 14x + 8$

(b) $h(x) = -x^3 + 4x^2 + x - 4$

(c) $f(x) = x^3 + 3x^2 - 10x$

(d) $y = x^3 - 5x^2 + 6$

(e) $f(x) = x^3 - 16x$

TOPIC: CALCULUS (Lesson 8)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Sketching graphs of cubic functions (third degree polynomial)**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Stationery points, point of inflection, local maximum, local minimum, concavity				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Intercepts, turning points, minimum and maximum point

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to relate point of inflection and concavity.

METHODOLOGY

The **cubic function** has the general equation $f(x) = ax^3 + bx^2 + cx + d$.

The effects of a on a cubic graph:

$a > 0$:  Graph increasing twice.

$a < 0$:  Graph decreasing twice

The graph of a cubic function has **two stationary points** called **turning points** (local maximum and minimum) as well as a **point of inflection.**

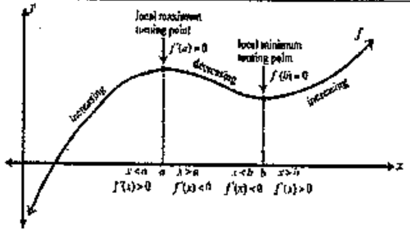
To determine the coordinates of the stationary point(s) of f(x):

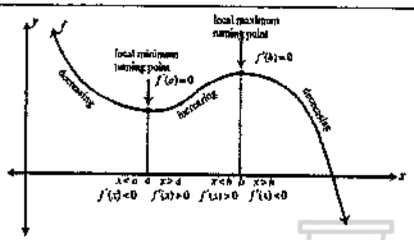
- Determine the derivative $f'(x)$
- Equate the derivative to zero ($f'(x) = 0$) and solve for the x-coordinates of the stationary points.
- Substitute values of x into the original/given function $f(x)$ to calculate the y-coordinates of the stationary points.
- If the function has **two stationary points**, establish whether they are **maximum or minimum** turning points by referring to the shape ($a > 0$ or $a < 0$).

NOTE:
For cubic functions, we refer to the turning (or stationary) points of the graph as **local minimum or local maximum turning points.**

To determine the coordinates of the point of inflection of f(x):

- Determine the second derivative $f''(x)$.
- Equate the second derivative to zero ($f''(x) = 0$) and solve for x –coordinate.
- If the cubic function has only **one stationary point**, this point will be a point of inflection that is also a stationary point.





To sketch the graph of a cubic function:

- Consider the sign of a and determine the general shape of the graph.
- Determine the y-intercept by letting x = 0.
- Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x.
- Find the x-coordinates of the turning points of the function by letting $f'(x) = 0$ and solving for x.
- Plot the points and draw a smooth curve.

x –intercepts of a cubic function $f(x) = ax^3 + bx^2 + cx + d$

- Use the **factor theorem** to find a factor of f(x) by trial and error
-

Example:

1. Sketch the graph of the function $f(x) = x^3 - 6x^2 + 9x$

y-intercept: $x = 0$

$f(0) = 0^3 - 6(0)^2 + 9(0) = 0$

(0; 0)

x-intercept: $y = 0$

$x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$ … common factor

$x(x-3)(x-3) = 0$ … factors of quadratic

$x = 0$ or $x = 3$

(0; 0) or (3; 0)

Stationery points:

$f'(x) = 3x^2 - 12x + 9$

$3x^2 - 12x + 9 = 0$ … $f'(x) = 0$

$x^2 - 4x + 3 = 0$ … common factor

$(x-3)(x-1) = 0$ … factors

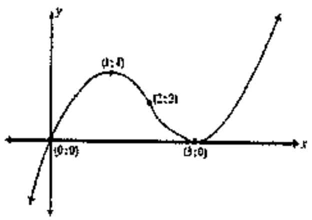
$x = 3$ or $x = 1$

$f(3) = 3^3 - 6(3)^3 + 9(3) = 0$

Local Minimum: (3; 0)

$f(1) = 1^3 - 6(1)^2 + 9(1) = 4$

Local maximum: (1:4)



Point of inflection:

$f''(x) = \frac{d}{dx} f''(x) = 6x - 12$

$6x - 12 = 0$ … $f''(x) = 0$

$x = 2$

$f(2) = 2^3 - 6(2)^2 + 9(2) = 2$

Point of inflection: (2; 2)

NOTE: The x-coordinate of the point of inflection can be determined by adding the x-coordinate of the turning points and dividing by 2

ACTIVITIES/ ASSESSMENT

For each of the following functions given below:

1. Determine the intercepts with the axes.
2. Determine the coordinates of the stationary points and establish whether they are maximum or minimum turning points, or points of inflection.
3. Determine the coordinates of the point of inflection which is not a stationary point.
4. Now sketch the graph of the function on a set of axes. Write down the values of x for which the function increases and/or decreases.

(a) $g(x) = x^3 - 3x^2 - 4x$.

(b) $f(x) = x^3 - 12x^2 + 36x$

(c) $f(x) = -x^3 + 6x^2 - 9x$

(d) $f(x) = 6x^2 - x^3$

TOPIC: CALCULUS (Lesson 9)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Sketching graphs of polynomials of third degree				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Intercepts, turning points or stationery points, point of inflection, local maximum, local minimum, concavity, minimum and maximum point.

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to relate concavity and point of inflection. Incorrect x-intercepts

METHODOLOGY

Examples:

1. Sketch the graph of the function $f(x) = -x^3 - 4x^2 + 3x + 18$.

y-intercept: $x = 0$

$f(0) = -0^3 - 40^2 + 3(0) + 18 = 18$

(0; 18)

x-intercepts: $y = 0$

$-x^3 - 4x^2 + 3x + 18 = 0$

$x^3 + 4x^2 - 3x - 18 = 0$

$(x-2)(x^2 + bx + 9) = 0$ ... factor theorem $(x+2)$

$b = 4 - (-2) = 6$

$(x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$x = 2$ 0r $x = -3$

$(2; 0)$ *or* $(-3; 0)$

Stationery point:

$f'(x) = -3x^2 - 8x + 3$

$-3x^2 - 8x + 3 = 0$

$3x^2 + 8x - 3 = 0$

$(3x - 1)(x + 3) = 0$

$x = \frac{1}{3}$ or $x = -3$

$f\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 18 = 18\frac{14}{27}$

$\left(\frac{1}{3}; 18\frac{14}{27}\right)$

$f(-3) = -(-3)^3 - 4(-3)^2 + 3(-3) + 18 = 0$

(-3; 0)

Point of Inflection:

$f''(x) = -6x - 8$

$-6x - 8 = 0$

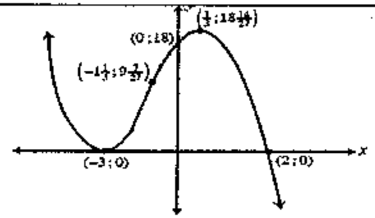
$-6x = 8$

$x = -\frac{4}{3}$

$f\left(-\frac{4}{3}\right) = -\left(-\frac{4}{3}\right)^3 - 4\left(-\frac{4}{3}\right)^2 + 3\left(-\frac{4}{3}\right) + 18$

$= 9\frac{7}{27}$

$\left(-\frac{4}{3}; 9\frac{7}{27}\right)$



2. Sketch the graph of $g(x) = -x^3 - 1$.

y-intercept: $x = 0$

$g(0) = -0^3 - 1 = -1$

(0; -1)

x-intercept: $y = 0$

$0 = -x^3 - 1$

$x^3 = -1$

$x = -1$

$f(-1) = -(-1)^3 - 1 = 0$

(-1; 0)

Stationery points:

$f'(x) = -3x^2$

$-3x^2 = 0$

$x = 0$

$f(0) = -0^3 - 1 = -1$

(0; -1)

Point of inflection:

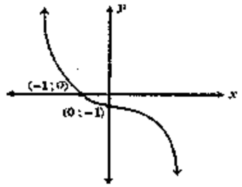
$f''(x) = -6x$

$-6x = 0$

$x = 0$

$f(0) = -0^3 - 1 = -1$

(0; -1)



ACTIVITIES/ ASSESSMENT

1. For each of the following functions given below:
   - Determine the intercepts with the axes.
   - Determine the coordinates of the stationary points and establish whether they are maximum or minimum turning points, or points of inflection.
   - Determine the coordinates of the point of inflection which is not a stationary point.
   - Now sketch the graph of the function on a set of axes. Write down the values of x for which the function increases and/or decreases.

(a) $f(x) = x^3 - 3x^2 + 4$

(b) $f(x) = x^3 - 12x - 16$

(c) $f(x) = x^3 - 2x^2 - 4x + 8$

(d) $f(x) = -2x^3 + x^2 + 8x - 4$

(e) $g(x) = 1 - x^3$

(f) $g(x) = x^3 - 3x^2 + 3x - 1$

(g) $f(x) = 2x^3 - 3x^2 + 2x - 3$

(h)

TOPIC: CALCULUS (Lesson 10)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Interpreting graphs of cubic functions: Determining the equation of a cubic function**				
RELATED CONCEPTS/ TERMS/VOCABULARY	$y = a(x - x_1)(x - x_2)(x - x_3)$				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Intercepts, turning points or stationery points, point of inflection

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

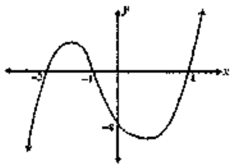
Failing to find the product of 3 linear expressions

Failing to notice that repeated x-intercept becomes the turning point of the graph.

METHODOLOGY

Examples:

1. Determine the equation of
$g(x) = ax^3 + bx^2 + cx + d$
if g passes through the points $(-2; 0)$, $(-1; 0)$, $(4; 0)$ and $(0; -8)$.
The diagram is not drawn to scale.



$g(x) = a(x - x_1)(x - x_2)(x - x_3)$, where x_1, x_2 *and* x_3 are x-intercepts.

$= a(x - (-2))(x - (-1))(x - 4)$

$= a(x + 2)(x + 1)(x - 4)$

$-8 = a(0 + 2)(0 + 1)(0 - 4)$ … substitute (0; -8)

$-8 = -8a$

$a = 1$

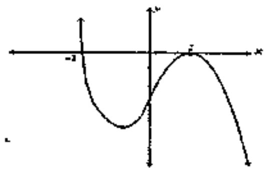
$g(x) = 1(x + 2)(x + 1)(x - 4)$

$= (x + 2)(x^2 - 3x - 4)$

$= x^3 - 3x^2 - 4x + 2x^2 - 6x - 8$

$\therefore g(x) = x^3 - x^2 - 10x - 8$

2. Determine the equation of
$f(x) = -x^3 + ax^2 + bx + c$



$f(x) = -(x - x_1)(x - x_2)(x - x_3)$

$= -(x - (-3))(x - 2)(x - 2)$

$= -(x + 3)(x^2 - 4x + 4$

$= -(x^3 - 4x^2 + 4x + 3x^2 - 12x + 12$

$= -(x^3 - x^2 - 8x + 12$

$\therefore f(x) = -x^3 + x^2 + 8x - 12$

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3. (2; 9) is a turning point on the graph of $f(x) = ax^3 + 5x^2 + 4x + b$. Determine the value of a and b and hence the equation of the cubic function.

$f(x) = ax^3 + 5x^2 + 4x + b$

$f'(x) = 3ax^2 + 10x + 4$

$f'(2) = 3a(2)^2 + 10(2) + 4$ … At the turning point (2; 9), we know that $f'(2) = 0$

$0 = 12a + 24$

$12a = -24$

$a = -2$

$f(x) = -2x^3 + 5x^2 + 4x + b$

$9 = -2(2)^3 + 5(2)^2 + 4(2) + b$

$9 = -16 + 20 + 8 + b$

$b = 9 + 16 - 20 - 8 = -3$

$\therefore f(x) = -2x^3 + 5x^2 + 4x - 3$

ACTIVITIES/ ASSESSMENT

1. $f(x) = ax^3 + bx^2 + cx + d$ is the graph of a cubic function passing through the points (6; 0), (0; −12), (1; 0), (−2; 0). Determine the values of a, b, c and d.

2. If the graph of $f(x) = -x^3 + ax^2 + bx - 12$ passes through (−3; 0) and has a turning point at (2; 0) determine the values of a and b.

3. Determine a and b if $y = -13$ is a tangent to the graph of the function $y = -ax^3 - 4x^2 + 8x + b$ at $x = -2$.

4. The graph of $f(x) = ax^3 + bx^2$ has a local turning point at (2; − 4). Find the value of a and b.

5. The graph of $f(x) = -x^3 + ax^2 + bx$ has a local turning point at (2; −32) . Find the value of a and b.

6. The graph of $y = mx^3 - 3x^2 - 12x + n$ has a local minimum turning point at (2; − 3). Find the value of m and n.

7. The line $y = -6x - 2$ is a tangent to the graph of $f(x) = ax^3 + bx^2$. Find the value of a and b if the point of tangency is (−1; 4) .

8. The line $2y + 3x = 6$ is a tangent to the curve $g(x) = 2ax^3 - bx^2$ at $x = 1$. Determine the value of a and b.

9. The gradient of the tangent to $y = ax^3 + bx^2$ at the point (1; 5) is 12.
   (a) Determine a and b.
   (b) Calculate the coordinates of the points on the curve where the tangent to the curve is parallel to the x-axis.

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TOPIC: CALCULUS (Lesson 11)	Weighting	35 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Interpreting graphs of cubic functions			
RELATED CONCEPTS/ TERMS/VOCABULARY	$y = a(x - x_1)(x - x_2)(x - x_3$			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Intercepts, turning points or stationery points, point of inflection

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to find the product of 3 linear expressions

Failing to notice that repeated x-intercept becomes the turning point of the graph.

METHODOLOGY

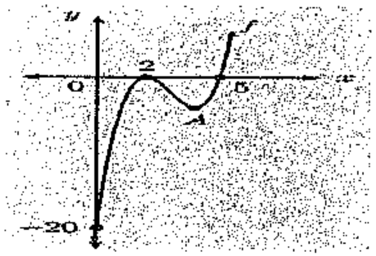
Example:

The sketch shows the graph of a cubic function, f, with a turning point at (2; 0), going through (5; 0) and (0; -20).

Determine:

(a) the equation of f.

(b) the coordinates of turning point A.



(a) $y = a(x-2)(x-2)(x-5)$ ... substitute x-intercepts

$-20 = a(0-2)(0-2)(0-5)$ ... substitute y-intercept

$-20 = -20a$

$a = 1$

$y = (x-2)(x-2)(x-5)$

$= (x^2 - 4x + 4)(x-5)$

$= x^3 - 4x^2 + 4x - 5x^2 + 20x - 20$

$\therefore f(x) = x^3 - 9x^2 + 24x - 20$

(b) $f'(x) = 3x^2 - 18x + 24$

$3x^2 - 18x + 24 = 0$ ... $f'(x) = 0$

$x^2 - 6x + 8 = 0$ ... 3 is a common factor

$(x-2)(x-4) = 0$

$x = 2$ or $x = 4$

$f(4) = 4^3 - 9(4)^2 + 24(4) - 20 = -4$

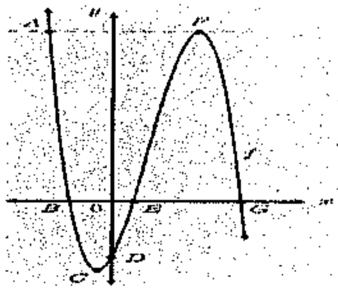
$\therefore A(4; -4)$

ACTIVITIES/ ASSESSMENT

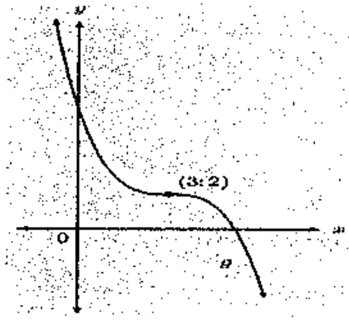
1. The sketch below shows the curve of $f(x) = -(x+2)(x-1)(x-6)$ with turning points at C and F. AF is parallel to the x-axis.

   Determine the:

   (a) length of OB, OE, EG and OD
   (b) coordinates of C and F
   (c) length AF
   (d) average gradient between E and F.
   (e) the equation of the tangent at E.



2. Given the graph of a cubic function with the stationary point (3; 2), sketch the graph of the derivative function if it is also given that the gradient of the graph is -5 at x = 0.



3. Use the information below to sketch a graph of each cubic function (do not find the equations of the functions).

   (a) $g(-6) = g(-1{,}5) = 0$
   $g'(-4) = g'(1) = 0$
   $g'(x) > 0$ for $x < -4$ or $x > 1$
   $g'(x) < 0$ for $-4 < x < 1$

   (b) $h(-3) = 0, h(0) = 4, h(-1) = 3, h'(-1) = 0$ and $h''(-1) = 0$
   $h'(x) > 0$ for all x values axcept $x = -1$

TOPIC: CALCULUS (Lesson 12)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Interpreting graphs of cubic functions: The graph of the derivative of a function**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Decreasing and Increasing Function Graph of the derivative of a cubic function				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Intercepts, turning points or stationery points, point of inflection

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

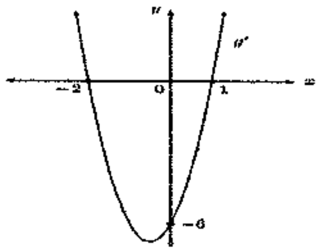
Failing to differentiate the meaning: $f(x) < 0, f'(x) < 0$ and $f''(x) < 0$

METHODOLOGY

Example:

Consider the graph of the derivative of g(x).

(a) For which values of x is g(x) decreasing?
(b) Determine the x-coordinate(s) of the turning point(s) of g(x).
(c) Given that $g(x) = ax^3 + bx^2 + cx$, calculate a, b and c.



(a) $g'(x)$ describes the gradient of $g(x)$.
To determine where the cubic function is decreasing, we must find the values of x for which $g'(x) < 0$:
$x \in (-2; 1)$ or $-2 < x < 1$

(b) To determine the turning points of a cubic function, we let $g'(x) = 0$ and solve for the x-values.
These x-values are the x-intercepts of the parabola and are indicated on the given graph:
$x = -2$ or $x = 1$

(c) $g(x) = ax^3 + bx^2 + cx$
$g'(x) = 3ax^2 2bx + c$
$= 3ax^2 + 2bx - 6$ … From the graph, we see that the y-intercept of $g'(x)$ is -6.
$g'(-2) = 3a(-2)^2 + 2b(-2) - 6$ … substitute $x = -2$
$0 = 12a - 4b - 6$ … (1)

$g'(1) = 3a(1)^2 + 2b(1) - 6$ … substitute $x = 1$
$0 = 3a + 2b - 6$ … (2)

(2) × 2: $0 = 6a + 4b - 12$ …. (3)
(3) + (2): $18a - 18 = 0$ (1): $0 = 12a - 4b - 6$

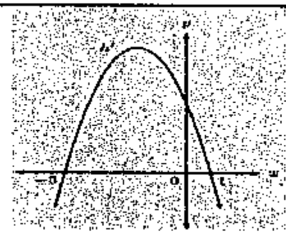
$18a = 18$
$a = 1$

$0 = 12(1) - 4b - 6$ … substitute $a = 1$
$4b = 6$
$b = \frac{3}{2}$

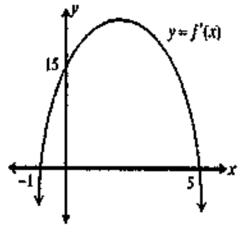
ACTIVITIES/ ASSESSMENT

1. The sketch below shows the graph of $h'(x)$ with x-intercepts at -5 and 1.
Draw a sketch graph of $h(x)$ if

$h(-5) = 2$ and $h(1) = 6$

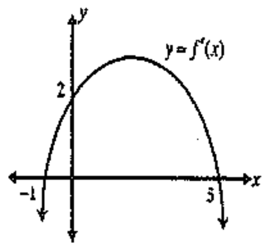


2. In the diagram alongside, the graph of $y = f'(x)$ is represented where $f(x) = ax^3 = bx^2 + cx$ represents a cubic function.



(a) Determine the equation of $f'(x)$ and $f(x)$ where $f(x) = ax^3 = bx^2 + cx$
(b) For which values of x will the graph of $f(x)$
i. Increase?
ii. Decrease?
(c) At which values of x will the graph of f have its turning points?

3. In the diagram alongside, the graph of $y = f'(x)$ is represented, which is the graph of the derivative of the cubic function $f(x)$.



(a) What is the gradient of the tangent to the graph of $f(x)$ at $x = 0$?
(b) For which value of x will there be a tangent to the curve of f which will be parallel to the tangent in (a)?
(c) For which values of x will $f(x)$ be decreasing?
(d) Write down the x-coordinates of the turning points of f and state whether they are local maximum or minimum turning points.
(e) If it is further given that the x-intercepts of the graph of f are -2, 2 and 7, use the information at your disposal and draw the graph of f.
It is unnecessary to determine the y-coordinates of the turning points.

TEST 2: CALCULUS

FROM: PAST PAPERS

MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [11 Marks]

Given $f(x) = x^3 - 3x^2 + 4$

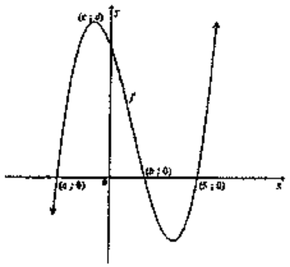
1.1 Determine the intercepts with the axes. (4)

1.2 Determine the coordinates of the turning points of f. (4)

1.3 Draw a neat sketch graph of f. Clearly show all the turning points and intercepts. (3)

QUESTION 2 [14 Marks]

The graph below, not drawn to scale, represents the cubic function f with equation $f(x) = x^3 - 4x^2 - 11x + 30$. The graph intersects the x-axis at (5; 0).



2.1 the values of a and b, the x-intercepts of f (4)

2.2 the values of c and d where (c; d) is the turning point of f (4)

2.3 the values of x for which:

2.3.1 $f'(x) < 0$ (1)

2.2.2 $f(x) \geq 0$ (1)

2.4 The graph of g with equation $g(x) = mx + c$ is a tangent to f at the point (5; 0). Calculate the value of m. (2)

2.5 Determine the values of p for which $x^3 - 4x^2 - 11x + 30 = p$ will have only one negative solution. (2)

TOPIC: CALCULUS (Lesson 13)		Weighting	35 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Problems involving maximum and minimum values: Two dimensional problems**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Optimisation				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Area of polygons, subject of the formula

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Do no finish reading the given statement or failing to comprehend the given statement.

METHODOLOGY

We have seen that **differential calculus** can be used to determine the **stationary points** of functions, in order to sketch their graphs. Calculating stationary points also lends itself to the solving of problems that require some variable to be **maximised or minimised**. These are referred to as **optimisation problems.**

Determining maximum or minimum values

The **quantity that is to be minimised or maximised** must be expressed in terms of only **one variable.** In order to maximize or minimize an object $a(x)$, determine $a'(x)$ and equate it to zero and solve for x. The value(s) of x obtained can then be investigated to establish whether they will yield maximum or minimum values of that given object.

Examples:

1. Michael wants to start a vegetable garden, which he decides to fence off in the **shape of a rectangle** from the rest of the garden. Michael has only **160 m of fencing**, so he decides to use a **wall as one border** of the vegetable garden.
   Calculate the **width and length** of the garden that corresponds to the largest possible area that Michael can fence off.

   Area of a rectangle is given by formula: $A = l \times w$
   Perimeter of a rectangle is given by the formula: $P = 2(l = w)$

   The fencing is only required for 3 sides and the three sides must add up to 160 m.

$$l + l + w = 160$$
$$2l + w = 160$$
$$w = 160 - 2l$$

Area= $l(160 - 2l)$ ... substitute $w = 160 - 2l$ into area formula

$= 160l - 2l^2$

$\frac{dA}{dl} = 160 - 4l$ ... maximising the area of the garden

$160 - 4l = 0$ ... equate the derivative to 0

$160 = 4l$

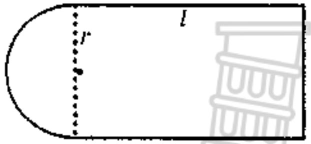
$40 = l$ ∴ the length of the garden is 40 m

$w = 160 - 2l$

$= 160 - 2(40)$ ... Substitute $l = 40$ to solve for the width

$= 80$ ∴ the width of the garden is 80 m

2. The area of the figure given is 20 m^2.



(a) Find an expression for the total area in terms of π, r and l

(b) Show that $l = \frac{40-\pi r^2}{4r}$

(c) Show that the perimeter P of the figure is given by:

$P = \frac{20}{r} + 2r + \frac{\pi}{2}r$

(d) Calculate the width of the figure when P is a minimum. Round off your answer to one decimal place.

(a) A = area of rectangle + area of semi-circle

$= l \times 2r + \frac{1}{2}\pi r^2$

$\therefore A = 2lr + \frac{1}{2}\pi r^2$

(b) $A = 2lr + \frac{1}{2}\pi r^2$

$20 = 2lr + \frac{1}{2}\pi r^2$ ... area of the diagram is 20 m^2

$40 = 4lr + \pi r^2$ ... multiply all terms by LCD

$40 - \pi r^2 = 4lr$

$\frac{40-\pi r^2}{4r} = l$ ... divide by $4r$ on both sides

(c) $P = l + l + 2r + \frac{1}{2} \times 2\pi r$

$= 2l + 2r + \pi r$

$= 2\left(\frac{40-\pi r^2}{4r}\right) + 2r + \pi r$

$= \frac{40-\pi r^2}{2r} + 2r + \pi r$

$= \frac{40}{2r} - \frac{\pi r^2}{2r} + 2r + \pi r$

$= \frac{20}{r} - \frac{\pi r}{2} + 2r + \pi r$

$= \frac{20}{r} - \frac{\pi r}{2} + \pi r + 2r$

$\therefore P = \frac{20}{r} + 2r + \frac{\pi}{2}r$

ACTIVITIES/ ASSESSMENT

1. A farmer has 100 metres of wire fencing from which to build a rectangular chicken run. He intends using two existing adjacent walls for two sides of the rectangular enclosure. Determine the dimensions which give a maximum enclosed area.

2. A toll-road company is planning to build a picnic site along a major highway. It is to be rectangular in shape with an area of 5000 m^2 and is to be fenced off on the three sides adjacent to the highway. Let the length be x and the width be y.

(a) Determine the total length, L, of fencing required in terms of x and y.

(b) Show that the total length of fencing required is also given by the formula: $L = x + \frac{10000}{x}$

(c) Determine the least amount of fencing that will be needed to complete the job.

3. The owner of a plot of land wishes to fence a rectangular area of 432 m^2 on his plot. One side of this rectangular area borders on his neighbour's property. The neighbour is prepared to pay half of the cost of the fence on the boundary line.

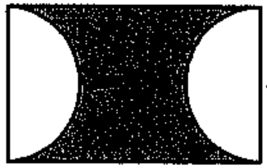
(a) If one side of the rectangular area is x metres and the cost of erecting the fence is R15 per metre, press the total cost (T) to the owner of the plot in terms of x.

(b) Hence calculate the dimensions of the rectangular area if the owner of the plot wishes to pay the minimum for erecting the fence.

4. A new design for a rectangular table place mat has been implemented by a company manufacturing dining room items. On the two width sides of the rectangular place mat, two white semi-circles are pasted as shown in the diagram below. The remaining area has been shaded with a multi-coloured pattern. The length of the place mat remains fixed at 100mm, while the width of the mat can vary in length.

(a) Show that the shaded area is given by: $A(x) = 100x - \frac{\pi}{4}x^2$

(b) Determine the maximum shaded area possible.



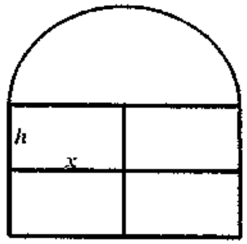
5. A metal frame consisting of four congruent rectangles and a semi-circle must be manufactured. The total length of the material that is to be used for the whole frame is 36 metres.

(a) If the length of a rectangle is x metres and the width is h metres, show that the total length of the material is given by: $L = (6h + 6x + \pi x)\ m$.

(b) Hence show that the area of the frame is given by:

$$A = \left(24x - 4x^2 - \frac{1}{6}\pi x^2\right)\ m^2$$

(c) Hence determine the length of the base of the frame if the frame is to have a maximum area.



TOPIC: CALCULUS (Lesson 14)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Problems involving maximum and minimum values: Three dimensional problems**				
RELATED CONCEPTS/ TERMS/VOCABULARY	Optimum point				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Surface area, Volume

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

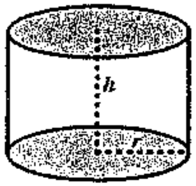
Do no finish reading the given statement or failing to comprehend the given statement.
Failing to use correct formula, failing to differentiate surface area and volume.

METHODOLOGY

Finding the optimum point:
Let $f'(x) = 0$ and solve for x to find the optimum point.

Examples:

1. A cylinder, closed at both ends, is to have a volume of $(2000\pi)\ cm^3$. What should its dimensions be if its surface area is to be as small as possible?



Area $= 2\pi r^2 + 2\pi rh$

Volume $= \pi r^2 h$

$2000\pi = \pi r^2 h$

$\frac{2000\pi}{\pi r^2} = h$

$\therefore h = \frac{2000}{r^2}$

$A = 2\pi r^2 + 2\pi r\left(\frac{2000}{r^2}\right)$

$= 2\pi r^2 + \frac{4000\pi}{r}$

$A = 2\pi r^2 + 4000\pi r^{-1}$

$\frac{dA}{dr} = 4\pi r - 4000\pi r^{-2}$

$4\pi r - 4000\pi r^{-2} = 0$ ... derivative = 0

$4\pi r - \frac{4000\pi}{r^2} = 0$

$4\pi r^3 - 4000\pi = 0$ ... multiply all terms by r^2 (LCD)

$4\pi r^3 = 4000\pi$

$r^3 = 1000$ ... divide all terms by 4π

$r = 10$

$h = \frac{2000}{r^2}$

$h = \frac{2000}{10^2} = 20\ cm$

The minimum surface area will be:

$A = 2\pi 10^2 + 2\pi(10)(20)$

$= 200\pi + 400\pi$

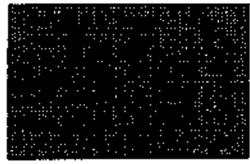
$\therefore A = (600\pi) cm^2$

COMPILED BY PINETOWN DISTRICT MATHEMATICS ADVISORS:
M.B. MPISI, Z.I. SHANGASE, P.T.C. ZUNGU & S.M. MQADI

ACTIVITIES/ ASSESSMENT

1. A piece of rectangular sheet metal with dimensions 80 mm by 50 mm is used to create a rectangular jewellery box. Square corners of x mm are cut out of the sheet metal and the edges are then folded up so as to form a box without a lid of depth x mm.

80mm



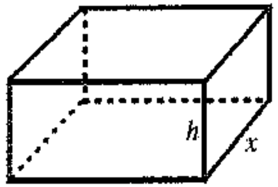
50mm

(a) Show that the volume of the jewelry box is given by the formula: $V(x) = 4x^3 - 260x^2 + 4000x$

(b) Determine the value of x so that the volume of the box is a maximum.

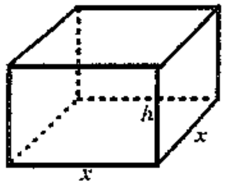
2. Bricks are to be painted with a special paint for use under water. The bricks are rectangular in shape and the length of each brick is three times its breadth. The volume of cement in each brick is 972 cm^3. If h is the height of a brick and x is its breadth:

(a) express h in terms of x.
(b) express the total surface area of each brick in terms of x.
(c) calculate the dimensions of a brick which will minimize the amount of paint required.

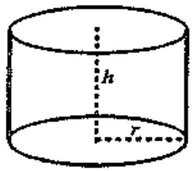


3. A rectangular water tank is to be manufactured. It must contain $32m^3$ of water. It has a square base (each side equal to x metres). The top of the water tank is open and its height is h metres.

(a) Determine the area (A) of the material that will be used in terms of x and h.
(b) Prove that $A = x^2 + \frac{128}{x}$
(c) Determine, in terms of x, an expression for C, the total cost of the material, if the material is bought at a price of R10 per m^2.
(d) Determine the values of x and h for which C will be a minimum.

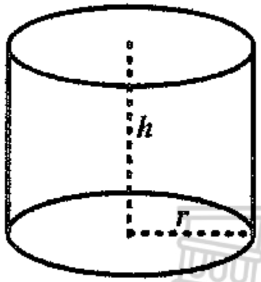


4. A solid cylinder is cast from 10 litres of molten metal. This cylinder is then covered by a layer of rust-proof paint. Calculate the radius and the height of the cylinder (in cm) such that the minimum quantity of paint is to be used.



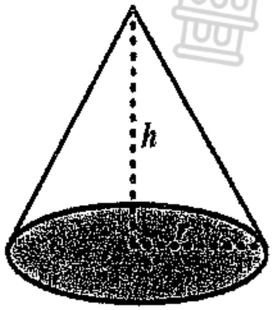
COMPILED BY PINETOWN DISTRICT MATHEMATICS ADVISORS:
M.B. MPISI, Z.I. SHANGASE, P.T.C. ZUNGU & S.M. MQADI

5. The total exterior surface area of an empty cylinder, which is closed on one end, is 462 cm^2.
a) Find the height (h) of the cylinder in terms of π and radius (r).
(b) Hence find the value of r for which the volume of the cylinder is a maximum.



6. The diagram represents a right circular cone of height, h cm, and a base radius of r cm. If the sum of the height and base radius is 12 cm,

(a) express r in terms of h.
(b) express the volume of the cone in terms of h.
(c) determine the maximum volume of the cone.

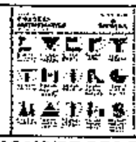


TOPIC: CALCULUS (Lesson 15)		Weighting	35 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Rate of change				
RELATED CONCEPTS/ TERMS/VOCABULARY	Average rate of change, instantaneous rate of change				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Derivative of a function, average gradient

RESOURCES



ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to understand the question or to interpret a question

METHODOLOGY

It is very useful to determine how **fast (the rate** at which) things are changing.
This **rate of change** is described by the gradient of the graph and can therefore be determined by calculating the derivative.

We have learnt how to determine the **average gradient** of a curve and how to determine the **gradient of a curve at a given point.**

These concepts are also referred to as the **average rate of change** and the **instantaneous rate of change.**

Average rate of change $= \frac{f(x+h)-f(x)}{x+h-x}$

Instantaneous rate of change $= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Example:

1. The height (in metres) of a golf ball t seconds after it has been hit into the air, is given by $H(t) = 20t - 5t^2$. Determine the following:

(a) The average vertical velocity of the ball during the first two seconds.
(b) The vertical velocity of the ball after 1;5 s.
(c) The time at which the vertical velocity is zero.
(d) The vertical velocity with which the ball hits the ground.
(e) The acceleration of the ball.

(a) Ave. velocity $= \frac{H(2)-H(0)}{2-0}$ ... initial velocity = 0 sec. and final velocity = 2 sec
$= \frac{20-0}{2}$ ... substitute $t = 0$ and $t = 2$ into the given equation
$= 10m/s$

(b) Instantaneous vertical velocity: $H'(t) = 20 - 10t$

$$H'(1{,}5) = 20 - 10(1{,}5)$$
$$\therefore v(t) = 5m/s$$

(c) $20 - 10t = 0$

$20 = 10t$

$t = 2\ sec$ Therefore, the velocity is zero after 2 sec

(d) The ball hits the ground when H (t) = 0

$20t - 5t^2 = 0$

$5t(4 - t) = 0$

$5t = 0\ or\ 4 - t = 0$

$t = 0\ or\ t = 4$

The ball hits the ground after 4 s. The velocity after 4 s will be:

$H'(4) = 20 - 10(4)$

$v(4) = -20m/s$

Therefore, the ball hits the ground at a speed of 20m/s

(e) Acceleration $= v'(t) = H''(t)$

$H''(t) = -10$

$\therefore a = -10m/s^2$

ACTIVITIES/ ASSESSMENT

1. In an experiment, the number of germs in a test tube at any time t, in seconds, is given by the equation $g(t) = 3t^2 + 2$. Determine:

   (a) the number of germs in the test tube after 4 seconds.
   (b) the rate of change in the number of germs after 4 seconds.
   (c) the rate of change in the number of germs after 6 seconds.

2. The volume of air in a spherical balloon change with respect to its radius. Determine the rate of change when the radius is 50 mm in length.

3. A soccer ball is kicked vertically into the air and its motion is represented by the equation: $D(t) = 1 + 18t - 3t^2$, where D = distance above the ground (in metres)
   t = time elapsed (in seconds)

   (a) Determine the initial height of the ball at the moment it is being kicked.
   (b) Find the initial velocity of the ball.
   (c) Determine the velocity of the ball after 1;5 s.
   (d) Calculate the maximum height of the ball.
   (e) Determine the acceleration of the ball after 1 second and explain the meaning of the answer.
   (f) Calculate the average velocity of the ball during the third second.
   (g) Determine the velocity of the ball after 3 seconds and interpret the answer.
   (h) How long will it take for the ball to hit the ground?
   (i) Determine the velocity of the ball when it hits the ground.

TEST 3: CALCULUS

FROM: PAST PAPERS

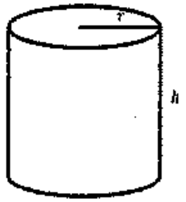
MARKS: 25 DURATION: 30 MIN.

INSTRUCTIONS

1. Answer ALL questions
2. Round off correct to TWO decimal places
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [18 Marks]

1.1 A 340 ml can of cool drink with height h and radius r is shown below.

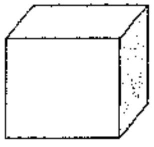


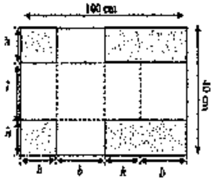
1.1.1 Determine the height of the can in terms of the radius r. (3)

1.1.2 Show that the surface area of the can be written as $SA = 2\pi r^2 + \frac{680}{r}$. (2)

1.1.3 Determine the radius of the can in cm, if the surface area of the can has to be as small as possible. (4)

1.2 A box is made from a rectangular piece of cardboard, 100 cm x 40 cm, by cutting out the shaded area and Folding along the dotted lines as shown in the diagram.





1.2.1 Express the length l in terms of the height h. (1)

1.2.2 Hence, prove that the volume of the box is given by: $V = h(50 - h)(40 - 2h)$. (3)

1.2.3 For which values of h will the volume of the box be a maximum? (5)

QUESTION 2 [7 Marks]

The flight of a cricket ball, hit by a batsman, is modelled by the equation $s(t) = -\frac{1}{3}t^2 + 2t + 1$ where $s(t)$ is the height of the ball in metres, t seconds after it was hit. A (0,3; 1,57) and B (5,7; 1,57) are two points which satisfy the equation.

2.1 What was the height of the ball when the batsman hit it? (1)

2.2 What was the average speed of the ball during the first 3 seconds? (2)

2.3 The ball was caught on its way down, at a height of 1,57 m.

2.3.1 After how many seconds was the ball caught? (1)

2.3.2 What was the speed of the ball at the moment it was caught? (3)

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 1)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Simple and Compound Increase and Decrease				
RELATED CONCEPTS/ TERMS/VOCABULARY	Simple interest, Compound Interest, Effective interest rate, Nominal interest rate				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Simple Interest, Compound Interest, Inflation, Simple Depreciation, Compound Depreciation, Effective and Nominal Interest rate

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to take into account compounding

METHODOLOGY

APPRECIATION

Simple interest is interest calculated on the principal amount of a loan/original contributing to a savings account. Interest is gained on the principal amount only.

$A = P(1 + i.n)$ ... Simple Interest

Compound Interest is the addition of interest to the principal deposit or loan. Interest is gained on the principal amount and its previously accumulated interest.

$A = P(1 + i)^n$ ... Compound Interest:
also known as inflation

A is the final/accumulated amount
P is the original/principal amount
i is the interest rate per period
n is the number of periods

DEPRETIATION

$A = P(1 - i.n)$ ... Simple Depreciation
Straight-line depreciation

$A = P(1 - i)^n$ ... Compound Depreciation
Reducing balance depreciation

A is the final/depreciated amount
P is the original amount
i is the depreciation rate per period
n is the number of periods

Nominal interest rate is also defined as a *stated* interest rate. This interest works according to the

simple interest and does not take into account the compounding periods. The nominal interest rate is the **periodic interest rate times the number of periods per year**. In general, stated or nominal interest rate is **less than the effective one.**

Examples of nominal interest rate:

- 8% per year compounded monthly
- 14% per year compounded semi-annually
- 4% per quarter compounded monthly

The effective interest rate is always calculated as if compounded annually. It is the one which caters the **compounding periods** during a payment plan.

Examples of effective interest rate:

- 12% per year compounded yearly
- 1% per month compounded monthly
- 3,5% per quarter compounded quarterly

$1 + i_{eff} = \left(\frac{1+i_{nom}}{m}\right)^m$ ... Effective and Nominal Interest Conversion

i_{eff} is the effective annual interest rate
i_{nom} is the nominal interest rate (rate p.a. compounded more than once)
m is the number of times interest is compounded in one year

Compound interest calculations, where n is an exponent in the formula, we need to use our knowledge of logarithms to determine the value of n.

NOTE: **Rounding off** should be done on the **final answer only**

Examples:

1. Thembile invests R 3500 into a savings account which pays 7,5% per annum **compounded yearly**. After an unknown period of time his account is worth R 4044,69. For how long did Thembile invest his money?

$P = R3\,500$ $\quad i = 7{,}5\%$ $\quad A = R4\,044{,}69$ $\quad n = ?$

$A = P(1 + i)^n$ ... compound interest formula

$4\,044{,}69 = 3\,500(1 + 7{,}5\%)^n$ ... substitute values
$\frac{4\,044{,}69}{3\,500} = (1{,}075)^n$
$1{,}1556 = (1{,}075)^n$
$n = \log_{1{,}075} 1{,}1556$
$n = 1{,}9997$
Therefore, the R 3500 was invested for 2 years.

2. Amanda deposits R10 000 into a savings account. Interest is calculated at 12% p.a. **compounded monthly**. How long will it take for her savings to grow to R15 000?

$i = \frac{12\%}{12} = 0{,}01$ ...divide by 12 because it is compounded monthly, $P = R10\,000$, $A = R15\,000$
$A = P(1 + i)^n$ ... Compound interest formula

$15\,000 = 10\,000\left(1+\frac{12\%}{12}\right)^n$

$\frac{15\,000}{10\,000} = (1{,}01)^n$

$\frac{3}{2} = (1{,}01)^n$

$n = \log_{1{,}01}\frac{3}{2}$

$n = 40{,}74890715609$ … number of compounding periods i.e., months

$\frac{40{,}74890715609}{12} = 3{,}39574226301\ years$ … divide compounding by 12 to convert to years

3. A company purchases a truck for R800 000. The value of the truck depreciates at 20% p.a. on the reducing balance. How long will it take the truck's value to depreciate to R500 000?

$P = R800\,000$ $\quad A = R500\,000$ $\quad i = 20\%$ $\quad n = ?$

$A = P(1-i)^n$ … Compound Depreciation

$500\,000 = 800\,000\,(1-20\%)^n$

$\frac{500\,000}{800\,000} = (0{,}8)^n$

$\frac{5}{8} = (0{,}8)^n$

$n = \log_{0{,}8}\frac{5}{8}$

$n = 2{,}10628371951$

Therefore, approximately 2 years and 1 month

ACTIVITIES/ ASSESSMENT

1. Nzuzo invests R 80 000 at an interest rate of 7,5% per annum compounded yearly. How long will it take for his investment to grow to R 100 000?
2. Determine how many years it will take an investment of R2 000 to earn an amount of R1 920 in interest, if the investment was made at an interest rate of 13% per annum compounded monthly.
3. Determine the time taken for a certain sum of money to double if the interest rate is 12% per annum compounded semi-annually.
4. R3 000 depreciates at 9% per annum on a straight – line basis to an amount of R1 872,10 over a period of time. Determine the depreciation time.
5. A car cost R300 000 and depreciated over a period of five years to half its original amount.
   (a) Calculate the annual rate of depreciation if it is based on the reducing balance method.
   (b) How long will it then take for the car to depreciate once again to half its value? Assume that the depreciation rate is the same as in (a).
6. Determine how many years it would take for the value of a car to depreciate to 25% of its original value, if the rate of depreciation, based on the reducing balance method, is 16% per annum.
7. How long will it take for the value of an investment to treble, if interest is calculated at 22% p.a. compounded semi-annually?
8. How long will it take for a bowling machine to halve in value if its value depreciates at 25% p.a. on the reducing balance?

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 2)	Weighting	15 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Future Value Annuity			
RELATED CONCEPTS/ TERMS/VOCABULARY	Annuity, future value, payment intervals/regular equal payments			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Nominal and effective interest rate

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Using compound interest formula for future value
- Not taking compounding into account e.g., dividing interest rate by 4 if quarterly compounding

METHODOLOGY

Annuity is a number of **equal payments** made at **regular intervals** for a certain period of time subject to a rate of interest over a period of time.

- All payments are equal
- Payments are made at regular intervals
- Interest rate remains fixed
- Compounding period for interest is the same as the payment intervals

Future value annuity – collective value of all regular equal payments made into a savings account including the interest at the end of the time period. The amount accumulating in the fund earns compound interest at a certain rate.

For future value annuities, we regularly **save the same amount of money** into an account, which earns a certain rate of compound interest, so that we have money for the future.

The formula used to calculate the Future Value annuity is: $F = \frac{x[(1+i)^n - 1]}{i}$

$F = future\ value$

$x = equal\ payments\ per\ period$

$i = interst\ rate = \frac{r}{100}$

$n = number\ of\ payments$

Examples:

Determining the value of an investment:

1. Ciza decides to start saving money for the future. At the end of each month, she deposits R 900 into an account at Harringstone Mutual Bank, which earns 8,25% interest p.a. compounded monthly.

   (a) Determine the balance of Ciza's account after 29 years.
   (b) How much money did Ciza deposit into her account over the 29-year period?

(c) Calculate how much interest she earned over the 29-year period.

$x = R900$ $\quad i = \frac{8,25\%}{12}$ $\quad n = 29 \times 12\ months = 348\ months$ $\quad F = ?$

(a) $F = \frac{x[(1+i)^n - 1]}{i}$

$= \frac{900[(1+\frac{8,25\%}{12})^{348} - 1]}{\frac{8,25\%}{12}}$

$= R1\ 289\ 665,06$

(b) Total Deposits $= 900 \times 348 = R313\ 20$

(d) Total Interest $=$ *Final amountat the end of the period − total amount deposited*

$= R1\ 289\ 665,06 - R313\ 200 = R976\ 465,06$

2. Thomas starts saving money in a Unit Trust fund. He immediately deposits R800 into the fund. Thereafter, at the end of each month, he deposits R800 into the fund and continues to do this for ten years. Interest is 8% per annum compounded monthly. Calculate the future value of his investment at the end of the ten-year period.

$x = R800$ $\quad i = \frac{8\%}{12} =$ $\quad n = 10 \times 12\ months + 1(immediately) = 121\ months$ $\quad F = ?$

$F = \frac{x[(1+i)^n - 1]}{i}$

$= \frac{800[(1+\frac{8\%}{12})^{121} - 1]}{\frac{8\%}{12}}$

$= R148\ 132,54$

Calculating the monthly payments:

4. Dayna has just turned 20 years old and has a dream of saving R8 000 000 by the time she reaches the age of 50. She starts to pay equal monthly amounts into a retirement annuity which pays 18% per annum compounded monthly. Her first payment starts on her 20th birthday and her last payment is made on her 50th birthday. How much will she pay each month?

$F = R80\ 000$ $\quad n = 30 \times 12 months + 1 = 361\ months$ $\quad i = \frac{18\%}{12}$ $\quad x = ?$

$F = \frac{x[(1+i)^n - 1]}{i}$

$8000\ 000 = \frac{x[(1+\frac{18\%}{12})^{361} - 1]}{\frac{18\%}{12}}$

$8000\ 000 \times \frac{18\%}{12} = x(214,8943376)$ ... multiply by the denominator on both sided

$120000 = x(214,8943376)$ ... we only round off the final answer

$x = \frac{120000}{214,8943376} = R558,41$ Therefore, Dyna will pay R558,41 each month

Calculating the number of payments:

4. Linda wants to save R30 000 to buy a gaming laptop. She can afford to save R3 000 every six months. Interest is calculated at 10,2% p.a. compounded **semi-annually**. How many payments will she have to make to have at least R30 000 in savings?

Semi-annually means **half yearly** or **twice a year**, therefore we divide the interest rate by 2.

$F = R30\ 000$ $\quad x = R3\ 000$ $\quad i = \frac{10,2\%}{2}$ $\quad n = ?$

$F = \frac{x[(1+i)^n - 1]}{i}$

$30\ 000 = \frac{3\ 000[(1+\frac{10,2\%}{2})^n - 1]}{\frac{10,2\%}{2}}$

$1\ 530 = 3\ 000[(1,051)^n - 1$ ... multiply by the denominator on both sides

$0,51 = (1,051)^n - 1$ ... divide by 3 000 on both sides

$1,51 = (1,051)^n$ ... transpose 1

$n = \log_{1,051} 1,51$ ... to solve for n, apply logs

$n = 8,284928018$ Therefore, Linda will have to make 9 payments to have at least R30 000.

ACTIVITIES/ ASSESSMENT

1. Shelly decides to start saving money for her son's future. At the end of each month, she deposits R 500 into an account at Durban Trust Bank, which earns an interest rate of 5;96% per annum compounded monthly.
   (a) Determine the balance of Shelly's account after 35 years.
   (b) How much money did Shelly deposit into her account over the 35 year period?
   (c) Calculate how much interest she earned over the 35 year period.

2. Griffen wants to move out of his parents' house when he is one with school. He decides to save R2 500 per quarter for the next two years. How much will he have saved up by the end of two years if interest is calculated at 9% p.a. compounded quarterly.

3. In order to supplement his pension after retirement, Mpho (aged 20) takes out a retirement annuity. He makes monthly payments of R2 000 into the fund and the payments start immediately. The payments are made in advance, which means that the last payment of R2 000 is made one month before the annuity pays out. The interest rate for the annuity is 15% per annum compounded monthly. Calculate the future value of the annuity when he turns 60.

4. Lerato plans to buy a car in five and a half years' time. She has saved R 30 000 in a separate investment account which earns 13% per annum compound interest. If she doesn't want to spend more than R 160 000 on a vehicle and her savings account earns an interest rate of 11% p.a. compounded monthly, how much must she deposit into her savings account each month?

5. R500 is invested each month, starting in one month's time, into an account paying 16% per annum compounded monthly. The fund accumulates to R10 000. How many payments of R500 will be made?

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 3)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Future Value Annuity				
RELATED CONCEPTS/ TERMS/VOCABULARY	Annuity, future value, regular equal payment				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

x = equal payments, nominal and effective interest rate

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Not taking compounding into account e.g., dividing interest rate by 2 for half-yearly compounding

METHODOLOGY

Payments made in advance:

1. Hannes invest R1 000 at the **beginning of each month** for 4 years. No payment is made at the end of the last month of the four-year period. The interest rate is 9,5% p.a. compounded monthly. How much money will he have at the end of the 4 years?

$x = R1\,000$ $\quad n = 4 \times 12\ months = 48\ months$ $\quad i = \frac{9{,}5\%}{12}$ $\quad F = ?$

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$= \frac{1000[(1+\frac{9{,}5\%}{12})^{48} - 1]}{\frac{9{,}5\%}{12}}$$

$$= 58\,117{,}67316$$

Amount at T_{48}:

$$A = P(1+i)^n$$

$$= 58\,117{,}67316\left(1+\frac{9{,}5\%}{12}\right)^1 = R58\,577{,}77$$

2. In order to supplement his state pension after retirement, a school teacher aged 30 takes out a retirement annuity. He makes monthly payments of R1 000 into the fund and the payments start immediately. The payments are made **in advance**, which means that the last payment of R1 000 is made one month before the annuity pays out. The interest rate for the annuity is 12% per annum compounded monthly. Calculate the future value of the annuity in twenty-five years' time.

$x = R1\,000$ $\quad i = \frac{12\%}{12}$ $\quad n = 25 \times 12 = 300$ $\quad F = ?$

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$= \frac{1000[(1+\frac{12\%}{12})^{300} - 1]}{\frac{12\%}{12}}$$

$$= 187\,884{,}626$$

At the end of the last month: $A = P(1+i)^n$

$$= 1\,878\,846{,}626\left(1+\frac{12\%}{12}\right)^1$$

$$= R1\,897\,635{,}09$$

OR $F = \frac{x[(1+i)^n - 1]}{i}(1+i)$

$$= \frac{1000\left[(1+\frac{12\%}{12})^{300} - 1\right]}{\frac{12\%}{12}}\left(1+\frac{12\%}{12}\right)$$

$$= R1\,897\,635{,}09$$

ACTIVITIES/ ASSESSMENT

1. Karabo invests R750 at the beginning of each month into a savings account for 6 years. No payment is made at the end of the last month of the six-year period. The interest rate is calculated at 11,25 p.a. compounded monthly. How much money will Karabo have at the end of 6 years?

2. In order to supplement his pension after retirement, Mpho (aged 20) takes out a retirement annuity. He makes monthly payments of R2 000 into the fund and the payments start immediately. The payments are made in advance, which means that the last payment of R2 000 is made one month before the annuity pays out. The interest rate for the annuity is 15% per annum compounded monthly. Calculate the future value of the annuity when he turns 60.

3. Joyce invests R2 550 per quarter for 4 years. Her first payment is made immediately and her last payment is made 3 months before the end of the 4 years. The interest rate is 10,5% p.a. compounded quarterly. How much money will Joyce have saved after 4 years?

5. R500 is invested each month, starting in one month's time, into an account paying 16% per annum compounded monthly. The fund accumulates to R10 000. How many payments of R500 will be made?

6. R1 000 is invested every three months, starting in three month's time, into an account paying 14% per annum compounded quarterly. The fund accumulates to R25 000. How many payments of R1 000 will be made?

7. R2 000 is immediately deposited into a savings account. Six months later and every six months thereafter, R2 000 is deposited into the account. The interest rate is 16% per annum compounded half-yearly. The future value of the savings is R100 000. How many payments of R2 000 will be made?

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 4)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Present Value Annuity				
RELATED CONCEPTS/ TERMS/VOCABULARY	Present value, loan				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Reducing balance, interest rate, regular equal payments

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Confuse P value with x value

Putting n on the formula instead of $-n$

METHODOLOGY

Present value annuity - regular equal payments/instalments are made to pay back a loan (student loan to study, vehicle loan to buy a car, loan to buy a house) over a given time period. Money is available in the present i.e.; first money received and pay later.

In bank loans, interest is paid on the **reducing balance**. The advantage of these loans is that any **additional payments** into the loan will reduce the amount owed. Hence, **reducing the duration** of the loan as well as **saving a lot of interest**.

$A = P(1+i)^n$

$P = A(1+i)^{-n}$ ... make P the subject of the formula

The formula used to calculate the Future Value annuity is: $P = \frac{x[1-(1+i)^{-n}]}{i}$

Examples:

1. James takes out a one-year bank loan to pay for an expensive laptop. The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan. Show that he borrowed an amount of R18 000.

$n = 1year \times 12months = 12\ months$ $\quad i = \frac{18\%}{12}$ $\quad x = R1\ 650{,}24$ $\quad P = ?$

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$= \frac{1\ 650{,}24[1-\left(1+\frac{18\%}{12}\right)^{-12}]}{\frac{18\%}{12}}$$

$= R18\ 000$

2. Malibongwe takes out a bank loan to pay for his new car. He repays the loan by means of monthly

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payments of R5 000 for a period of five years starting one month after the granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the purchase price of his new car.

$x = R5\ 000$ $\quad n = 5years \times 12months = 60\ months$ $\quad i = \frac{24\%}{12}$ $\quad P = ?$

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$= \frac{5000[1-\left(1+\frac{24\%}{12}\right)^{-60}]}{\frac{24\%}{12}}$$

$= R173\ 804{,}43$

3. History wants to buy a small wine farm worth R 8 500 000. He plans to sell his current home for R 3 400 000 which he will use as a deposit for the purchase of the farm. He secures a loan with HBP Bank with a repayment period of 10 years and an interest rate of 9,5% compounded monthly.

(a) Calculate what History's monthly repayments will be.
(b) Determine how much interest History will have paid on his loan by the end of the 10 years.

$P = R8\ 500\ 000 - R3\ 400\ 000 = R5\ 100\ 000$ $\quad n = 10 \times 12\ months = 120\ months$ $\quad i = \frac{9{,}5\%}{12}$

(a) $P = \frac{x[1-(1+i)^{-n}]}{i}$

$$5\ 100\ 000 = \frac{x[1-\left(1+\frac{9{,}5\%}{12}\right)^{-120}]}{\frac{9{,}5\%}{12}}$$

$40\ 375 = x(0{,}6118095902)$ ... multiply by denominator on both sides

$x = \frac{40\ 375}{0{,}6118095902} = R65\ 992{,}75$

Therefore, History must pay R 65 992,75 per month to repay his loan over the 10 year period.

(b) Total amount repaid for loan $= R65\ 992{,}75 \times 120 = R7\ 919\ 130$
Total interest paid $= R7\ 919\ 130 - 5\ 100\ 000 = R2\ 819\ 130$

4. Karel has to pay off a loan of R75 000. He can afford to pay R1500 per month with the interest rate of 16,2% p.a. compounded monthly. How many payments will he have to make?

$P = R75\ 000$ $\quad x = R1\ 500$ $\quad i = \frac{16{,}2\%}{12}$ $\quad n = ?$

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$75\ 000 = \frac{1\ 500[1-\left(1+\frac{16{,}2\%}{12}\right)^{-n}]}{\frac{16{,}2\%}{12}}$$

$1012{,}5 = 1\ 500[1-(1{,}0135)^{-n}]$ ... multiply by denominator on both sides

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$0,675 = 1 - (1,0135)^{-n}$ ... divide by 1 500 on both sides

$-0,325 = -(1,0135)^{-n}$ ... transpose 1

$-n = \log_{1,0135} 0,325$ ... apply logs

$-n = -83,81479032$

$n \approx 83,81$

Karel will make 84 payments (83 payments of R1 500 and one lesser payment).

ACTIVITIES/ ASSESSMENT

1. (a) Ziyanda arranges a bond for R 17 000 from Langa Bank. If the bank charges 16;0% p.a. compounded monthly, determine Ziyanda's monthly repayment if she is to pay back the bond over 9 years.
   (b) What is the total cost of the bond?

2. How much can Lerato borrow from a bank if she repays the loan by means of equal quarterly payments of R2 000, starting in three months time? The interest rate is 18% per annum compounded quarterly and the duration of the loan is ten years.

3. Otto repays a loan over a period of 5 years by means of equal semi-annually payments of R15 000, starting 6 months after the loan was granted. What is the value of the loan if the interest is 14,8% p.a. compounded semi-annually?

4. Brenda takes out a twenty-year loan of R400 000. She repays the loan by means of equal monthly payments starting one month after the granting of the loan. The interest rate is 16% per annum compounded monthly. Calculate the monthly repayments.

5. Lerato plans to buy a car for R125 000. She pays a deposit of 15% and takes out a bank loan for the balance. The bank charges 12,5% per annum compounded monthly. Calculate the monthly repayment on the car if the loan is repaid over the six-year period.

6. Stefan and Marna want to buy a flat that costs R 1;2 million. Their parents offer to put down a 20% payment towards the cost of the house. They need to get a mortgage for the balance. What is the monthly repayment amount if the term of the home loan is 30 years and the interest is 7;5% p.a. compounded monthly?

7. Dullstroom Bank offers personal loans at an interest rate of 15;63% p.a. compounded twice a year. Lubabale borrows R 3000 and must pay R 334;93 every six months until the loan is fully repaid.
   (a) How long will it take Lubabale to repay the loan?
   (b) How much interest will Lubabale pay for this loan?

TEST 1: FINANCE, GROWTH AND DECAY

MARKS: 25 DURATION: 30 Min.

INSTRUCTIONS TO CANDIDATE:

1. Answer ALL the questions
2. Choose relevant formula from the FORMULA SHEET
3. Round off FINAL answers correct to TWO decimal places.

QUESTION 1 [5Marks]

1.1 Mrs. Bella deposits R9 000 into a savings account. Calculate how long it will take her to double her money if the interest rate is 6% p.a. compounded quarterly. (3)

1.2 The computers in a school's computer lab originally cost R 70 000. These computers were sold for R30 000 a few years ago. Calculate how long it took the computers to depreciate if the interest rate of depreciation was 12% p.a. on a reducing balance. (2)

QUESTION 2 [10 Marks]

2.1 Gugu wants to save up to R250 000 in 5 years' time in order to purchase a car. She starts making monthly payments into an account paying 13% p.a. compounded monthly, starting immediately. How much will she pay each month? (6)

2.2 R2 000 is invested each month into an account paying 16% p.a. compounded semi-annually, starting in 6 months' time. How long will it take to accumulate R 100 000? (4)

QUESTION 3 [10 Marks]

3.1 How much can be borrowed from the bank if the repayment of the loan is made by means of 30 equal monthly payments of R1 250. The interest rate is 14% p.a. compounded monthly. (6)

3.2 The retirement fund does not pay out R2,5 million when Mark retires. Instead, he will be paid monthly amounts for a period of 20 years, starting one month after his retirement. If the interest rate that he earns over this period is calculated at 7% p.a. compounded monthly, determine the monthly payments he will receive. (4)

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 5)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Sinking Fund				
RELATED CONCEPTS/ TERMS/VOCABULARY	Lump sum, sinking fund				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Compound interest formula, reducing balance, straight line basis, future value formula, compounding

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Do not read question with understanding.

METHODOLOGY

Sinking fund is money set aside in order to buy or replace outdated **equipment** e.g., photocopying machine, computer, car and other similar asserts. This equipment is usually sold at **scrap value.**
A **sinking fund** is nothing more than a normal **savings plan** or **future value annuity.**

Regular deposits, and sometimes **lump sum deposits**, are made into these accounts so that enough money will have accumulated by the time a new machine or vehicle needs to be purchased.
This savings plan is a future value annuity.

To set up a sinking fund, the following must be calculated:

- The value of the old equipment (scrap value/depreciation value/book value) – **Depreciation formula**
  Either reducing balance: $A = P(1-i)^n$ or straight-line basis: $A = P(1 - i.n)$
- The value of the new equipment – **Inflation:** $A = P(1+i)^n$
- Amount to be saved: $sinking\ fund = inflated\ value - depreciated\ value$

Examples:

1. Wellington Courier Company buys a delivery truck for R 296 000. The value of the truck depreciates on a reducing-balance basis at 18% per annum. The company plans to replace this truck in seven years' time and they expect the price of a new truck to increase annually by 9%.

(a) Calculate the **book value** of the delivery truck in seven years' time.

$P = R296\ 000 \qquad i = 18\% \qquad n = 7 \qquad A = ?$

$A = P(1-i)^n$ … reducing balance

$= 296\ 000(1-18\%)^7$

$= R73\ 788,50$

(b) Determine the minimum balance of the sinking fund in order for the company to afford a new truck in seven years' time.

Price of the new truck in 7 years: $A = P(1+i)^n$

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$= 296000(1+9\%)^7$

$= R541\ 099,58$

Sinking fund $= R541\ 099,58 - R73\ 788,50 = R467\ 311,08$

(c) Calculate the required monthly deposits if the sinking fund earns an interest rate of 13% per annum compounded monthly.

$F = R467\ 311,08 \qquad i = \frac{13\%}{12} \qquad n = 7 \times 12\ months = 84\ months$

$F = \frac{x[(1+i)^n - 1]}{i}$

$467\ 311,08 = \frac{x[\left(1+\frac{13\%}{12}\right)^{84} - 1]}{\frac{13\%}{12}}$

$5062,5367 = x(1,472194323$ … multiply by denominator on both sides

$x = R3\ 438,77$ Therefore, the company must deposit R 3438;77 each month.

(d) Suppose that Wellington Courier Company decides to service the truck at the end of each year for the 7-year period. R4 000 will be withdrawn from the sinking fund at the end of each year starting one year after the original machine was bought.

(1) Calculate the reduced value of the sinking fund at the end of the 7e-year period due to these withdrawals.

The sinking fund will not only lose the seven amounts of R4 000, it will also lose the interest earned on these amounts at the end of the seven- year period.

Future value of the withdrawals:

$4000\left(1+\frac{13\%}{12}\right)^{72} + 4000\left(1+\frac{13\%}{12}\right)^{60} + 4000\left(1+\frac{13\%}{12}\right)^{48} + 4000\left(1+\frac{13\%}{12}\right)^{36} +$
$4000\left(1+\frac{13\%}{12}\right)^{24} + 4000\left(1+\frac{13\%}{12}\right)^{12} = R38\ 662,26$

The reduced value of the sinking fund will be: $R467\ 311,08 - R38\ 662,26 = R428\ 648,82$

(2) Calculate the increased **monthly payment into the sinking fund** which will yield the original sinking fund amount as well as allow for withdrawals from the fund for the services.

If we add $R38\ 662,26$ to the original sinking fund amount of $R467\ 311,08$, then it will be possible to not only receive the sinking fund amount of $R467\ 311,08$ at the end of the 7- year period, but also be able to make the service withdrawals at the end of each year for the 7- year period.

Amount to be saved $= R467\ 311,08 + R38\ 662,26 = R505\ 973,34$

$F = \frac{x[(1+i)^n - 1]}{i}$

$505\ 973,34 = \frac{x[\left(1+\frac{13\%}{12}\right)^{84} - 1]}{\frac{13\%}{12}}$

$5841,37785 = x(1,472194323$

$x = R3\ 967,80$

ACTIVITIES/ ASSESSMENT

1. Mfethu owns his own delivery business and he will need to replace his truck in 6 years' time. Mfethu deposits R 3100 into a sinking fund each month, which earns 5;3% interest p.a. compounded monthly.

(a) How much money will be in the fund in 6 years' time, when Mfethu wants to buy the new truck?

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(b) If a new truck costs R 285 000 in 6 years' time, will Mfethu have enough money to buy it?

2. Atlantic Transport Company buys a van for R 265 000. The value of the van depreciates on a reducing-balance basis at 17% per annum. The company plans to replace this van in five years' time and they expect the price of a new van to increase annually by 12%.
   (a) Calculate the book value of the van in five years' time.
   (b) Determine the amount of money needed in the sinking fund for the company to be able to afford a new van in five years' time.
   (c) Calculate the required monthly deposits if the sinking fund earns an interest rate of 11% per annum compounded monthly.

3. A car wash business purchases large washing equipment for R140 000. The cost of the new is expected to rise by 18% per annum while the rate of depreciation is 20% per annum on the reducing- balance. The life span of the equipment is six years.
   (a) Find the scrap value of the original equipment.
   (b) Find the cost of new equipment in six years' time.
   (c) Find the value of the sinking fund that will be required to purchase the new equipment in six years' time, if the proceeds from the sale of the old equipment (at scrap value) will be used.
   (d) The business sets up a sinking fund to pay for the new equipment. Payments are to be made into an account paying 13,2% per annum compounded monthly. Find the monthly payments, if they are to commence one month after the purchase of the old equipment and cease at the end of the six -year period.
   (e) Suppose that the business decides to service the equipment at the end of each year for the six-year period. R4 000 will be withdrawn from the sinking fund at the end of each year starting one year after the original machine was bought.
      (1) Calculate the reduced value of the sinking fund at the end of the six -year period due to these withdrawals.
      (2) Calculate the increased monthly payment into the sinking fund which will yield the original sinking fund amount as well as allow for withdrawals from the fund for the services.

4. Due to load shedding, a restaurant buys a large generator for R227 851. It depreciates at 23% per annum on a reducing-balance. A new generator is expected to appreciate in value at a rate of 17% per annum. A new generator will be purchased in five years' time.
   (a) Find the scrap value of the old generator in five years' time.
   (b) Find the cost of a new machine in five years' time.
   (c) The restaurant will use the money received from the sale of the old machine (at scrap value) as part payment for the new one. The rest of the money will come from a sinking fund that was set up when the old generator was bought. Monthly payments, which started one month after the purchase of the old generator, have been paid into a sinking fund account paying 11,4% per annum compounded monthly. The payments will finish three months before the purchase of the new machine. Calculate the monthly payments into the sinking fund that will provide the required money for the purchasing of the new machine.

5. Cleaning equipment is bought for R120 000. The value of the equipment depreciates at 15% p.a. on the reducing balance. The inflation rate is 9% p.a. in 5 years' time. the old cleaning equipment will be sold at scrap value. The proceeds of the sale, together with money saved up in a sinking fund will be used to purchase replacement cleaning equipment.
   (a) Calculate the scrap value of the old equipment after 5 years.
   (b) What will the new equipment cost in 5 years' time?
   (c) What amount should the company budget for in 5 years' time?
   (d) Calculate the monthly payment to be paid into a sinking fund paying 12,5% p.a. compounded monthly, in order to have enough money to replace the equipment in 5 years' time.

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 6)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Outstanding Balance				
RELATED CONCEPTS/ TERMS/VOCABULARY	Loan, repayment				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Compound interest formula and future value formula

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Could not differentiate between number of payments outstanding and number of payments made

METHODOLOGY

The outstanding balance of a loan at a given moment is the **amount that has to be paid** to settle the loan. It is the actual remaining amount.

$$\textit{Outstanding Balance} = \text{loan with interest to date} - \text{repayments with interest to date}$$
$$= \text{money owed (compound interest)} - \text{money paid (future value)}$$

Therefore, $OB = P(1+i)^n - \frac{x[(1+i)^n-1]}{i}$ ... n = number of payments already made.

Examples:

1. James takes out a one-year bank loan of R18 000 to pay for an expensive laptop. The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan.

$P = R18\ 000$ $\quad\quad i = \frac{18\%}{12} = 0{,}015$ $\quad\quad x = R1\ 650{,}24$

(a) Calculate his balance outstanding after he has paid the sixth instalment.

$$OB = P(1+i)^n - \frac{x[(1+i)^n-1]}{i}$$
$$= 18\ 000(1+0{,}015)^6 - \frac{1\ 650{,}24[(1+0{,}015)^6-1]}{0{,}015}$$
$$= R9\ 401{,}72$$

(b) Calculate his balance outstanding after he has paid the ninth instalment.

$$OB = P(1+i)^n - \frac{x[(1+i)^n-1]}{i}$$
$$= 18\ 000(1+0{,}015)^9 - \frac{1650{,}24[(1+0{,}015)^9-1]}{0{,}015}$$
$$= R4\ 805{,}83$$

2. Yandisa takes a loan from a bank to start the business. The monthly repayments are R30 428 a month for 15 years. The interest rate is 9% p.a. compounded monthly.

$x = R30\ 428$ $\quad n = 15 \times 12 = 180\ months$ $\quad i = \frac{9\%}{12} = 0{,}0075$ $\quad P = ?$

(a) Determine how much Yandisa initially borrowed from the bank.

$P = \frac{x[1-(1+i)^{-n}]}{i}$ ... present value formula

$= \frac{30\ 428[1-(1+0{,}0075)^{-180}]}{0{,}0075}$

$= R3\ 000\ 000{,}24$

(b) Determine the balance of the loan at the end of 5 years.

$n = 5years \times 12 = 60\ months$

$OB = P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$

$= 3\ 000\ 000{,}24(1+0{,}0075)^{60} - \frac{30\ 428[(1+0{,}0075)^{60}-1]}{0{,}0075}$

$= R2\ 402\ 037{,}82$

ACTIVITIES/ ASSESSMENT

1. Siphokazi takes out a 20 -year loan from the bank to buy a house which costs R850 000. She pays a deposit of 12% of the selling price of the house. The bank charges an interest rate of 9% p.a. compounded monthly.
   (a) Determine the amount granted by the bank to Siphokazi.
   (b) Calculate the monthly instalments of the loan.
   (c) How much interest will she pay at the end of 20 years?
   (d) Calculate the outstanding balance of the loan after the 85th instalment.

2. Kevin takes out a bank loan to pay for his new car. He repays the loan by means of monthly payments of R3 000 for a period of 6 years starting one month after the granting of the loan. The interest rate is 18% p.a. compounded monthly.
   (a) Calculate the purchase price of the new car.
   (b) Calculate the balance outstanding after the 20th payment.
   (c) Calculate the balance outstanding after the 60th payment

3. Nolusizo took out a loan of R1 500 000 to buy a house. She will repay the loan with monthly instalments over 20 years. The interest rate is 8% p.a. compounded quarterly.
   (a) Calculate the monthly instalments of the loan.
   (b) Calculate the balance of the loan after 12 years.

4. Lerato takes out a loan and repays the loan by means of equal quarterly payments of R2 000, starting in three months' time. The interest rate is 18% p.a. compounded quarterly and the duration of the loan is 10 years.
   (a) Calculate the amount borrowed
   (b) Calculate the balance outstanding after the 30th payment has been made.

5. Thabo pays off a loan of R80 000 over a period of 5 years. He makes half-yearly payments, starting 6 months after the loan was granted and ending at the end of the 5-year period. The interest rate is 10,8% p.a. compounded half-year.
   How much does he owe immediately after the 10th payment?

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TOPIC: FINANCE, GROWTH AND DECAY (Lesson 7)	Weighting	15 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Final/Last Payment			
RELATED CONCEPTS/ TERMS/VOCABULARY	Full payment			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Outstanding balance.

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Manipulation of the formula when calculating P, i or n

METHODOLOGY

Final payment is the amount made one period after the last full payment.

To calculate the final payment:

- Calculate the number of payments of the loan (n)
- Calculate the outstanding balance after the last full payment

Final payment $=$ ***outstanding balance after the last full payment*** $\times (1+i)$

Examples:

1. Jill negotiates a loan of R300 000with a bank which has to be paid by means of R5 000. The repayments start one month after the granting of the loan. The interest rate is fixed at 18% p.a. compounded monthly.

$P = R300\ 000$ $\quad x = R5\ 000$ $\quad i = \frac{18\%}{12} = 0{,}015$

(a) Determine the number of payments required to settle the loan.

$P = \frac{x[1-(1+i)^{-n}]}{i}$ ... present value formula

$300\ 000 = \frac{5\ 000[1-(1+0{,}015)^{-n}]}{0{,}015}$

$0{,}9 = 1 - (1{,}015)^{-n}$ ... multiply 300 000 by denominator and divide by 5 000

$-0{,}1 = -(1{,}015)^{-n}$ ... transpose 1

$-n = \log_{1{,}015} 0{,}1$ ... multiply/divide by (-) on both sides and apply logs

$-n = -154{,}6541086$

Jill will have 155 payments [54 payments of R5 000 and one lesser payment (0,6541086)]

(b) Calculate the balance outstanding after Jill has paid the last R5 000.

$OB = P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$

$= 300\ 000(1+0{,}015)^{154} - \frac{5\ 000[(1+0{,}015)^{154}-1]}{0{,}015}$

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$= R3\ 230,50$

(c) Calculate the value of the final payment made by Jill to settle the loan.

Final payment = ***outstanding balance after the last full payment*** $\times (1+i)$

$= 3\ 230,50 \times (1+0,015)$

$= R3\ 278,96$

2. George has to repay a loan of R375 000. The interest rate is 14% p.a. compounded monthly. He pays back R7 500 per month.
   (a) How many payments will George have to make?

$P = R375\ 000$ $\qquad i = \frac{14\%=0,11666667}{12}$ $\qquad x = R7\ 500$

$P = \frac{x[1-(1+i)^{-n}]}{i}$ … present value formula

$375\ 000 = \frac{7\ 500[1-(1+0,011666667)^{-n}]}{0,011666667}$

$0,5835 = 1-(1,011666667)^{-n}$ … multiply 375 000 by 0,011666667 and divide by 7 500

$-\frac{5}{12} = -(1,011666667)^{-n}$ … transpose 1

$-n = \log_{1,011666667} 0,416666667$ … multiply/divide by (-) on both sides and apply logs

$-n = -75,4770656342$

George will make 76 payments [75 payments of R7 500 and 1 lesser payment (0,4770656342)]

(c) What will his last payment be?

Final payment = ***outstanding balance after the last full payment*** $\times (1+i)$

$= P(1+i)^n - \frac{x[(1+i)^n-1]}{i} \times (1+i)$

$= [375\ 000(1+0,011666667)^{75} - \frac{7\ 500[(1+0,011666667)^{75}-1]}{0,011666667}] \times (1+0,011666667)$

$= 3\ 547,457499 \times (1,011666667)$

$= R3\ 588,84$

ACTIVITIES/ ASSESSMENT

1. Lindiwe secured a loan of R85 000 from the bank. She has agreed to repay in monthly instalments of R2 000 each, starting in three months' time. Interest is charged at 21% p.a. compounded monthly.
   (a) What is the value of the loan at the end of 2 months?
   (b) How long will it take before the loan is completely paid off?
   (c) Calculate Lindiwe's last instalment in the 85th month which is less than R2 000.

2. Alicia wants to pay off a loan of R450 000. She can afford to pay back R11 000 per month. Interest is calculated at 12,8% p.a. compounded monthly.
   (a) How many payments will Alicia have to make?
   (b) What is the outstanding balance of the loan immediately after the last full payment?
   (c) Determine the value of the last payment, made one month after the last full payment.
   (d) How much interest will Alicia pay on the loan?

3. Petro pays back R7 500 per quarter on a loan of R 75 000. The interest rate on the loan is 8,2% p.a. compounded quarterly.
   (a) How many payments will Petro have to make?
   (b) Calculate Petro's last payment, made one quarter after her last payment of R7 5000.

TOPIC: FINANCE, GROWTH AND DECAY (Lesson 8)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Deferred Payments and Missed Payments				
RELATED CONCEPTS/ TERMS/VOCABULARY	Defer, accumulated interest				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Compound interest formula, present value formula, outstanding balance

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Confuse P value with x value

METHODOLOGY

1 Deferred Payment

In many situations, the repayment of a loan begins one month after the loan has been granted. However, circumstances arise where the repayments are **deferred** for an agreed period of time. To defer a payment is to make an arrangement to pay later. These are **delayed** payments.

NOTE: If you skip an instalment/payment, the loan will **accumulate the interest** at the time the payment is not made.

Example:

A car that costs R 130 000 is advertised as follows: 'No deposit necessary and first payment is due three months after the date of purchase'. The interest rate is 18% p.a. compounded monthly. Peter bought this car on 1 March 2019 and made his first payment on 1 June 2019. Thereafter, he made another 53 equal payments on the first day of each month. Calculate his monthly repayments.

$P = R130\ 000$ $\qquad i = \frac{18\%}{12} = 0,015$

The amount owed after two months: $A = P(1+i)^n$

$= 130\ 000(1+0,015)^2$

$= R133\ 929,25$ … amount owed by Peter for skipping 2 months

Monthly instalment: $P = \frac{x[1-(1+i)^{-n}]}{i}$ … present value formula

$133\ 929,25 = \frac{x[1-(1+0,015)^{-54}]}{0,015}$

$2008,93875 = x(0,5524580822)$ … multiply by denominator on both sides

$x = R3\ 636,36$

2. **Missed payments** happen due to financial difficulty and as soon as the situation improves, payments will resume.
The loan will **accumulate the interest** at the time the payments are not made.

To calculate the new payment:

- Calculate the balance outstanding immediately after the last payment
- Calculate compound interest of this outstanding balance

Example:

James takes out a loan of R100 000. He is to repay the loan by means of equal payments, starting one month after the loan was granted. The loan is to be paid in 4 years, at an interest rate of 10% p.a. compounded monthly.

$P = R100\,000 \qquad n = 4 \times 12 = 48 \text{ months} \qquad i = \frac{10\%}{12} = 0{,}008333\ldots$

(a) What is his monthly payment?

$$P = \frac{x[1-(1+i)^{-n}]}{i} \ldots \text{ present value formula}$$

$$100\,000 = \frac{x\left[1-\left(1+\frac{10\%}{12}\right)^{-48}\right]}{\frac{10\%}{12}}$$

$$100\,000 \times \frac{10\%}{12} = x(0{,}3285680008)$$

$$x = R2\,536{,}26$$

(b) James lost his job and is unable to make the 20th, 21st and 22nd payments. He wishes to still repay the loan by the end of the original 4 years. What will his new monthly payment be?

$$OB = P(1+i)^n - \frac{x[(1+i)^n-1]}{i} \ldots \text{ after the } 19^{th} \text{ payment}$$

$$= 100\,000\left(1+\frac{10\%}{12}\right)^{19} - \frac{2\,536{,}26\left[\left(1+\frac{10\%}{12}\right)^{19}-1\right]}{\frac{10\%}{12}}$$

$$= R65\,099{,}21$$

$A = P(1+i)^n$ …compound interest of the outstanding balance (3 months not paid)

$$= 65\,099{,}21\left(1+\frac{10\%}{12}\right)^3$$

$$= R66\,740{,}29$$

New payment: $P = \frac{x[1-(1+i)^{-n}]}{i}$ … present value formula

$66\,740{,}29 = \frac{x\left[1-\left(1+\frac{10\%}{12}\right)^{-26}\right]}{\frac{10\%}{12}}$ … left with 26 months as 22 months of 48 have been catered for

$$66\,740{,}29 \times \frac{10\%}{12} = x(0{,}1940784496$$

$x = R2\,865{,}69$ … new instalment

ACTIVITIES/ ASSESSMENT

1. Edward takes out a loan of R840 000 from the bank to start his own business. The loan is paid off by monthly instalments for a period of ten years after it was granted. The repayment started 12 months after the loan was granted. The interest rate is calculated at 18% p.a. compounded monthly. Calculate the monthly repayments.

2. Magda takes out a home loan of R100 000. She will repay the home loan by means of equal monthly payments starting 6 months after the loan was granted and ending 20 years after she took out the loan. The interest rate is 9,8% p.a. compounded monthly.

   (a) Calculate her monthly payment.
   (b) Calculate the outstanding balance on the loan immediately after the 100th payment.

3. Francois takes out a loan of R50 000, over a period of 2 years to renovate his home. The loan is to be repaid by means of equal monthly payments, starting one month after the loan was granted. The interest rate is calculated at 14,5% p.a. compounded monthly.

   (a) Calculate his monthly payment.
   (b) Francois is unable to make the 11th, 12th, 13th, and 14th payments. He still wants to repay the loan by the end of the original 2 years. Calculate his new monthly payment.

4. Lesego is granted a loan of R150 000. Payments are to be made half-yearly, starting 6 months after the loan was granted and ending 5 years after the loan was granted.

   (a) Calculate her half-yearly payments
   (b) Lesego misses the 3rd and the 4th payments. Calculate her new monthly payment if she still wants to repay the loan in the original time.

TEST 2: FINANCE, GROWTH AND DECAY

MARKS: 25 DURATION: 30 MIN.

Instructions to Candidates:

1. Answer ALL questions
2. Round off FINAL answer correct to TWO decimal places.
3. Choose relevant formula from the FORMULA SHEET

QUESTION 1 [8 Marks]

Mr. Brighton is a businessman in Pretoria; South Africa. In view of load shedding by Eskom, He decided to buy a large generator for R 235 652. It depreciates at 12% p.a. on a reducing balance. He wants to buy a new generator in 5 years' time. The old generator will be sold at a scrap value after five years. He sets up a sinking fund in order to save for the new generator. The proceeds from the sale of the old generator will be used together with the sinking fund to buy the new generator.

1.1 Determine the scrap value of the old machine in 5 years' time. 2

1.2 If the price of new machine increases by 17% per annum, determine how much a new machine will cost in 5 years' time. 2

1.3 Determine the equal monthly payments made into the sinking fund if the Interest earned is 13,2% p.a., compounded monthly and the first payment is made at the end of the month. 4

QUESTION 2 [17 Marks]

2.1 Clive applies for a student loan of R120000 to cover the costs of his first two years at University. The loan is approved at an interest rate of 14, 25% p.a. compounded monthly. He prefers to repay the loan in 24 equal monthly payments. These payments start 1 month after receiving the loan. Calculate:

2.1.1 his monthly repayments. 4

2.1.2 the outstanding balance immediately after his 16th payment has been made. 3

2.2 A loan of R50 000 is repaid over a period of 5 years by equal monthly payments, at an interest rate of 5.8% per annum compounds monthly. If payments start 6 months after the loan is granted, what monthly payments are required to repay the loan? 5

2.3 Mpho deposits R2 000 into a savings account at the end of every quarter for 16 years.

The account offers an interest rate of 8.5% p.a. compounded quarterly. Due to financial difficulty, he is unable to make his last two payments. How much will he have in his account after 16years? 5

TOPIC: DATA HANDLING (Lesson 1)	Weighting	20 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Ungrouped Data			
RELATED CONCEPTS/ TERMS/VOCABULARY	Five-Number-Summary, Box-and-Whisker diagram, outlier			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Measure of Central Tendency, Measure of Dispersion, stem-and-leaf plot, Bar graph

RESOURCES



ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confusing bar graph and a histogram
- Failing to understand relationship between quartiles and percentiles
- Failing to differentiate between position and value of the quartile
- Failing to differentiate between variance and standard deviation on the calculator

METHODOLOGY

Calculator skills are very important in this chapter. Methods for SHARP and CASIO calculators are shown but practical demonstration may be required.

Statistics is the branch of Mathematics dealing with collection, organization, presentation, analysis, and interpretation of **data**. Data can either be **discrete or continuous.**

Discrete data is counted (number of learners in a class, number of people in the household)
Continuous data is measured (height of a person, time in a race)

UNGROUPED DATA

Ungrouped data can be represented in different forms, e.g., frequency tables, bar graphs, pie charts, stem-and-leaf, etc.

A MEASURE OF CENTRAL TENDENCY (also referred to as measures of centre or central location) is a summary measure that attempts to describe a whole set of data with a single value that represents the middle or centre of its distribution. There are **three** main measures of central tendency:

1. Mean is the sum of the values of each observation in data set divided by the number of observations.

- The mean can be used for both continuous and discrete numeric data.
- The mean is influenced by the outliers and skewness of data.

Formula: $\bar{x} = \frac{\sum x}{n}$ ... ungrouped data

$\bar{x}$ = Mean $\sum x$ = sum of the data values and n = sample space

2. The **median** is the middle value in the distribution when the values are arranged in ascending or descending order. The median **divides the distribution in half.**

- **Median** is less affected by outliers than the mean. Use the median position to determine the estimated median.

3. **Mode** is the most commonly occurring value/item in a distribution.

- The **mode** has the advantage over the median and mean as it can be found in both quantitative and qualitative data.
- There can be more than one mode in the distribution, thus one single measure of central tendency cannot be identified.

An outlier is an **extremely high** or **extremely** low value in a data i.e., data point that is very far from other points.
Always include an outlier in your calculations, unless a question a question specifically states that you have to omit it.

MEASURES OF DISPERSION

In statistics, dispersion (also called variability, scatter, or spread) is the extent to which a distribution is stretched or squeezed. Common examples of statistical dispersion are the **variance, standard deviation, and interquartile range.**

1. **Variance** is the average of the squared differences from the mean.

Formula: Variance(σ^2) $= \frac{\Sigma(x-\bar{x})^2}{n}$

2. **Standard deviation** is a statistic that measures the dispersion of a dataset relative to the mean. It shows how **much data is spread out around the mean.**

- Low standard deviation indicates that the values are closer to the mean of the data.
- High standard deviation indicates that the values are spread out, data points are further from the mean, thus the more spread out the data, the higher the standard deviation.

The standard deviation (SD) is calculated as the square root of the variance.

Formal: $SD(\sigma) = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$

3. **The interquartile range** (IQR) is a measure of variability, based on dividing a data set into quartiles. **Quartile** are the values that divide a list of numbers into quarters. The values that separate parts are called first quartile ($Q1$), second quartile ($Q2$), and third quartile ($Q3$).

$IQR = Q_3 - Q_1$

FIVE NUMBER SUMMARY AND BOX AND WHISKERS PLOT

The five-number summary includes five items:
1. The minimum
2. The lower quartile (Q_1)
3. The median (Q_2)
4. The upper quartile (Q_3)
5. The maximum

For the five-number summary to exist, the data must meet the following two requirements:
- Data must be univariate
- Data must be ordinal, interval, or ratio.

BOX AND WHIKERS PLOT

The box and whiskers plot is a visual representation of the five-number summary
- $\bar{x} > Q_2$, data is positively skewed/ skewed to the right ... **(Q_2 is the Median)**
- $\bar{x} < Q_2$, data is negatively skewed/ skewed to the left

Example:

Data below represents marks (out of 50) of learners in a Mathematics Test:

17 38 10 17 44 26 20 22 9 17 20 38 10
22 20 9 44 38 17 10 22 20 9 44 5

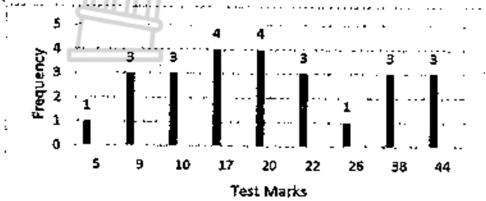
1. Represent the given data in a frequency table, bar graph and stem-and-leaf.
2. Determine the mean, median and the mode.
3. Calculate the standard deviation of the test marks.
4. Write down the Five-number summary of the data.
5. Draw box-and-whisker diagram, using intervals of 5.
6. Comment on the skewness of the data.

Firstly, write data in ascending order: 5 9 9 9 10 10 10 17 17 17 17 20 20
20 20 22 22 22 26 38 38 38 44 44 44

1. Frequency table:

Marks	Tally	Frequency
5	/	1
9	///	3
10	///	3
17	////	4
20	////	4
22	///	3
26	/	1
38	///	3
44	///	3
		$n = 25$

Bar graph:



Stem-and-leaf:

0	5 9 9 9
1	0 0 0 7 7 7 7
2	0 0 0 0 2 2 2 6
3	8 8 8
4	4 4 4

2. Mean: Use calculator = 21,92   Median = 20   Mode = 17 and 20

3. Standard deviation: use calculator = 11,97

4. Five-Number-Summary:

5 9 9 9 10 10 10 17 17 17 17 20 20
20 20 22 22 22 26 38 38 38 44 44 44

$Minimum = 5,\ \ Q_1 = \frac{10+10}{2} = 10,\ \ Median = 20,\ \ Q_3 = \frac{26+38}{2} = 32\ \ and\ \ Maximun = 44$

5. Box-and-Whisker:

TO BE DRAWN

6. Standard deviation is less than the median, therefore, data is negatively skewed.

ACTIVITIES/ ASSESSMENT

1. The Kwazulu-Natal Comrades marathon run annually between the cities of Durban and Pietermaritzburg is the world's largest and oldest ultramarathon race of approximately 89 km. Below is the summary of number of wins by some athletes to date:

4 5 9 3 5 3 2 7 4 5 3 5

(a) Calculate the mean for this data.

(b) What is the mode for the number of wins per athlete?

(c) Draw a box and whisker plot for this data.

(d) Comment on the distribution of the data in terms of skewness.

(e) Calculate the standard deviation for this data (one decimal place).

(f) How many athletes lie outside the first standard deviation interval?

(g) Why is number of wins 9 an outlier? Give a mathematical reason to justify your answer.

(h) Draw a box and whisker plot for the data highlighting the outlier.

2. The ordered data below lists the fees (in thousands of Rands) paid by 30 students for their first year of study at university.

11	11,9	12	12,5	13,13	13,13	13,15	13.2	13,25	13,25
13,25	13,25	13,4	13,5	13,7	13,7	14	14,2	14,2	14,5
14,5	15	15,5	15,6	16,1	17,3	17,5	17,8	19	19,7

(a) Calculate the mean and standard deviation of the given data.

(b) Draw the box and whisker diagram for the given data.

(c) Comment on the skewness of the data.

3. The following data set has a mean of 14,7 and a variance of 10,01.

18 11 12 a 16 11 19 14 b 13

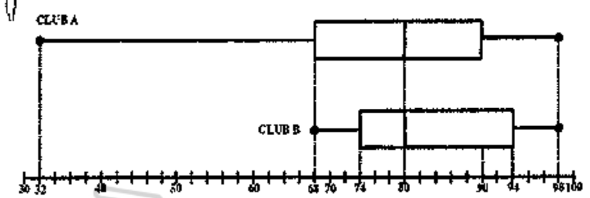
Calculate the values of a and b.

4. There are 184 students taking Mathematics in a first-year university class. The marks, out of 100, in the half-yearly examination are normally distributed with a mean of 72 and a standard deviation of 9.

(a) What percentage of students scored between 72 and 90 marks?

(b) Approximately how many students scored between 45 and 63 marks?

5. The box and whisker plots below represent the distances run by athletes from two different running clubs over a period of one month.



(a) What features are the same for both clubs?
(b) It seems that there is no significant difference in the performance between the two clubs. Is this conclusion valid? Support your answer with reasons.
(c) Comment on the distribution of distances for Club A if the mean is 76.
(d) Determine whether the minimum or maximum values for Club A are outliers.
(e) Does Club B have any outliers? Explain.

TOPIC: DATA HANDLING (Lesson 2)	Weighting	20 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Grouped Data			
RELATED CONCEPTS/ TERMS/VOCABULARY	Modal class, midpoints of intervals, ogive (cumulative frequency curve)			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Frequency, histogram

RESOURCES



ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confusing a bar graph with a histogram.
- Confusing position and the value of the median
- Failing to read modal class correctly

METHODOLOGY

GROUPED DATA

Grouped data is data that has been **grouped together in categories**. Histograms, bar graphs and frequency tables are some of the tools to use to represent grouped data

- About grouped data, measures of central tendency are not exact values but **estimated** ones.
- In a case of a mode, learners are expected to only determine the modal class and not estimated mode.
  Modal class is the group/class/interval with the highest frequency.
- Median position formula will help to identify the interval in which the median lies.
  $Median_{position} = \frac{n+1}{2}$, where n is the total frequency
- Estimated mean can be calculated using the midpoints of intervals.

$\bar{x} = \frac{\Sigma f.x}{\Sigma f}$, where x is the class midpoint and f is class frequency

Example:

Alex timed 21 people in the sprint race, to the nearest second. The information is represented in the frequency table below:

SECOND	FREQUENCY
51 - 55	2
56 - 60	7
61 - 65	8
66 - 70	4

1. Write down the modal class.

Remember that modal class is the one with the highest frequency. Therefore, the modal class is 61 - 65

2. Determine the estimated mean.

Estimated mean can be calculated using the midpoints of intervals, therefore, we need a table

SECOND	FREQUENCY (f)	MIDPOINT INTERVAL (x)	Frequency × midpoint ($f \times x$) = fx
51 – 55	2	53	106
56 – 60	7	58	406
61 – 65	8	63	504
66 - 70	4	68	272
	21		1288

$Midpoint = \frac{lower\ limit + upper\ limit}{2}$ e.g., $\frac{51+55}{2} = 53$

Estimated Mean: $\bar{x} = \frac{\Sigma f.x}{\Sigma f}$

$= \frac{1288}{21} = 61{,}33$

3. Determine the estimated median.

$Median_{position} = \frac{n+1}{2}$

$= \frac{21+1}{2} = 11$ and the estimated median is in the range 61-6.

OGIVE (CUMULATIVE FREQUENCY GRAPH)

Ogive is a graphical representation of cumulative frequency.

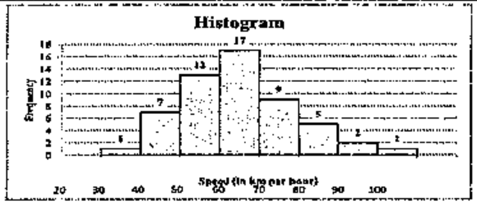
Cumulative frequency distribution is the sum of the class and all classes below it in a frequency distribution.

DRAWING OGIVE (CUMULATIVE FREQUENCY GRAPH)

- Complete the cumulative frequency column
- Plot the points of the cumulative column on the upper limit of the intervals
- Join the points for an ogive
- Grounding on the lower limit of the first class must also be drawn.
- Please take note that interpretation of drawn ogive to determine estimated quartiles, mean, and median is very important.

Example:

The speeds of 55 cars passing through a certain section of a road are monitored for one hour. The speed limit on this section of road is 60 km per hour. A histogram is drawn to represent of this data.



1. Identify the modal class of the data.

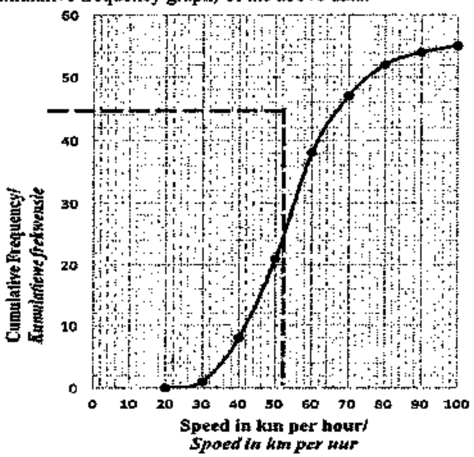
   $50 < x \le 60$ **OR** $50 \le x < 60$ **OR** between 50 and 60

2. Use the histogram to:

   2.1 Draw and complete the cumulative frequency table.

CLASS	FREQUENCY	CUMMULATIVE FREUENCY
$20 < x \le 30$	1	1
$30 < x \le 40$	7	8
$40 < x \le 50$	13	21
$50 < x \le 60$	17	38
$60 < x \le 70$	9	47
$70 < x \le 80$	5	52
$80 < x \le 90$	2	54
$90 < x \le 100$	1	55

   2.2 Draw an ogive (cumulative frequency graph) of the above data.



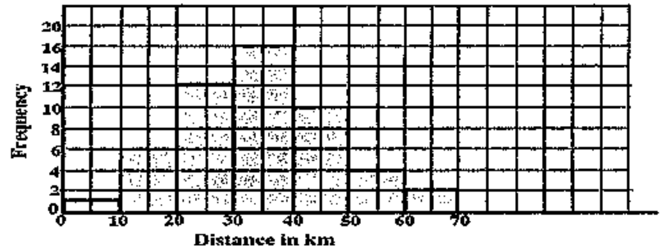
3 The traffic department sends speeding fines to all motorists whose speed exceeds 66 km per hour. Estimate the number of motorists who will receive a speeding fine.

$55 - 44 = 11 \therefore \approx 11$ motorists

$55 - 43$ and $55 - 45$ also accepted.

ACTIVITIES/ ASSESSMENT

1. The following histogram represents the distance run by a few Comrades Marathon runners who didn't complete the 89 km race.



(a) Redraw and complete the following table:

CLASS INTERVAL	FREQUENCY	CUMMULATIVE FREQUENCY
$0 < x \le 10$		
$10 < x \le 20$		
$20 < x \le 30$		
$30 < x \le 40$		
$40 < x \le 50$		
$50 < x \le 60$		
$60 < x \le 70$		

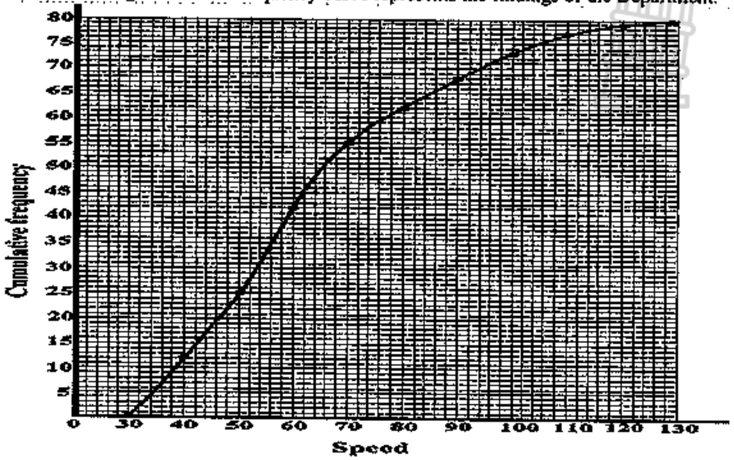
(b) Calculate the estimated mean.
(c) State the modal class.
(d) Draw the cumulative frequency curve for this data.
(e) Determine estimates for the quartiles.
(f) Draw a box and whisker diagram if the minimum distance is 10 km and the maximum distance is 70 km.
(h) Comment on the distribution of the data in terms of skewness.

2. A company has 132 employees in their Polokwane branch. The distance (x) in kilometres which they travel to work each day, is summarised in the following frequency table:

DISTANCE (in kms)	FREQUENCY
$5 < x \le 10$	12
$10 < x \le 15$	29
$15 < x \le 20$	48
$20 < x \le 25$	27
$25 < x \le 30$	13
$30 < x \le 35$	3

(a) Determine the estimated mean distance covered by the employees.
(b) Determine the standard deviation for the data, to two decimal places.
(c) Draw an ogive for this data on a set of axes provided in the answer book.
(d) Determine the median distance travelled, showing on the graph where your answer is read off.
(e) By referring to the relationship between the mean and median, discuss the distribution of the data.

3. Residents in Knysna complained to the Police department that too many drivers were exceeding the speed limit of 60 km/h on a busy road near a school. The department recorded the speeds of drivers along this road over one week. The speed limit is 60 km/h.
The following cumulative frequency curve represents the findings of the Department.



(a) How many drivers were there in total?
(b) How many drivers exceeded the speed limit of 60 km/h?
(c) How many drivers were within the speed limit?
(d) What is the median speed for this data?
(e) How many drivers had a speed of less than 80 km/h?
(f) What percentage of drivers had a speed above 100 km/h?
(g) Complete the following table:

SPEED (in km/h)	FREQUENCY	CUMMULATIVE FREQUENCY
$30 \leq x < 40$		12
$40 \leq x < 50$		
$50 \leq x < 60$		
$60 \leq x < 70$		
$70 \leq x < 80$		
$80 \leq x < 90$		
$90 \leq x < 100$		
$100 \leq x < 110$		
$110 \leq x < 120$		
$120 \leq x < 130$		

(h) Calculate an estimated mean speed.
(i) Calculate an estimated value for the standard deviation of this data.
(Hint: Use the midpoints of the class intervals as the x-values to enter in your calculator)

TOPIC: DATA HANDLING (Lesson 3)		Weighting	20 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Scatterplots: Least Squares Regression Line				
RELATED CONCEPTS/ TERMS/VOCABULARY	Bivariate, scatterplot, correlation, line of best fit/regression line				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Dependent variables, independent variables, y-intercept, gradient, straight line, outlier

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

- Confuse dependent variable with independent variable
- Could not relate least squares regression line with real life context

METHODOLOGY

Scatterplot is a plot of bivariate data plotted along two axes, one variable on the x-axis and the other on the y-axis.

Bivariate data is a data for two variables (usually two types of related data). The other variable is a **dependent** and the other is **independent** variable.

It is important to identify the **dependent** and **independent variables** in case they are not indicated in the information.

Scatterplots are used to **compare two sets of data** that might have a relationship or **correlation.**

Correlation is a measure of how things are related.

The data in a scatterplot could follow a **linear, quadratic or exponential** trend.

When data follows a **linear trend**, a **line of best** fit can be drawn on the scatterplot diagram.

Line of best fit is the **straight line** that best represents the relationship between two variables in a linear trend, it is also called the **least squares regression line.**

The equation of the least squares regression line is written in the form: $\hat{y} = a + bx$, where

$a = y - intercept$ $\qquad$ $b = gradient$

The line of best fit or least squares regression line always passes through the point: $(\bar{x}; \bar{y})$

Therefore, to draw the least squares regression line, plot the **y-intercept** and **mean point** $(\bar{x}; \bar{y})$ and draw the straight line through these points.

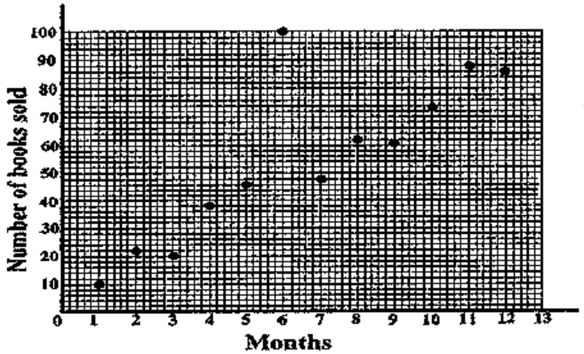
A **calculator** can be used to determine the values of $(\bar{x}; \bar{y})$ and the values of a and b. **Always include outliers in your calculations, unless a question states that you have to omit them.**

Example:

Consider the following scatterplot of information obtained by a publishing company that recorded the number of books sold over a period of 12 months.
The results are recorded in the following table:

No. of Months	1	2	3	4	5	6	7	8	9	10	11	12
No. of Books Sold	10	22	20	38	46	100	48	62	61	74	88	86

(a) Represent the data in a scatterplot.



(b) Determine the values of $(\bar{x};\bar{y})$ **if the outlier is excluded.**

$$\bar{x} = \frac{1+2+3+4+5+7+8+9+10+11+12}{11} = 6{,}54545454 \approx 6{,}5$$

$$\bar{y} = \frac{10+22+20+38+46+48+62+61+74+88+86}{11} = 50{,}45454545 \approx 50{,}5$$

NOTE: Outlier was not included because it is too far away from other points.

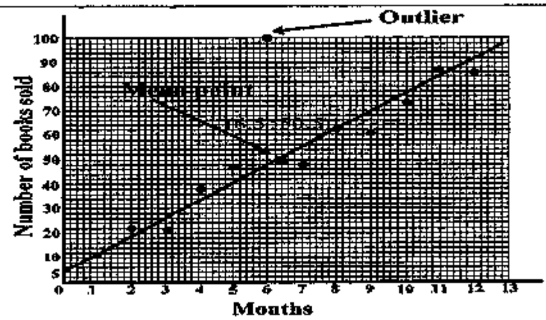
(c) Determine the equation of the least squares regression line **if the outlier is excluded.**

Use a calculator: CASIO $fx-82ES\ PLUS$
$MODE \to 2 \to 2 \to AC \to SHIFT\ 1 \to 5 \to 1$ *to get the value of* $a \to AC \to SHIFT\ 1 \to 5 \to 2$ *to get the value of* b.

$a = 5{,}315923567$ and $b = 6{,}896178344$

$\therefore\ \hat{y} = 5{,}315923567 + 6{,}896178344x$

(d) Draw the least squares regression line on the scatterplot.



(e) Write down outlier(s), if exists.

(6; 100)

ACTIVITIES/ ASSESSMENT

1. A medical researcher recorded the growth in the number of bacteria over a period of 10 hours.
The results are recorded in the following table:

Time in hours	0	1	2	3	4	5	6	7	8	9	10	11
Number of Bacteria	5	10	75	13	10	20	30	35	5	65	80	I

(a) Draw a scatterplot to represent this data.
(b) State the type of relationship (linear, quadratic or exponential) that exists between the number of hours and the growth in the number of bacteria.
(c) Are there any outliers? Write them down if they exist.

2. The table below shows the average maintenance cost in rands of a certain model of car compared to the age of the car in years.

Age (x)	1	3	5	6	8	9	10
Cost (y)	1000	1500	1600	18000	2000	2400	2600

(a) Draw the least squares regression line.
(b) Determine the equation of the least squares regression line. Round a and b to two decimal places.
(c) Use your equation to estimate what it would cost to maintain this model of car in its 15th year.
(d) Use your equation to estimate the age of the car in the year where the maintenance cost totals over R 3000 for the first time.

3. The table below represents the distance in metres required by a car to apply brakes and reach a standstill when it is travelling at a given speed.

Speed km/h	20	40	60	80	100	120	140
Breaking Distance	6	16	30	48	70	80	110

(a) Draw a scatter plot to represent this data.
(b) Explain whether a linear, quadratic or exponential curve would be a line or curve of best fit.

(c) Determine the equation of the regression line.
(d) Draw the regression line on the scatterplot diagram.
(e) Use your regression line to estimate the breaking distance at a speed of 130 km/h

4. The ages and kilometres on the clock of ten second hand cars are given in the table below.

Age in years (x)	1	2	3,5	4	5	5	6	6,5	7	8
KM in thousands (y)	15	33	50	65	73	95	84	100	95	110

(a) Represent this data by means of a scatterplot.
(b) Determine the equation of the least squares regression line.
(c) Sketch the least squares regression line on the scatterplot.

5. The table below shows the acidity of eight dams near an industrial plant in Limpopo and their distance from it.

Distance (km)	4	34	17	60	6	52	42	31
Acidity (pH)	3,0	4,4	3,3	7,0	3,2	6,8	5,2	4,8

(a) Draw a scatterplot to represent this data.
(b) Determine the equation of the regression line.
(c) Draw a line of best fit on the diagram.
(d) Use your line of best fit to predict the acidity of the dam at a distance of 25 kilometres.

6. A Gauteng motor company did research on how the speed of a car affects the fuel consumption of the vehicle. The following data was obtained:

Speed in km/h	60	75	115	85	110	95	120	100	70
Fuel consumption in ℓ/100 km	11,5	10	8,4	9,2	7,8	8,9	8,8	8,6	10,2

(a) Draw a scatter plot to represent this bivariate data.
(b) Suggest whether a linear, quadratic or exponential function would best fit the data.
(c) What advice can the company give about the driving speed in order to keep the cost of fuel to a minimum?

TOPIC: DATA HANDLING (Lesson 4)	Weighting	20 ± 3	Grade	12
Term		Week no.		
Duration	1 hour	Date		
Sub-topics	Correlation			
RELATED CONCEPTS/ TERMS/VOCABULARY	Correlation coefficient, positive correlation, negative correlation			

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Straight line, calculator usage, least squares regression line, gradient

RESOURCES



ERRORS/MISCONCEPTIONS/PROBLEM AREAS

No virtual understanding of correlation coefficient

METHODOLOGY

Correlation is a measure of how things are related i.e., the **strength of the linear relationship** between two variables in a scatterplot. It depends on how close the data points are to the line of best fit.

When the **points are too close** to the least squares regression line, the **relationship** between two variables is **strong** and if the **points are far away** from the line of best fit, the **relationship** between the variables is **weak**.

Correlation between two variables can be **positive**, **negative** and **the gradient** of the line of best fit indicates whether the association is positive or negative.

Correlation coefficient (r) is a value that gives an indication of the strength of the linear relationship between two variables. It is a numerical **value between – 1 and 1**, where a coefficient of **+1** indicates a **perfect positive correlation** and a coefficient of **-1** indicates a **perfect negative correlation**.

Correlation coefficient can be calculated using a calculator.

The following table provides different examples of r and how to interpret these values.

If $r =$	Interpretation
1	**Perfect** Positive Linear Correlation
0,9	**Strong** Positive Linear Correlation
0,5	**Moderate** Positive Linear Correlation
0,2	**Weak** Positive Linear Correlation
Between – 0,2 and 0,2	**No Correlation** when $r = 0$
– 0,2	**Weak** Negative Linear Correlation
– 0,5	**Moderate** Negative Linear Correlation
– 0,9	**Strong** Negative Linear Correlation
-1	**Perfect** Negative Linear Correlation

Example:

The data below shows the marks obtained by ten grade 12 learners from each of the 2 different mathematics classes.

Class A	16	36	20	38	40	30	35	22	40	24
Class B	45	70	44	56	60	48	75	60	63	38

(a) Calculate the correlation coefficient.

$MODE \to 2 \to 2 \to Enter\ data\ points \to AC \to SHIFT \to 1 \to 5 \to 3: r = 0{,}6624986603$

$\therefore r = 0{,}66$

(b) Comment on the strength of the relationship.

Moderate Positive Correlation

ACTIVITIES/ ASSESSMENT

1. A taxi driver recorded the number of kilometres his taxi travelled per trip and his fuel cost per kilometre in Rands.
Examine the table of his data below and answer the questions that follow.

Distance (x)	3	5	7	9	11	13	15	17	20	25	30
Cost (y)	2,8	2,5	2,46	2,42	2,4	2,36	2,32	2,3	2,25	2,22	2

(a) Draw a scatter plot of the data.
(b) Use your calculator to determine the equation of the least squares regression line and draw this line on your scatter plot. Round a and b to two decimal places in your final answer.
(c) Using your calculator, determine the correlation coefficient to two decimal places.
(d) Describe the relationship between the distance travelled per trip and the fuel cost per kilometre.
(e) Predict the distance travelled if the cost per kilometre is R 1;75.

2. The table shows the number of kilometres still to go after different ten-minute periods. Calculate the correlation coefficient and comment on the strength of the linear association.

Class A	16	36	20	38	40	30	35	22	40	24
Class B	45	70	44	56	60	48	75	60	63	38

3. A survey was conducted indicating the number of bees that visited flowers over a period of 12 days. The information is represented in the table and in the scatter plot below.

No. of Flowers	4	10	7	12	1	6	2	5	11	9	8	3
No. of Bees	30	22	20	38	65	160	48	62	61	74	88	86

(a) Write down the coordinates of the outlier.
(b) Represent the data on a scatterplot.
(c) Determine the equation for the least squares regression line.
(d) Calculate the correlation coefficient if the outlier is excluded.
(e) Comment on the trend of the data.

4. The table below gives the years of experience of employees in the IT Department of a company and their monthly salaries.

No. of Flowers	4	10	7	12	1	6	2	5	11	9	8	3
No. of Bees	30	22	20	38	65	160	48	62	61	74	88	86

(a) Represent the data by means of a scatterplot.
(b) Identify the outlier in the data set.
(c) Determine the equation of the least squares regression line.
(d) Determine the correlation coefficient.
(e) Estimate the monthly salary that a person who has 15 years of experience would earn.
(f) Determine the least squares regression line and the correlation coefficient if the outlier is removed from the data set.
(g) Use the least squares regression line in (f) to predict how much a person would earn if they have 15 years experience.

5. The table below compares the number of hours spent by 7 Mathematics learners and the learners' performance in the test.

Hours	1	3	5	6	8	10	11
Marks (%)	35	55	60	65	75	70	80

(a) Represent the data in a scatter plot.
(b) Identify the outlier.
(c) Determine the equation of the line of best fit for the data.
(d) Hence, draw the line of best fit on the scatter plot.
(e) Determine the correlation coefficient of the data. Hence, comment about the strength of the relationship between the two variables.

TEST: DATA HANDLING

MARKS: 25 DURATION: 30 MIN.

INSTRUCRION

1. Answer ALL questions
2. Unless stated or otherwise, round off answers correct to TWO decimal places
3. You may use an approved scientific calculator

QUESTION 1 [9 Marks]

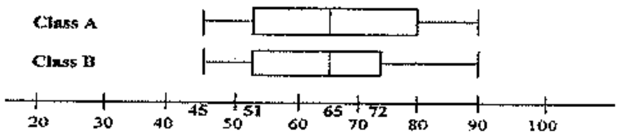
The ages of 500 people attending a concert are given in the table below.

Age in Years	Number of People	Cumulative Frequency
$0 \le A < 10$	20	
$10 \le A < 20$	130	
$20 \le A < 30$	152	
$30 \le A < 40$	92	
$40 \le A < 50$	86	
$50 \le A < 60$	18	
$60 \le A < 70$	2	

1.1 Complete the cumulative frequency column above. (2)
1.2 Draw an ogive (cumulative frequency graph) of the above data. (3)
1.3 Use your cumulative frequency graph to estimate:
1.3.1 the median age (1)
1.3.2 the percentage of people at the concert who are 16 years and older. (3)

QUESTION 2 [16 Marks]

2.1 Mrs Smith has two classes, each having 30 learners. Their final marks (out of 100) are represented in the box-and-whisker diagram below:



2.1.1 Determine the interquartile range of Class B. (2)

2.1.2 Explain the significance in the difference of the length of the boxes in the diagram. (2)

2.1.3 Mrs Smith studied the results and made the comment that there was no significance difference in the performance of the two classes. Give TWO reasons you think Mrs Smith will use to prove her statement. (2)

2.2 Eight couples entered a dance competition. The performances were scored by two judges. The scores (out of 20) are given in the table below.

COUPLE	1	2	3	4	5	6	7	8
JUDGE 1	18	4	6	8	5	12	10	14
JUDGE 2	15	6	3	5	5	14	8	15

2.2.1 Represent the data in a scatterplot. (2)
2.2.2 Determine the equation of the least squares regression line of the scores given by the judges. (3)
2.2.3 Hence, draw the line of best fit. (2)
2.2.4 Determine the correlation coefficient of the data. Hence, comment about the strength of the relationship between the two judges. (3)

TOPIC: COUNTING AND PROBABILITY (Lesson 1)		Weighting	15 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Revision**				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Venn Diagrams, Tree, Diagrams, Contingency Table, Independent Events

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Failing to understand that Venn diagram must total to sample space.
Putting events as they are without subtracting intersection from given events.
Do not start with the intersection of all events.

METHODOLOGY

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Alternative Notation: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $P(A \cap B) \neq 0$

Mutually Exclusive Events are events that cannot happen in the same trial.

$$P(A \text{ or } B) = P(A) + P(B), \text{ where } P(A \text{ and } B) = 0$$

Alternative Notation: $P(A \cup B) = P(A) + P(B), \text{ where } P(A \cap B) = 0$

Complementary events are two mutually exclusive events that together contain all the outcomes in the sample space.

- Complementary events are **mutually exclusive**
- And $P(A) + P(B) = 1$ (Exhaustive).

$P(\text{not } A) = P(B)$ **OR** $P(\text{not}A) = 1 - P(A)$

$P(\text{not}A)$ can be written as $P(A')$

Venn diagrams are a useful tool for recording and visualising the counts and it may be used for any number of events.

Tree diagrams are useful for organising and visualising the different possible outcomes of a sequence of events. Drawing a tree diagram provides **a systematic way of generating all the possible outcomes**

A contingency table is another tool for keeping a record of the counts or percentages in a probability problem.

Independent events are events where the outcome of the second event is not affected by the outcomes of the first event.

If event A and event B are independent, then $P(A \text{ and } B) = P(A) \times P(B)$

ACTIVITIES/ ASSESSMENT

1. Let A and B be two events in a sample space. Suppose that P(A) = 0.4; P(A or B) = 0.7 and P(B) = k.

   (a) For what value of k are A and B mutually exclusive?
   (b) or what value of k are A and B independent?

2. In a random experiment it was found that: P(A) = 0.25; P(B) = 0.5 and P(A or B) = 0.625

   (a) Calculate P(A and B)
   (b) Determine, giving reasons, if events A and B are:

      1) mutually exclusive or inclusive.
      2) complementary

3. At a school for boys there are 240 learners in Grade 12. The following information was gathered about participation in school sport.
   - 122 boys play rugby (R)
   - 58 boys play basketball (B)
   - 96 boys play cricket (C)
   - 16 boys play all three sports
   - 22 boys play rugby and basketball
   - 26 boys play cricket and basketball
   - 26 boys do not play any of these sports

   Let the number of learners who play rugby and cricket only be x

   (a) Draw a Venn diagram to represent the above information.
   (b) Determine the number of boys who play rugby and cricket.
   (c) Determine the probability that a learner in Grade 12 selected at random:
      1) does not play cricket
      2) participates in at least 2 of these sports.

4. There are 20 boys and 15 girls in a class. The teacher chooses individual learners at random to deliver a speech.

   (a) Draw a tree diagram to represent the situation if the teacher chooses three learners, one after the other. Indicate on your diagram ALL possible outcomes.
   (b) Calculate the probability that all three learners chosen are girls.
   (c) What is the probability that 5 learners chosen are of the same gender?

TOPIC: COUNTING AND PROBABILITY (Lesson 2)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Fundamental Counting Principles				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Multiplication, $n(E)$, possible outcome

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Can't differentiate between $n(E)$ and P (E)

METHODOLOGY

The **Fundamental Counting Principle** is the guiding rule for finding the number of ways to accomplish two or more tasks.
The fundamental counting principle states that if there are m possible outcomes in event A and n possible outcomes in event B, then total possible number of outcomes for both events is $m \times n$.

Examples:

1. A take-away has a 4-piece lunch special which consists of a sandwich, soup, dessert and drink for R 25;00. They offer the following choices for:
Sandwich: chicken mayonnaise, cheese and tomato, tuna mayonnaise, ham and lettuce
Soup: tomato, chicken noodle, vegetable
Dessert: ice-cream, piece of cake
Drink: tea, coffee, Coke, Fanta, Sprite

(a) How many possible meals are there?

Number of possible meals = 4 x 3 x 2 x 5 = 120

(b) If a particular person wishes to have Coke as a drink. How many possible meals can be chosen?

Number of possible meals = 4 x 3 x 2 x 1 = 24

2. Khanyi has 3 pairs of shoes, 4 trousers and 6 jerseys. Use counting principles to

Determine how many different combinations he has.

Number of different combinations = 3 x 4 x 6 = 72

3. Brent wants to buy running shoes. There are two types (road running or trail running shoes), 9 brands and 4 colours available at the shopping centre. How many choices des Brent have?

Number of choices = 2 x 9 x 4 = 72

ACTIVITIES/ ASSESSMENT

1. Tarryn has five different skirts, four different tops and three pairs of shoes. Assuming that all the colours complement each other, how many different outfits can she put together?

2. A party pack of three items can be made up by selecting one item from each of the following choices:

Choice 1: Smarties, Astros, Jelly Tots, Wine Gums
Choice 2: Coke, Fanta, Sprite, Ginger Beer, Crème Soda
Choice 3: Doughnut, Chelsea Bun, Cheese Roll

How many different party packs can be made?

3. Ronald has two pairs of Jeans and five t-shirts and three hats. How many different outfits does Ronald has?

4. Thembi has three lipsticks, six shades of eye shadow and two tubes of mascara in her drawer.
(a) How many looks can she create if she uses only one of each?
(b) How many looks can she create if she loses one lipstick?

5. The Menu at Mama Mary's Kitchen offers three starters, four main dishes and five desserts.
The Menu at Tauriq's Bistro offers four starters, six main dishes and eight desserts.
(a) Find the difference in the number of choices customers have between the two restaurants if they select one starter, one main dish and one dessert.
(b) Another restaurant advertises: 'More than 300 options on our Menu!'. How can they make this claim if they only offer starters, main dishes and desserts?

6. During the time of registration of a student at the University of South Africa, three subject groups are presented to him for the B. Com course that he wants to follow.

SUBJECT GROUP A	SUBJECT GROUP B	SUBJECT GROUP C
Accounting	Financial Management	Applied Mathematics
Audit	Financial Mathematics	Physics
Taxation	Office Management	Chemistry
Statistics	Psychology	Biochemistry
Mathematics		Physiology

(a) If a student decides to choose one subject from group A and one subject from group B, how many different combinations are possible?

(b) If one subject is chosen from each of the three subject groups and you know that you will definitely not choose Biochemistry and Chemistry, how many combinations will there be to choose from?

TOPIC: COUNTING AND PROBABILITY (Lesson 3)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Fundamental Counting Principles				
RELATED CONCEPTS/ TERMS/VOCABULARY	Repeat, options				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Events, exponents

RESOURCES


ERRORS/MISCONCEPTIONS/PROBLEM AREAS

METHODOLOGY

Fundamental Counting principle is used **to solve problems,** when a problem involves two or more separate events.

NOTE:

- In all events, the **order of events** matters
- Sometimes **repetition** is allowed and sometimes not.
- Make sure you understand the difference between '**repetition allowed**' and '**repetition not allowed**'

REPETITION ALLOWED

In cases where repetition is allowed:

$$\textbf{Number of possibilities} = n^r$$

n = number of options to choose from
r = number of times chosen from the options

Examples:

1. A school plays a series of 6 soccer matches. For each match there are 3 possibilities: a win, a draw or a loss. How many possible results are there for the series?

Solution

- There are 3 outcomes for each match: win, draw or lose.
- There are 6 matches, therefore the number of events is 6.
- There are 3 possible outcomes for each of the 6 events.

∴ the total number of possible outcomes for the series of matches is:
$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$

2. How many different ways are there of predicting the results of 5 rugby matches where each match can end in either a **win, lose or draw**?

Solution

- 3 outcomes
- 5 matches
- 3 possible outcomes for each of 5 matches

∴ total different ways of predicting is $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$

3. A code consists of 3 vowels, followed by two prime numbers less than 10. How many codes can you form if you may use a vowel or a prime number **more than once**?

Solution:

- There are 5 vowels in total (a, e, i, o, u)
- There are 4 prime numbers less than 10 (2, 3, 5, 7)
- Therefore, vowels are 5 x 5 x 5 (choose from 5 vowels 3 times)
- Prime numbers are arranged as 4 x 4 (choose from 4 prime numbers 2 times)

∴ the total number of codes that can be formed is $\underline{5} \times \underline{5} \times \underline{5} \times \underline{4} \times \underline{4} = 2\ 000$

REPETITION NOT ALLOWED

4. A three-letter code consists of vowels. How many 3-letter codes can be formed if vowels are not used more than once?

Solution:

The first letter will be chosen from 5 vowels and 4 vowels are left
The second letter will be chosen from 4 vowels and 3 vowels are left
The third letter will be chosen from 3 vowels and two vowels are left untouched
Therefore, number of 3-letter codes = 5 x 4 x 3 = 60

5. A code consists of 3 vowels, followed by two prime numbers less than 10. How many codes can be formed if it is not allowed to use a vowel or a prime number more than once?

Solution:

5 x 4 x 3 x 4 x 3 = 720
Vowels x Numbers

ACTIVITIES/ ASSESSMENT

1. How many different ways are there of predicting the results of six soccer matches where each match can end in either a win or a lose?

2. A party pack of three items can be made up by selecting one item from each of the following choices:

Choice 1: Smarties, Astros, Jelly Tots, Wine Gums
Choice 2: Coke, Fanta, Sprite, Ginger Beer, Crème Soda
Choice 3: Doughnut, Chelsea Bun, Cheese Roll

How many different party packs can be made?

3. In a multiple-choice question paper of 20 questions the answers can be A, B, C or D. How many different ways are there of answering the question paper?

4. How many three-digit numbers can be made from the digits 1 to 6 if:
   (a) repetition is not allowed?
   (b) repetition is allowed?

5. A debit card requires a five-digit personal identification number (PIN) consisting of digits from 0 to 9 How many possible PINs are there?
   (a) if digits may be repeated
   (b) If digits may not be repeated

6. A password is to be made up using any 3 digits from 0 to 9 and any 2 letters of the alphabet. The digits may be repeated as well as the letters of the alphabet.
   How many different passwords can be formed?
   (a) if the digits may be repeated as well as the letters of the alphabet.
   (b) If the digits may not be repeated as well as the letters of the alphabet
   (c) The digits and letters may be repeated but the number 0 and the vowels must be excluded.
   (d) The digits and letters may not be repeated and the number 0 and the vowels must be excluded.
   (e) The digits may be repeated but must be prime numbers and the letters may be repeated excluding the first five letters and the last five letters.

7. Using the digits 1, 2, 3, 4, 5, 6 and 7, how many two-digit numbers can be formed
   (a) if digits may be repeated.
   (b) if repetition of digits is not allowed.

8. Consider the word NKULULEKO.
   (a) How many ways can the letters of the word be arranged if the letters may be repeated?
   (b) How many ways can the letters of the word be arranged if the letters may not be repeated?
   (c) How many 5 letter words can be formed from the letters of the word if letters may be used more than once?
   (d) How many 5 letter words can be formed from the letters of the word if letters may be used once?

9. Eight athletes take part in a 400 m race. In how many different ways can the first three places be arranged?

10. A head boy, a deputy head boy, a head girl and a deputy head girl must be chosen out of a student council consisting of 18 girls and 18 boys. In how many ways can they be chosen?

11. Twenty different people enter a golf competition. Only the first six of them can win prizes. In how many different ways can the prizes be won?

12. Pool balls are numbered from 1 to 15. You have only one set of pool balls. In how many different ways can you arrange:
    (a) all 15 balls. Write your answer in scientific notation, rounding off to two decimal places.
    (b) four of the 15 balls.

TOPIC: COUNTING AND PROBABILITY (Lesson 4)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	Fundamental Counting Principles: Factorial and Arrangement with Restrictions				
RELATED CONCEPTS/ TERMS/VOCABULARY	Factorial				

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Descending order, $n(E)$

RESOURCES

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Writing a factorial as a product of a natural number and a factorial e.g., 8! As 2 x 4! If 8! Is divided by 4!

METHODOLOGY

Factorial Notation (!) is when a series of descending natural numbers are multiplied together till, we get to 1.

Examples:

1. Eight athletes take part in a 400 m race. In how many different ways can all 8 places in the race be arranged?
   Solution:
   Any of the 8 athletes can come first in the race. Now there are only 7 athletes left to be second, because an athlete cannot be both second and first in the race. After second place, there are only 6 athletes left for the third place, 5 athletes for the fourth place, 4 athletes for the fifth place, 3 athletes for the sixth place, 2 athletes for the seventh place and 1 athlete for the eighth place. Therefore, the number of ways that the athletes can be ordered is as follows: $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40\ 320$

The number of arrangements of n different things taken in n ways is: $n!$

2. How many ways can the letters of the word ACTION be arranged? Letters may not repeat.
   Solution:
   There are 6 letters in the word ACTION. We will start with 6 options and then reduce it by 1 for each digit after that: $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

3. Determine $\frac{4!}{8!}$

$$= \frac{4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1} = \frac{1}{8\times7\times6\times5} = \frac{1}{1680} \quad \text{OR} = \frac{4!}{8\times7\times6\times5\times4!} \ldots$$ Calculator may be used

ARRANGEMENT WITH RESTRICTIONS

RESTRICTIONS ON POSITION

Examples:

1. A code consists of 3 vowels followed by 2 prime numbers less than 10. The code must start with an E and end with 3. How many codes can be formed if a vowel or a number

(a) may be used more than once (may be repeated).

E __ __ __ 3 which is **1(option) x 5 (options) x 5 (options) x 4 (options) x 1 (option)**
VOWELS NUMBERS

$1 \times 5 \times 5 \times 4 \times 1 = 100$

(b) May not be used more than once (may not be repeated).

$1 \times 4 \times 3 \times 3 \times 1 = 36$

2. How many ways can the letters of the word EQUATIONS be arranged if:

(a) the first letter must be E?

$\underline{1} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 1 \times 8! = 40\ 320$

(b) the first letter must be E and the last letters must be S?

$\underline{1} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times \underline{1} = 1 \times 1 \times 7! = 5\ 040$

(c) the second letter must be a Q and the seventh letter must be an O?

$7 \times \underline{1} \times 6 \times 5 \times 4 \times 3 \times \underline{1} \times 2 \times 1 = 1 \times 1 \times 7! = 5\ 040$

(d) the first letter must be a vowel?

$5 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 8! = 201\ 600$

(e) the first and the last letters must be a vowel?

$5 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 4 = 5 \times 4 \times 7! = 100\ 800$

3. If you take the word 'BASSOON', how many letter arrangements can you make if:

(a) repeated letters are treated as different?

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5\ 040$

(b) repeated letters are treated as identical?

If repeated letters are **treated as identical characters**, there are two S's and two O's. This is similar to the previous worked example except now we have more than one letter repeated. **When more than one letter is repeated, we have to divide the total number of possible arrangements by the product of the factorials of the number of times each letter is repeated.**

$$\frac{7!}{2! \times 2!} = 1\ 260$$

(c) the word starts with an O and repeated letters are treated as identical?

If the word starts with an 'O', there are still 6 letters left of which two are S's.

$$\frac{6!}{2!} = 360$$

(d) the word starts and ends with the same letter and repeated letters are treated as identical?

If the word **starts and ends with the same letter**, there are **two possibilities** because there are two S's and two O's.

O __ __ __ __ __ O OR S __ __ __ __ __ S

$\frac{5!}{2!} = 60$ $\frac{5!}{2!} = 60$

Therefore, number of arrangements = 60 +60 = 120

GROUPINGS

4. In how many ways can seven boys of different ages be seated on a bench if:

(a) the youngest boy sits next to the oldest boy?

If **the youngest and oldest boys are treated as a single object, there are six different objects to arrange** so there are 6! different arrangements. However, the youngest and oldest boys can be arranged in **2! different ways** and still be together:

$$2! \times 6! = 1\ 440$$

(b) the youngest and the oldest boys must not sit next to each other?

The arrangements where the youngest and oldest must not sit together is the **total number of arrangements minus the number of arrangements where the oldest and youngest sit together.** Therefore, there are:

$$7! - (2! \times 6!) = 3\ 600$$

5. A group of eight people consists of four couples: four men and four women. Each couple consists of a man and a woman. These eight people are to be seated on a bench. How many possible seating arrangements are there if

(a) All the men have to sit together?

$\underline{M_1M_2M_3M_4}$ $\underline{W_1}$ $\underline{W_2}$ $\underline{W_3}$ $\underline{W_4}$ $4! \times 5! = 2\ 880$
1 2 3 4 5

(b) All the men have to sit together and all the women have to sit together?

$\underline{M_1M_2M_3M_4}$ $\underline{W_1W_2W_3W_4}$ $4! \times 4! \times 2! = 1\ 152$
1 2

(c) Each couple has to sit together?

$\underline{M_1W_1}$ $\underline{M_2W_2}$ $\underline{M_3W_3}$ $\underline{M_4W_4}$ $\underline{M_5W_5}$ $4! \times 2! \times 2! \times 2! \times 2! = 384$

(d) They have to alternate between men and women?

M	W	M	W	M	W	M	W	OR	W	M	W	M	W	M	W	M
4	4	3	3	2	2	1	1		4	4	3	3	2	2	1	1

4! X 4! X 2! = 1 152

ACTIVITIES/ ASSESSMENT

1. Three Mathematics books and five Science books are to be arranged on a shelf.
   (a) In how many ways can these books be arranged if they are treated as separate books?
   (b) In how many ways can these books be arranged if they are treated as identical books?

2. There are two different red books and three different blue books on a shelf.
   (a) In how many different ways can these books be arranged?
   (b) If you want the red books to be together, in how many different ways can the books be arranged?
   (c) If you want all the red books to be together and all the blue books to be together, in how many different ways can the books be arranged?

3. Consider the word WINNING.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) How many word arrangements can be made with this word if the word starts with W and ends with the G?
   (e) How many word arrangements can be made with this word if the word starts with the letter I?
   (f) How many word arrangements can be made with this word if the word ends with the letter N?

4. There are two different Mathematics books, three different Natural Sciences books, two different Life Sciences books and four different accounting books on a shelf. In how many different ways can they be arranged if:
   (a) the order does not matter?
   (b) all the books of the same subject stand together?
   (c) the two Mathematics books stand first?
   (d) the accounting books stand next to each other?

5. Three girls and three boys are to be seated in the front row at assembly.
   How many different ways
   (a) can these six learners be seated?
   (b) can they be seated if two particular learners must be seated together?
   (c) can they be seated if all the girls must be seated together and all the boys must be seated together.
   (d) can they be seated if they have to alternate between girls and boys?

6. A band is planning a concert tour with one performance in each of the cities: Johannesburg, Pretoria, Durban, Bloemfontein, Cape Town and Port Elizabeth.
   How many different way can they can they plan their tour if
   (a) there are no restrictions?
   (b) the first performance must be in Johannesburg and the last performance must be in Cape Town?
   (c) the performance in the non-coastal cities (Johannesburg, Pretoria and Bloemfontein) must be grouped together?

TOPIC: COUNTING AND PROBABILITY (Lesson 5)		Weighting	15 ± 3	Grade	12
Term		Week no.			
Duration	1 hour	Date			
Sub-topics	**Fundamental Counting Principles: Application to Probability Problems**				
RELATED CONCEPTS/ TERMS/VOCABULARY					
PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE					
Probability formula, factorial, arrangement using factorial, complementary events					
RESOURCES					

ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Forgetting to divide by the sample space the outcome of the arrangement
Could not relate complementary events to real life context

METHODOLOGY

The **probability** may be determined **using the fundamental counting principle**. The probability of the event, E, is the total number of arrangements of the event divided by the total number of arrangements of the sample space.

$$Probability = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}} = \frac{n(E) \rightarrow E \text{ is the event}}{n(S) \rightarrow S \text{ is the sample space}}$$

Examples:

1. Every client of a certain bank has a personal identification number (PIN) which consists of four randomly chosen digits from 0 to 9.

   (a) How many PINs can be made if digits can be repeated?

   If digits can be repeated: you have 10 digits to choose from and you have to choose four times,
   Therefore, the number of possible PINs $= 10 \times 10 \times 10 \times 10 = 10^4 = 10\ 000 = n(S)$.

   (b) How many PINs can be made if digits cannot be repeated?

   If digits cannot be repeated: you have 10 digits for your first choice, nine for your second, eight for your third and seven for your fourth.
   Therefore, the number of possible PINs $= 10 \times 9 \times 8 \times 7 = 5\ 040$

   (c) If a PIN is made by selecting four digits at random, and digits can be repeated, what is the probability that the PIN contains at least one eight?

   Let B be the event that at least one eight is chosen. Therefore, the complement of B is the event that no eights are chosen. If no eights are chosen, there are only nine digits to choose from.
   Therefore, $n(\text{not } B) = 9 \times 9 \times 9 \times 9 = 9^4 = 6\ 561$
   The total number of arrangements in the set, as calculated in (a), is 10 000 (sample space).
   Therefore: $P(B) = 1 - P(not\ B)$
   $= 1 - \frac{n(not\ B)}{n(S)} = 1 - \frac{6\ 561}{10\ 000}$

(d) If a PIN is made by selecting four digits at random, and digits cannot be repeated, what is the probability that the PIN contains at least one eight?

Let B be the event that at least one eight is chosen. Therefore, the complement of B, is the event that no eights are chosen.
If no eights are chosen, there are only 9 then 8 then 7 then 6 digits to choose from as we cannot repeat a digit once it is chosen. Therefore, n(not B) $= 9 \times 8 \times 7 \times 6 = 3\,024$
Therefore, $P(B) = 1 - P(not\ B)$
$$= 1 - \frac{n(not\ B)}{n(S)} = 1 - \frac{3\,024}{10\,000} = \frac{169}{625}$$

2. Vehicle number plates in the Free State province begin with three letters of the alphabet (excluding the vowels) followed by any three digits from 0 to 9. All the number plates have FS at the end. Letters and digits may be repeated in a number plate.

(a) How many different number plates are available in the Free State province?

Number of letters: 21 (vowels excluded)
Number of digits: 10
∴number of different number plates $= 21 \times 21 \times 21 \times 10 \times 10 \times 10 = 9\,261\,000 = n(S)$

(b) What is the probability that a number plate stars with an E?

P (begins with E) = 0, vowels are excluded

(c) What is the probability that a number plate has a G as the second letter?

P (second letter G) $= \frac{21\times1\times21\times10\times10\times10}{9\,261\,000} = \frac{1}{21}$

(d) What is the probability that the last digit of a number plate will be a 5?

P (last digit 5) $= \frac{21\times21\times21\times10\times10\times1}{9\,261\,000} = \frac{1}{10}$

(e) What is the probability that a number plate will have only one 5?

First calculate the possible number of different arrangements with only one 5.
The first digit can be 5 OR the second digit can be 5 OR the third digit can be 5.
(1 x 9 x 9) + (9 x 1 x 9) + (9 x 9 x 1) = 81 + 81 + 81

∴ P (only one 5) $= \frac{21\times21\times21\times(81+81+81)}{9\,261\,000} = \frac{243}{1000}$

(f) How many different number plates will be available if the letters are not repeated?

21 x 21 x 19 x 10 x 10 x 10 = 7 980 000

3. Consider the letters of the word NEEDED. The repeated letters are identical.
What is the probability that the word arrangement formed will start and end with the same letter?

$n(S) = \frac{6!}{3!\times2!} = 60$

Start and end with a D: D ----- ----- ----- ----- D $= \frac{4!}{3!} = 4$
Start and end with an E: E ----- ----- ----- ----- E $= \frac{4!}{2!} = 12$
∴ P(start and end with the same letter) $= \frac{4+12}{60} = \frac{16}{60} = \frac{4}{15}$

ACTIVITIES/ ASSESSMENT

1. A photographer places eight chairs in a row in his studio in order to take a photograph of the debating team. The team consists of three boys and five girls.

(a) In how many ways can the debating team be seated?
(b) What is the probability that a particular boy and a particular girl sit next to each other?

2. Seven boys and six girls are to be seated randomly in a row. What is the probability that:

(a) the row has a boy at each end?
(b) the row has boys and girls sitting in alternate positions?
(c) two particular girls land up sitting next to each other?
(d) all the girls sit next to each other?

3. A password is formed using three letters of the alphabet, excluding the letters A, E, I, O and U and using any three digits, excluding 0. The numbers and letters can be repeated. Calculate the probability that a password, chosen at random:

(a) starts with the letter B and ends with the number 4.
(b) has exactly one B.
(c) has at least one 4.

4. Consider the letters of the word MATHEMATICIAN. The repeated letters are identical. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:

(a) start and end with the same letter?
(b) end with the letter N?

5. A number plate is made up of three letters of the alphabet (excluding F and S) followed by three digits from 0 to 9. The numbers and letters can be repeated.
Calculate the probability that a randomly chosen number plate:

(a) starts with the letter D and ends with the digit 3.
(b) has precisely one D.
(c) contains at least one 5.

6. Determine the probability of getting a ten-digit cell-phone number if the first digit is even, none of the first three must be 0 and none of the digits may be repeated.

7. Vehicle registration number plate in Gauteng province begin with two letters of alphabet followed by two digits (0 to 9) and then two letters of the alphabet. Vowels are excluded. All the number plates have GP at the end, e.g., BD25CFGP. Letters and digits may be repeated.

(a) How many different number plates are available for vehicle registration in Gauteng province?
(b) What is the probability that a number plate has C as the first letter?

(c) What is the probability that a number plate has four Cs?
(d) What is the probability that the second digit on a number plate will be a 3?
(e) What is the probability that a number plate will have only one 3?
(f) How many different number plates will be available if the numbers may not repeat?

8. In the 13-digit identification (ID) numbers of South African citizens:
- The first six numbers are the birth date of the person in YYMMDD format.
- The next four digits indicate gender, with 5000 and above being male and 0001 to 4999 being female.
- The next number is the country ID; 0 is South Africa and 1 is not.
- The second last number used to be a racial identifier but it is now 8 for everybody.
- The last number is a control digit, which verifies the rest of the number.

Assume that the control digit is a randomly generated digit from 0 to 9 and ignore the fact that leap years have an extra day.
a) Calculate the total number of possible ID numbers.
b) Calculate the probability that a randomly generated ID number is of a South African male born during the 1980s. Write your answer correct to two decimal places.

9. The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are used to generate a four-digit codes randomly. Any digit may be used any number of times. Calculate the probability that

(a) the code will consist only of even digits
(b) the digits of the code will all be different
(c) the code will consist of different even digits

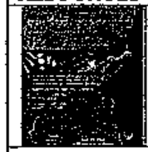
10. The digits 1 to 7 are used to make four-digit codes.
(a) How many different codes are possible if digits may not be repeated?
(b) What is the probability of having code less than 4 000 and divisible by 5 if digits may not be repeated?

TOPIC: COUNTING AND PROBABILITY (Lesson 6)		Weighting	15 ± 3	Grade	12
Term		**Week no.**			
Duration	1 hour	**Date**			
Sub-topics	**Fundamental Counting Principles: Application to Probability Problems**				
RELATED CONCEPTS/ TERMS/VOCABULARY					

PRIOR-KNOWLEDGE/ BACKGROUND KNOWLEDGE

Probability formula, factorial, arrangement using factorial, complementary events

RESOURCES



ERRORS/MISCONCEPTIONS/PROBLEM AREAS

Forgetting to divide by the sample space the outcome of the arrangement
Could not relate complementary events to real life context

METHODOLOGY

1. A cutlery set has 3 spoons, 4 knives and 5 forks. Each of the spoons, forks and knives are similar in design. Calculate:

(a) how many different arrangements of the set there are

Arrangement $= \frac{12!}{3!\times 4!\times 5!} = 27\ 720 = n(S)$

(b) how many different arrangements there are, so that each of the three cutlery types are together

Arrangement $= 3! = 6$

(c) the probability of selecting an arrangement where there is no grouping of cutlery items at all.

P (no grouping of cutlery) = 1 − P (grouping of cutlery)
$= 1 - \frac{6}{27\ 720} = \frac{27\ 714}{27\ 720} = \frac{4619}{4620} = 0{,}9997835498$

(d) The probability of selecting an arrangement where two forks are placed first, and two forks are placed last.

Arrangement $= \frac{8!}{3!\times 4!} = 280$

Probability $= \frac{280}{27\ 720} = \frac{1}{99} = 0{,}0101010101$

ACTIVITIES/ ASSESSMENT

1. Ryan packs his suitcase for his holiday with 3caps, 5 shirts, 3 pairs of jeans and 2 pairs of takkies:

(a) How many different outfits can he put together if when he dresses, he must wear a shirt, a pair of jeans, a pair of takkies and a cap?

Ryan reaches his destination and hangs all the 5 shirts and the three pairs of jeans (each item separately) on a different hanger, on the rail in the cupboard.

(b) How many different arrangements are possible?

(c) What is the probability that the shirts are all hanging together next to each other in the cupboard?

(d) While on holiday Ryan decides to buy a pair of sandals in addition to his outfit items, on a given day what is the probability that Ryan will wear a pair of sandals or a pair of takkies?

2. Eight different pairs of jeans and 5 different shirts hang on a rail.

(a) In how many different ways can the clothes be arranged on the rail?

(b) In how many ways can the clothing be arranged if all the jeans hang together and all the shirts hang together?

(c) What is the probability, correct to three decimal places, of the clothing being arranged on the rail with a shirt at one end and a pair of jeans at the other?

3. Tarryn has five different skirts, four different tops and three pairs of shoes.

(a) Assuming that all the colours complement each other, how many different outfits can she put together?

(b) In how many different ways can the clothing be arranged if tops are together?

(c) What is the probability that there is no grouping of skirts at all?

(d) Calculate the probability that the arrangement starts with a skirt and end with a pair of shoes.

TEST 2: PROBABILITY

MARKS: 25

DURATION: 30 MIN

INSTRUCTIONS

1. Answer ALL questions
2. Unless stated or otherwise, round off answers correct to TWO decimal places
3. You may use an approved scientific calculator

QUESTION 1 [9 Marks]

There are 7 different shirts and 4 different pairs of trousers in a cupboard. The clothes have to be hung on the rail.

1.2 In how many different ways can the clothes be arranged on the rail? (3)

1.2 In how many different ways can the clothes be arranged if all the shirts are to be hung next to each another and the pairs of trousers are to be hung next to each another on the rail? (3)

1.2 What is the probability that a pair of trousers will hang at the beginning of the rail and a shirt will hang at the end of the rail? (3)

QUESTION 2 [1 Marks]

2.1 In Gauteng number plates are designed with 3 alphabetical letters, excluding the 5 vowels, next to one another and then any 3 digits, from 0 to 9, next to one another. The GP is constant in all Gauteng number plates, for example, BBV 023 GP. Letters and digits may be repeated in a number plate.

2.1.1 How many unique number plates are available? (2)

2.1.2 What is the probability that a car's number plate will start with a Y? (3)

2.1.3 What is the probability that a car's number plate will contain only one 7? (3)

2.1.4 How many unique number plates will be available if the letters and numbers are not repeated? (3)

2.2 Consider the word CONCENTRATION (Repeating letters are regarded as identical)

2.2.1 How many different arrangements of the letters of this word are possible? (2)

2.2.2 If the letters of this word are arranged randomly, what is the probability that the word will start and end with the same letter? (3)